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Generalized interpolating refinable function vectors

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ABSTRACT

Interpolating scalar refinable functions with compact support are of interest in several applications such as sampling theory, signal processing, computer graphics, and numerical algorithms. In this paper, we shall generalize the notion of interpolating scalar refinable functions to compactly supported interpolating d-refinable function vectors with any multiplicity r and dilation factor d. More precisely, we are interested in a d-refinable function vector $\phi = [\phi_1, \dots, \phi_r]^T$ such that ϕ is an $r \times 1$ column vector of compactly supported continuous functions with the following interpolation property

$$\phi_\ell\left(\frac{m}{r} + k\right) = \delta_k \delta_{\ell-1-m}, \qquad \forall\, k \in \mathbb{Z}, m = 0, \dots, r-1, \ell = 1, \dots, r,$$

where $\delta_0 = 1$ and $\delta_k = 0$ for $k \neq 0$. Now for any function $f : \mathbb{R} \mapsto \mathbb{C}$, the function f can be interpolated and approximated by

$$\tilde{f} = \sum_{\ell=1}^{r} \sum_{k\in\mathbb{Z}} f\left(\frac{\ell-1}{r} + k\right) \phi_\ell(\cdot - k)$$
$$= \sum_{k\in\mathbb{Z}} \left[f(k), f\left(\frac{1}{r} + k\right), \dots, f\left(\frac{r-1}{r} + k\right)\right] \phi(\cdot - k).$$

Since ϕ is interpolating, $\tilde{f}(k/r) = f(k/r)$ for all $k \in \mathbb{Z}$, that is, $\tilde{f}$ agrees with f on $r^{-1}\mathbb{Z}$. Moreover, for $r \geqslant 2$ or $d > 2$, such interpolating refinable function vectors can have the additional orthogonality property: $\langle \phi_\ell(\cdot - k), \phi_{\ell'}(\cdot - k')\rangle = r^{-1}\delta_{\ell-\ell'}\delta_{k-k'}$ for all $k, k' \in \mathbb{Z}$ and $1 \leqslant \ell, \ell' \leqslant r$, while it is well-known that there does not exist a compactly supported scalar 2-refinable function with both the interpolation and orthogonality properties simultaneously. In this paper, we shall characterize both interpolating d-refinable function vectors and orthogonal interpolating d-refinable function vectors in terms of their masks. We shall study their approximation properties and present a family of interpolatory masks, for compactly supported interpolating d-refinable function vectors, of type (d, r) with increasing orders of sum rules. To illustrate the results in this paper, we also present several examples of compactly supported (orthogonal) interpolating refinable function vectors and biorthogonal multiwavelets derived from such interpolating refinable function vectors.

1. Introduction and motivation

Wavelet analysis has many applications in a broad range of scientific areas such as signal denoising, image processing, computer graphics, and numerical algorithms. In general, a wavelet is derived from a d-refinable function vector via a

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multiresolution analysis. Throughout this paper, d denotes a dilation factor which is just an integer with $|d| > 1$; for simplicity of presentation, we further assume in this paper that $d > 1$, and the results for a negative dilation factor can be obtained similarly. We say that $\phi := [\phi_1, \ldots, \phi_r]^{\mathsf{T}} : \mathbb{R} \mapsto \mathbb{C}^{r\times 1}$ is a *d-refinable function vector* if

$$\phi(x) = d \sum_{k\in\mathbb{Z}} a(k)\phi(dx-k), \quad \text{a.e. } x \in \mathbb{R}, \tag{1.1}$$

where $a : \mathbb{Z} \mapsto \mathbb{C}^{r\times r}$ is a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$, called the *(matrix) mask* with *multiplicity* r for the refinable function vector ϕ. When the multiplicity $r = 1$, the function vector ϕ is simply a scalar function and therefore for the case $r = 1$, ϕ is called *a scalar d-refinable function*. In the frequency domain, the matrix refinement equation in Eq. (1.1) can be rewritten as

$$\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi), \qquad \xi \in \mathbb{R}, \tag{1.2}$$

where $\hat{a}$ is the *Fourier series* of the mask a given by

$$\hat{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}, \qquad \xi \in \mathbb{R}, \tag{1.3}$$

where i denotes the imaginary unit such that $i^2 = -1$. The Fourier transform $\hat{f}$ of $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$ and can be extended to square integrable functions and tempered distributions. For simplicity, we also call $\hat{a}$ the mask for ϕ. A wavelet ψ is generally derived from the refinable function vector ϕ via

$$\hat{\psi}(d\xi) := \hat{b}(\xi)\hat{\phi}(\xi), \qquad \xi \in \mathbb{R} \tag{1.4}$$

for some $r\times r$ matrix $\hat{b}$ of 2π-periodic trigonometric polynomials. According to various requirements of problems in different applications, different desirable properties of the wavelet function vector ψ and the refinable function vector ϕ are needed. Since the wavelet ψ is derived from the refinable function vector ϕ, many properties of ψ are generally determined by those of ϕ and therefore, the construction of desirable refinable function vectors ϕ plays an important role in wavelet analysis. Two particular important families of scalar refinable functions are interpolating refinable functions and orthogonal refinable functions. We say that a compactly supported d-refinable function ϕ with mask a is *interpolating* if the function ϕ is continuous and $\phi(k) = \delta_k$ for all $k \in \mathbb{Z}$, where δ denotes the *Dirac sequence* such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \neq 0$. We say that a compactly supported d-refinable function ϕ with mask a is *orthogonal* if $\int_{\mathbb{R}} \phi(x-k)\overline{\phi(x)}\,dx = \delta_k$ for all $k \in \mathbb{Z}$. By the refinement equation (1.1), one can easily see that the mask a of a scalar interpolating d-refinable function must be *an interpolatory mask* with the dilation factor d: $a(dk) = d^{-1}\delta_k$ for all $k \in \mathbb{Z}$, or equivalently, $\sum_{m=0}^{d-1} \hat{a}(\xi + 2\pi m/d) = 1$. Similarly, the mask a for an orthogonal d-refinable function must be an *orthogonal mask* with dilation factor d: $\sum_{m=0}^{d-1} |\hat{a}(\xi+2\pi m/d)|^2 = 1$. A family of interpolatory masks $\{b_n\}_{n=1}^{\infty}$ with the dilation factor 2 has been obtained in [3] such that the interpolatory mask b_n is supported on $[1-2n, 2n-1]$ and $\widehat{b_n}(\xi)$ contains the factor $(1+e^{-i\xi})^{2n}$. Also $\widehat{b_n}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$ [3]. It is well-known [2] that the Daubechies orthogonal mask a_n of order n is closely related to the mask b_n via $|\widehat{a_n}(\xi)|^2 = \widehat{b_n}(\xi)$. That is, the Daubechies orthogonal mask $\widehat{a_n}$ of order n can be obtained from the interpolatory mask $\widehat{b_n}$ via the Riesz lemma.

Motivated by the wavelet applications in sampling theorems in signal processing, it is desirable to have compactly supported refinable functions that are both interpolating and orthogonal [14–16]. However, it has been observed in [14–16] that for the dilation factor $d = 2$, it is impossible to have a compactly supported scalar 2-refinable function such that it is both interpolating and orthogonal. In order to achieve both interpolation and orthogonality, it is natural to consider either the dilation factor $d > 2$ or the multiplicity $r > 1$. For $d = r = 2$, several interesting examples have been obtained in [14–16] to show that one indeed can achieve both the interpolation and orthogonality properties of a refinable function vector simultaneously. This paper is largely motivated in [14–16] on interpolating 2-refinable function vectors with multiplicity 2. In this paper, we would like to consider the general case of interpolating d-refinable function vectors and investigate their properties. More precisely, we are interested in a family of d-refinable function vectors with the following interpolation property. We say that a refinable function vector $\phi = [\phi_1, \ldots, \phi_r]^{\mathsf{T}} : \mathbb{R} \mapsto \mathbb{C}^{r\times 1}$ is *interpolating* if ϕ is continuous and

$$\phi_\ell\left(\frac{m}{r}+k\right) = \delta_k\delta_{\ell-1-m}, \qquad \forall\, k \in \mathbb{Z}, m = 0, \ldots, r-1, \ell = 1, \ldots, r. \tag{1.5}$$

For a function $f : \mathbb{R} \mapsto \mathbb{C}$, the function f can be interpolated and approximated by

$$\tilde{f} = \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} f\left(\frac{\ell-1}{r}+k\right)\phi_\ell(\cdot-k) = \sum_{k\in\mathbb{Z}}\left[f(k), f\left(\frac{1}{r}+k\right), \ldots, f\left(\frac{r-1}{r}+k\right)\right]\phi(\cdot-k).$$

Since ϕ is interpolating, $\tilde{f}(k/r) = f(k/r)$ for all $k \in \mathbb{Z}$, that is, $\tilde{f}$ agrees with f on $r^{-1}\mathbb{Z}$. The contributions of this paper lie in three parts. First, we generalize the notion of interpolating refinable function vectors from the special case $d = r = 2$ in [14–16] to the most general case $d > 1$ and $r \geqslant 1$. We notice that the papers [14–16] mostly concentrate on the design of some masks for orthogonal interpolating 2-refinable function vectors with multiplicity 2 and only some mathematical analysis has been provided in [16] for the case of orthogonal interpolating 2-refinable function vectors with multiplicity 2. Second, we provide in this paper a complete mathematical analysis for such interpolating refinable function vectors and orthogonal interpolating refinable function vectors. Third, for any dilation factor d, and multiplicity r, we propose a family of interpolatory masks for interpolating d-refinable function vectors with multiplicity r and with increasing orders of sum rules.

The structure of this paper is as follows. In Section 2, we shall characterize both compactly supported interpolating d-refinable function vectors and orthogonal interpolating d-refinable function vectors in terms of their masks. In Section 2, we also study the sum rule structure of the interpolatory masks of type (d, r) for interpolating d-refinable function vectors with multiplicity r, which will play a central role in our construction of interpolatory masks of type (d, r) with increasing orders of sum rules in Section 3. In Section 3, for any dilation factor d and multiplicity r, we shall construct a family of interpolatory masks of type (d, r) with increasing orders of sum rules. Some examples of (orthogonal) interpolating refinable function vectors will be presented in Section 4. Next, in Section 5, we shall discuss biorthogonal multiwavelets derived from interpolating refinable function vectors via the CBC (coset by coset) algorithm in [1,5,6] and some examples of biorthogonal multiwavelets will be presented in Section 5. We complete the paper by some conclusions and remarks in Section 6.

2. Characterization of interpolating refinable function vectors

In this section, we shall generalize interpolating 2-refinable function vectors with multiplicity 2 in [14–16] to the general setting of any dilation factors and multiplicities. Based on [6,8], we shall provide a complete characterization for a compactly supported d-refinable function vector with a finitely supported mask with multiplicity r in terms of its mask. We also study the approximation property and sum rules of such generalized interpolating refinable function vectors. As a consequence, we obtain a criterion for a compactly supported interpolating refinable function vector whose shifts are orthogonal.

Throughout the paper, for a smooth function f, $f^{(j)}$ denotes the jth derivative of the function f. For $0 < \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$, we say that $f \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}))$ if there is a constant C_f such that $\|f - f(\cdot - h)\|_{L_p(\mathbb{R})} \leqslant C_f h^\alpha$ for all $h > 0$. The L_p smoothness of a function $f \in L_p(\mathbb{R})$ is measured by

$$\nu_p(f) := \sup\{n + \alpha \ : \ n \in \mathbb{N} \cup \{0\}, 0 < \alpha \leqslant 1, f^{(n)} \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}))\}. \tag{2.1}$$

For a function vector $\phi = [\phi_1, \ldots, \phi_r]^{\mathrm{T}}$, we denote $\nu_p(\phi) := \min_{1\leqslant \ell \leqslant r} \nu_p(\phi_\ell)$.

By $(\ell_0(\mathbb{Z}))^{m\times n}$ we denote the linear space of all finitely supported sequences of $m \times n$ matrices on $\mathbb{Z}$. Similarly, $u \in (\ell_p(\mathbb{Z}))^{m\times n}$ for $1 \leqslant p \leqslant \infty$ means that u is a sequence of $m \times n$ matrices on $\mathbb{Z}$ and $\|u\|_{(\ell_p(\mathbb{Z}))^{m\times n}} := (\sum_{k\in\mathbb{Z}} \|u(k)\|^p)^{1/p} < \infty$ for $1 \leqslant p < \infty$ and $\|u\|_{(\ell_\infty(\mathbb{Z}))^{m\times n}} := \sup_{k\in\mathbb{Z}} \|u(k)\|$, where $\|\cdot\|$ denotes any matrix norm on $m \times n$ matrices.

Before proceeding further, let us recall a quantity $\nu_2(a, d)$ from [8], which will play an important role in our investigation of interpolating refinable function vectors. The convolution of two sequences u and v is defined to be

$$[u * v](j) := \sum_{k\in\mathbb{Z}} u(k)v(j - k), \qquad u \in (\ell_0(\mathbb{Z}))^{r\times m}, v \in (\ell_0(\mathbb{Z}))^{m\times n}.$$

Clearly, $\widehat{u * v} = \hat{u}\hat{v}$. For a matrix mask a with multiplicity r, we say that a satisfies the *sum rules* of order κ with a dilation factor d [6,8] if there exists a sequence $y \in (\ell_0(\mathbb{Z}))^{1\times r}$ such that $\hat{y}(0) \neq 0$ and

$$[\hat{y}(d\cdot)\hat{a}(\cdot)]^{(j)}(2\pi m/d) = \delta_m \hat{y}^{(j)}(0) \qquad \forall\, j = 0, \ldots, \kappa - 1 \quad \text{and} \quad m = 0, \ldots, d - 1. \tag{2.2}$$

For $y \in (\ell_0(\mathbb{Z}))^{1\times r}$ and a positive integer κ, as in [8], we define the space $\mathcal{V}_{\kappa,y}$ by

$$\mathcal{V}_{\kappa,y} := \{v \in (\ell_0(\mathbb{Z}))^{r\times 1} \ : \ [\hat{y}(\cdot)\hat{v}(\cdot)]^{(j)}(0) = 0 \ \forall\, j = 0, \ldots, \kappa - 1\}. \tag{2.3}$$

By convention, $\mathcal{V}_{0,y} := (\ell_0(\mathbb{Z}))^{r\times 1}$. Note that the above equations in Eqs. (2.2) and (2.3) depend only on the values $\hat{y}^{(j)}(0)$, $j = 0, \ldots, \kappa - 1$. For a mask a with multiplicity r, a sequence $y \in (\ell_0(\mathbb{Z}))^{1\times r}$ and a dilation factor d, we define

$$\rho_\kappa(a, d, y, p) := \sup\left\{\limsup_{n\to\infty} \|a_n * v\|^{1/n}_{(\ell_p(\mathbb{Z}))^{r\times 1}} \ : \ v \in \mathcal{V}_{\kappa,y}\right\}, \qquad \kappa \in \mathbb{N} \cup \{0\}, \tag{2.4}$$

where $\widehat{a_n}(\xi) := \hat{a}(d^{n-1}\xi) \cdots \hat{a}(d\xi)\hat{a}(\xi)$. For $1 \leqslant p \leqslant \infty$, define

$$\rho(a, d, p) := \inf\{\rho_\kappa(a, d, y, p) \ : \ (2.2) \text{ holds for some } \kappa \in \mathbb{N} \cup \{0\} \text{ and some } y \in (\ell_0(\mathbb{Z}))^{1\times r} \text{ with } \hat{y}(0) \neq 0\}. \tag{2.5}$$

As in [8, Page 61], we define the following important quantity:

$$\nu_p(a, d) := 1/p - 1 - \log_{|d|} \rho(a, d, p), \qquad 1 \leqslant p \leqslant \infty. \tag{2.6}$$

In the above definition of $\rho(a, d, p)$, it seems that the sequences y (more precisely, the vectors $\hat{y}^{(j)}(0), j = 0, \ldots, \kappa - 1$) are not uniquely determined. Up to a scalar multiplicative constant, we point out that the vectors $\hat{y}^{(j)}(0), j \in \mathbb{N} \cup \{0\}$ are quite often uniquely determined [8, Proposition 3.1].

The above quantity $\nu_p(a, d)$ plays a very important role in characterizing the convergence of a vector cascade algorithm in a Sobolev space and in characterizing the L_p smoothness of a refinable function vector. It was showed in [8, Theorem 4.3] (also see [9, Theorem 3.1]) that the vector cascade algorithm associated with mask a and dilation factor d converges in the Sobolev space $W_p^k(\mathbb{R}) := \{f \in L_p(\mathbb{R}) \ : \ f^{(j)} \in L_p(\mathbb{R}) \ \forall\, j = 0, \ldots, k\}$ if and only if $\nu_p(a, d) > k$. In general, $\nu_p(a, d)$ provides a lower bound for the L_p smoothness exponent of a refinable function vector ϕ with mask a and dilation factor d, that is, $\nu_p(a, d) \leqslant \nu_p(\phi)$ always holds. Moreover, if the shifts of the refinable function vector ϕ associated with mask a and dilation factor d are stable in $L_p(\mathbb{R})$, then $\nu_p(\phi) = \nu_p(a, d)$ (see [8] and [9, Theorem 4.1]). That is, in this case, $\nu_p(a, d)$ indeed characterizes the L_p smoothness exponent of a refinable function vector ϕ with mask a and dilation factor d. Interested readers should consult [4–8,11] and many references therein for more details on the convergence of vector cascade algorithms, smoothness of refinable function vectors and interpolating refinable functions.

Since the stability and linear independence of a refinable function vector will be needed in our proof of the main results in this paper, let us recall their definitions here. For an $r \times 1$ vector $\phi := [\phi_1, \dots, \phi_r]^T$ of compactly supported functions in $L_p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$, we say that the shifts of ϕ are *stable* in $L_p(\mathbb{R})$ if there exist two positive constants C_1 and C_2 such that

$$C_1 \sum_{\ell=1}^{r} \sum_{k \in \mathbb{Z}} |c_\ell(k)|^p \leqslant \left\| \sum_{\ell=1}^{r} \sum_{k \in \mathbb{Z}} c_\ell(k) \phi_\ell(\cdot - k) \right\|_{L_p(\mathbb{R})}^p \leqslant C_2 \sum_{\ell=1}^{r} \sum_{k \in \mathbb{Z}} |c_\ell(k)|^p$$

for all finitely supported sequences $c_1, \dots, c_r$ in $\ell_0(\mathbb{Z})$. For a compactly supported function vector $\phi = [\phi_1, \dots, \phi_r]^T$, we say that the shifts of ϕ are *linearly independent* if for any sequences $c_1, \dots, c_r : \mathbb{Z} \mapsto \mathbb{C}$ such that

$$\sum_{\ell=1}^{r} \sum_{k \in \mathbb{Z}} c_\ell(k) \phi_\ell(x - k) = 0, \qquad a.e.\ x \in \mathbb{R}, \tag{2.7}$$

then one must have $c_\ell(k) = 0$ for all $\ell = 1, \dots, r$ and $k \in \mathbb{Z}$. Note that since ϕ is compactly supported, for any fixed $x \in \mathbb{R}$, the summation in $\sum_{\ell=1}^{r} \sum_{k \in \mathbb{Z}} c_\ell(k) \phi_\ell(x - k) = 0$ is in fact finite. For a compactly supported function vector $\phi = [\phi_1, \dots, \phi_r]^T$ in $L_p(\mathbb{R})$, it is known in [13] that the shifts of ϕ are stable in $L_p(\mathbb{R})$ (or linearly independent) if and only if $\operatorname{span}\{\hat{\phi}(\xi + 2\pi k) : k \in \mathbb{Z}\} = \mathbb{C}^{r \times 1}$ for all $\xi \in \mathbb{R}$ (or for all $\xi \in \mathbb{C}$). Therefore, if the shifts of a compactly supported function vector ϕ in $L_p(\mathbb{R})$ are linearly independent, then the shifts of ϕ must be stable in $L_p(\mathbb{R})$.

For $1 \leqslant \ell \leqslant r$, let E_ℓ denote the ℓth unit coordinate column vector in $\mathbb{R}^{r \times 1}$, that is, E_ℓ is the $r \times 1$ column vector whose only nonzero entry is located at the ℓth component with value 1.

Now we have the following result characterizing a compactly supported interpolating d-refinable function vector in terms of its mask.

Theorem 2.1. *Let d and r be positive integers such that $d > 1$. Let $a : \mathbb{Z} \mapsto \mathbb{C}^{r \times r}$ be a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$. Let $\phi = [\phi_1, \dots, \phi_r]^T$ be a compactly supported d-refinable function vector such that $\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. Then ϕ is interpolating, that is, ϕ is a continuous function vector and (1.5) holds if and only if the following statements hold:*

(i) $[1, \dots, 1]\hat{\phi}(0) = 1$ *(This is a normalization condition on the refinable function vector ϕ).*

(ii) *a is an interpolatory mask of type (d, r):* $[1, \dots, 1]\hat{a}(0) = [1, \dots, 1]$ *and*

$$a(R_\ell + dj)E_{Q_\ell+1} = d^{-1}\delta_j E_{\ell+1}, \qquad \forall\, j \in \mathbb{Z};\ \ell = 0, 1, \dots, r-1, \tag{2.8}$$

where $R_\ell \in \mathbb{Z}$ and $Q_\ell \in \{0, 1, \dots, r-1\}$ are defined to be

$$R_\ell := \left\lfloor \frac{d\ell}{r} \right\rfloor \quad \text{and} \quad Q_\ell := r\left(\frac{d\ell}{r} - \left\lfloor \frac{d\ell}{r} \right\rfloor\right) = d\ell \bmod r, \qquad \ell = 0, \dots, r-1, \tag{2.9}$$

where $\lfloor x \rfloor$ denotes the largest integer that is not larger than x.

(iii) $\nu_\infty(a, d) > 0$.

Proof. Necessity: Suppose that (1.5) holds. Evidently, (1.5) can be equivalently rewritten as

$$\phi(\ell/r + j) = \delta_j E_{\ell+1} \qquad \forall\, j \in \mathbb{Z} \quad \text{and} \quad \ell = 0, \dots, r-1. \tag{2.10}$$

By the definition of R_ℓ and Q_ℓ in (2.9), we observe that

$$\frac{d\ell}{r} = \frac{Q_\ell}{r} + R_\ell, \qquad \ell = 0, \dots, r-1. \tag{2.11}$$

Now it follows from the refinement equation (1.1) and (2.10) that for $\ell = 0, \dots, r-1$,

$$\begin{aligned} \phi\left(\frac{\ell}{r} + j\right) &= d \sum_{k \in \mathbb{Z}} a(k) \phi\left(\frac{d\ell}{r} + dj - k\right) = d \sum_{k \in \mathbb{Z}} a(k) \phi\left(\frac{Q_\ell}{r} + R_\ell + dj - k\right) \\ &= d \sum_{k \in \mathbb{Z}} a(k) \delta_{R_\ell + dj - k} E_{Q_\ell+1} = d a(R_\ell + dj) E_{Q_\ell+1}. \end{aligned}$$

That is, by (2.10) again, we deduce that

$$\delta_j E_{\ell+1} = \phi\left(\frac{\ell}{r} + j\right) = d a(R_\ell + dj) E_{Q_\ell+1}.$$

Hence, (2.8) holds. If ϕ is interpolating, that is (2.10) holds, then it is easy to check that the shifts of ϕ must be linearly independent. In fact, suppose that (2.7) holds for some sequences $c_1, \dots, c_r$ on $\mathbb{Z}$. Since $\phi = [\phi_1, \dots, \phi_r]^T$ is interpolating, ϕ

must be continuous and therefore, (2.7) must hold for all $x \in \mathbb{R}$. By the interpolation property of ϕ in (2.10), setting $x = m/r+j$ with $j \in \mathbb{Z}$ and $m = 0, \ldots, r-1$ in (2.7), we have

$$0 = \sum_{\ell=1}^{r} \sum_{k\in\mathbb{Z}} c_\ell(k)\phi_\ell(m/r+j-k) = \sum_{\ell=1}^{r} \sum_{k\in\mathbb{Z}} c_\ell(k)\delta_{\ell-1-m}\delta_{j-k} = c_{m+1}(j).$$

That is, we must have $c_{m+1}(j) = 0$ for all $m = 0, \ldots, r-1$ and $j \in \mathbb{Z}$. So, the shifts of ϕ are linearly independent. That is, $\operatorname{span}\{\hat{\phi}(\xi + 2\pi k) : k \in \mathbb{Z}\} = \mathbb{C}^{r\times 1}$ for all $\xi \in \mathbb{C}$ [13]. Since $\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ and ϕ is continuous, it follows from [8, Proposition 3.1] that 1 must be a simple eigenvalue of $\hat{a}(0)$ and all its other eigenvalues of $\hat{a}(0)$ are less than 1 in modulus. Consequently, we must have $\hat{\phi}(0) \neq 0$; otherwise, ϕ must be identically zero by $\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. Since 1 is a simple eigenvalue of $\hat{a}(0)$ and $\hat{a}(0)\hat{\phi}(0) = \hat{\phi}(0)$ with $\hat{\phi}(0) \neq 0$, using the Jordan canonical form of the matrix $\hat{a}(0)$, we see that there exists a (unique) nonzero row vector y such that $y\hat{a}(0) = y$ and $y\hat{\phi}(0) = 1$. Now by $\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ and [8, Proposition 3.2], we must have $y\hat{\phi}(2\pi k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$. That is, combining with $y\hat{\phi}(0) = 1$, we have $y\sum_{k\in\mathbb{Z}}\phi(x-k) = 1$ for all $x \in \mathbb{R}$, since ϕ is continuous. Taking $x = 0/r, 1/r, \ldots, (r-1)/r$, since ϕ is interpolating, we deduce that

$$y = y\left[\phi(0), \phi\left(\frac{1}{r}\right), \ldots, \phi\left(\frac{r-1}{r}\right)\right] = y\left[\sum_{k\in\mathbb{Z}}\phi\left(\frac{0}{r}-k\right), \ldots, \sum_{k\in\mathbb{Z}}\phi\left(\frac{r-1}{r}-k\right)\right] = [1, \ldots, 1].$$

That is, $y = [1, \ldots, 1]$. It follows from $y\hat{a}(0) = y$ and $y\hat{\phi}(0) = 1$ that $[1, \ldots, 1]\hat{a}(0) = [1, \ldots, 1]$ and $[1, \ldots, 1]\hat{\phi}(0) = 1$. Therefore, both (i) and (ii) have been verified.

Since ϕ is continuous, it follows from the linear independence of ϕ that the shifts of ϕ are stable in $C(\mathbb{R})$. Since ϕ is continuous, in [8, Corollary 5.1], we must have $\nu_\infty(a, d) > 0$. So, (iii) holds.

Sufficiency: Let $y := [1, \ldots, 1]$ and $h(x) := \max\{0, 1-|x|\}$ the hat function. Define a function vector $f(x) := [h(rx), h(rx-1), \ldots, h(rx-(r-1))]^{\mathrm{T}}$. Then it is evident that $\hat{f}(\xi) = r^{-1}\hat{h}(\xi/r)[1, \mathrm{e}^{-\mathrm{i}\xi/r}, \ldots, \mathrm{e}^{-\mathrm{i}\xi(r-1)/r}]^{\mathrm{T}}$. Note that $\hat{h}(\xi) = 4\xi^{-2}\sin^2(\xi/2)$. Therefore, $\hat{h}(0) = 1$ and $\hat{h}(2\pi k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$. Now by calculation, we have $y\hat{f}(0) = r^{-1}\hat{h}(0)[1, \ldots, 1][1, \ldots, 1]^{\mathrm{T}} = 1$ and for $k \in \mathbb{Z}\setminus\{0\}$,

$$\begin{aligned} y\hat{f}(2\pi k) &= r^{-1}\hat{h}(2\pi k/r)[1, \ldots, 1][1, \mathrm{e}^{-\mathrm{i}2\pi k/r}, \ldots, \mathrm{e}^{-\mathrm{i}2\pi k(r-1)/r}]^{\mathrm{T}} \\ &= \frac{1}{r}\hat{h}\left(\frac{2\pi k}{r}\right)\left[\sum_{\ell=0}^{r-1}\mathrm{e}^{-\mathrm{i}\frac{2\pi k\ell}{r}}\right] = \begin{cases} 0, & k \in \mathbb{Z}\setminus r\mathbb{Z}, \\ \hat{h}(2\pi k/r) = 0, & k \in r\mathbb{Z}\setminus\{0\}. \end{cases} \end{aligned}$$

By our assumption in (ii), we have $y\hat{a}(0) = [1, \ldots, 1]\hat{a}(0) = [1, \ldots, 1] = y$. Therefore, f is a suitable initial function vector [8] in $C(\mathbb{R})$ and f is a continuous function vector. Now by our assumption in (iii), we have $\nu_\infty(a, d) > 0$. Therefore, in [8, Theorem 4.3], $\nu_\infty(a, d) > 0$ implies that the vector cascade algorithm associated with mask a and dilation factor d converges in $C(\mathbb{R})$. More precisely, there exists a compactly supported continuous function vector f_∞ such that $\lim_{n\to\infty}\|f_n - f_\infty\|_{(C(\mathbb{R}))^{r\times 1}} = 0$, where the cascade sequence f_n, $n \in \mathbb{N}\cup\{0\}$ is defined to be $f_0 := f$ and

$$f_n := Q_{a,d}f_{n-1} := d\sum_{k\in\mathbb{Z}} a(k)f_{n-1}(d\cdot -k), \qquad n \in \mathbb{N}. \tag{2.12}$$

Since $y\widehat{f_0}(0) = y\hat{f}(0) = 1$ and $y\hat{a}(0) = y$, by induction we deduce that

$$y\widehat{f_n}(0) = y\hat{a}(0)\widehat{f_{n-1}}(0) = y\widehat{f_{n-1}}(0) = \cdots = y\widehat{f_0}(0) = y\hat{f}(0) = 1.$$

Consequently, by $\lim_{n\to\infty}\|f_n - f_\infty\|_{(C(\mathbb{R}))^{r\times 1}} = 0$ and $\widehat{f_n}(d\xi) = \hat{a}(\xi)\widehat{f_{n-1}}(\xi)$, we have $\widehat{f_\infty}(d\xi) = \hat{a}(\xi)\widehat{f_\infty}(\xi)$ and $y\widehat{f_\infty}(0) = \lim_{n\to\infty} y\widehat{f_n}(0) = 1$. So, f_∞ is not identically zero. By the assumption in (i), we also have $y\hat{\phi}(0) = 1$. Therefore, we have

$$y[\widehat{f_\infty}(0) - \hat{\phi}(0)] = 0 \quad \text{and} \quad \widehat{f_\infty}(d\xi) = \hat{a}(\xi)\widehat{f_\infty}(\xi), \quad \hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi). \tag{2.13}$$

On the other hand, it is easy to verify that $\operatorname{span}\{\hat{f}(2\pi k) : k \in \mathbb{Z}\} = \mathbb{C}^{r\times 1}$. Since $\lim_{n\to\infty}\|f_n - f_\infty\|_{(C(\mathbb{R}))^{r\times 1}} = 0$ and $f_\infty \not\equiv 0$, in [8, Proposition 3.1], we conclude that 1 is a simple eigenvalue of $\hat{a}(0)$ and all its other eigenvalues are less than 1 in modulus. Consequently, up to a multiplicative constant, the solution to the refinement equation (1.1) is unique. Now by $y\hat{a}(0) = y$ and $\hat{a}(0)[\widehat{f_\infty}(0) - \hat{\phi}(0)] = \widehat{f_\infty}(0) - \hat{\phi}(0)$, since 1 is a simple eigenvalue of $\hat{a}(0)$, it follows from $y[\widehat{f_\infty}(0) - \hat{\phi}(0)] = 0$ in (2.13) that $\widehat{f_\infty}(0) = \hat{\phi}(0)$, which can be easily verified by considering the Jordan canonical form of the matrix $\hat{a}(0)$ and noting that 1 is a simple eigenvalue of $\hat{a}(0)$. Therefore, by (2.13), we have $\phi = f_\infty$. That is, we conclude that ϕ is a compactly supported continuous function vector and $\lim_{n\to\infty}\|f_n - \phi\|_{(C(\mathbb{R}))^{r\times 1}} = 0$.

Now we show by induction on n that

$$f_n\left(\frac{\ell}{r}+j\right) = \delta_j E_{\ell+1} \qquad \forall\, n \in \mathbb{N}\cup\{0\}, j \in \mathbb{Z}, \ell = 0, \ldots, r-1. \tag{2.14}$$

Note that $h(0) = 1$ and $h(k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$. Clearly, by our definition of the initial function vector f_0 (that is, f), (2.14) holds for $n = 0$. Suppose that (2.14) holds for $n-1$. Now we show that (2.14) holds for n. By the definition of f_n, we deduce that for $\ell = 0, \ldots, r-1$ and $j \in \mathbb{Z}$, by (2.11) and the induction hypothesis for $n-1$,

$$
\begin{aligned}
f_n\left(\frac{\ell}{r}+j\right) &= d\sum_{k\in\mathbb{Z}} a(k) f_{n-1}\left(\frac{d\ell}{r}+dj-k\right)\\
&= d\sum_{k\in\mathbb{Z}} a(k) f_{n-1}\left(\frac{Q_\ell}{r}+R_\ell+dj-k\right)\\
&= d\sum_{k\in\mathbb{Z}} a(k)\delta_{R_\ell+dj-k}E_{Q_\ell+1}\\
&= da(R_\ell+dj)E_{Q_\ell+1} = \delta_j E_{\ell+1},
\end{aligned}
$$

where we used (2.8) in the last identity. So, (2.14) holds for n. Now by induction, (2.14) holds for all $n \in \mathbb{N}\cup\{0\}$. Since $\lim_{n\to\infty}\|f_n-\phi\|_{(C(\mathbb{R}))^{r\times 1}} = 0$, in particular, we have $\lim_{n\to\infty} f_n(x) = \phi(x)$ for all $x \in \mathbb{R}$. Now (1.5) or equivalently (2.10) follows directly from (2.14). ■

As a consequence of Theorem 2.1, we have the following result characterizing compactly supported orthogonal interpolating refinable function vectors.

Corollary 2.2. *Let d and r be positive integers such that $d > 1$. Let $a : \mathbb{Z} \mapsto \mathbb{C}^{r\times r}$ be a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$ and ϕ be a compactly supported d-refinable function vector such that $\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. Then ϕ is an orthogonal interpolating function vector, that is, ϕ is continuous, (1.5) holds and*

$$
\int_{\mathbb{R}} \phi(x-j)\overline{\phi(x)}^{\mathrm{T}}\,\mathrm{d}x = \frac{1}{r}\delta_j I_r \qquad \forall\, j\in\mathbb{Z}, \tag{2.15}
$$

if and only if, (i)–(iii) *of Theorem 2.1 hold and a is an orthogonal mask:*

$$
\sum_{m=0}^{d-1}\hat{a}(\xi+2\pi m/d)\overline{\hat{a}(\xi+2\pi m/d)}^{\mathrm{T}} = I_r. \tag{2.16}
$$

Proof. Necessity: Suppose that ϕ is an orthogonal interpolating d-refinable function vector. Then in particular, ϕ is an interpolating d-refinable function vector. Hence, by Theorem 2.1, (i)–(iii) hold. Now we show that (2.15) implies (2.16). Since ϕ is a compactly supported d-refinable function vector satisfying the refinement equation (1.1), noting that the mask a is finitely supported, we deduce from (2.15) that

$$
\begin{aligned}
\frac{1}{r}\delta_j I_r &= \int_{\mathbb{R}} \phi(x-j)\overline{\phi(x)}^{\mathrm{T}}\,\mathrm{d}x\\
&= d^2\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}} a(k)\phi(dx-dj-k)\overline{\sum_{k'\in\mathbb{Z}} a(k')\phi(dx-k')}^{\mathrm{T}}\,\mathrm{d}x\\
&= d^2\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} a(k)\left[\int_{\mathbb{R}}\phi(dx-dj-k)\overline{\phi(dx-k')}^{\mathrm{T}}\,\mathrm{d}x\right]\overline{a(k')}^{\mathrm{T}}\\
&= d\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} a(k)\left[\int_{\mathbb{R}}\phi(x-(dj+k-k'))\overline{\phi(x)}^{\mathrm{T}}\,\mathrm{d}x\right]\overline{a(k')}^{\mathrm{T}}.
\end{aligned}
$$

Now by (2.15) again, we deduce

$$
\frac{1}{r}\delta_j I_r = d\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} a(k)\frac{1}{r}\delta_{dj+k-k'}I_r\overline{a(k')}^{\mathrm{T}} = \frac{d}{r}\sum_{k\in\mathbb{Z}} a(k)\overline{a(dj+k)}^{\mathrm{T}}.
$$

That is, (2.15) implies

$$
\sum_{k\in\mathbb{Z}} a(k)\overline{a(dj+k)}^{\mathrm{T}} = d^{-1}\delta_j I_r \qquad \forall\, j\in\mathbb{Z}. \tag{2.17}
$$

It is known and can be easily verified by a direct calculation that (2.16) is equivalent to (2.17).

Sufficiency: Since (i)–(iii) of Theorem 2.1 hold, by Theorem 2.1, we see that ϕ is continuous and (1.5) holds. To complete the proof, we show that (2.15) holds. Since (iii) of Theorem 2.1 holds, we have $\nu_\infty(a,d) > 0$. By [8, (4.7)], we have $\nu_\infty(a,d) \leqslant \nu_2(a,d)$. Consequently, we get $\nu_2(a,d) \geqslant \nu_\infty(a,d) > 0$. So, by [8, Theorem 4.3], the vector cascade algorithm associated with mask a and dilation factor d converges in $L_2(\mathbb{R})$.

Define $f_0 := [g(r\cdot), g(r\cdot -1), \ldots, g(r\cdot -(r-1))]^{\mathrm{T}}$ and f_n as in (2.12), where $g = \chi_{[0,1]}$, the characteristic function of the interval $[0, 1]$. Denote $y := [1, \ldots, 1] \in \mathbb{R}^{1\times r}$. By calculation, we have $\hat{g}(\xi) = \frac{1-\mathrm{e}^{-\mathrm{i}\xi}}{\mathrm{i}\xi}$. Therefore, $\hat{g}(0) = 1$ and $\hat{g}(2\pi k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$. By the same argument as in the proof of Theorem 2.1, we can check that $y\hat{f_0}(0) = 1$ and $y\hat{f_0}(2\pi k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$. Since $y\hat{a}(0) = y$ by (ii) of Theorem 2.1, f_0 is a suitable initial function vector in $L_2(\mathbb{R})$. On the other hand, by (i), we have $y\hat{\phi}(0) = 1$. Now by [8, Theorem 4.3] and $\nu_2(a, d) > 0$, we see that $\lim_{n\to\infty} \|f_n - \phi\|_{(L_2(\mathbb{R}))^{r\times 1}} = 0$. Now we prove by induction on n that

$$\int_{\mathbb{R}} f_n(x-j)\overline{f_n(x)}^{\mathrm{T}}\,\mathrm{d}x = \frac{1}{r}\delta_j I_r \qquad \forall\, j \in \mathbb{Z},\, n \in \mathbb{N} \cup \{0\}. \tag{2.18}$$

By the definition of $g = \chi_{[0,1]}$, it is obvious that $\int_{\mathbb{R}} g(x-j)\overline{g(x)}\,\mathrm{d}x = \delta_j$ for all $j \in \mathbb{Z}$. By the definition of the function vector $f_0 = [g(r\cdot), g(r\cdot -1), \ldots, g(r\cdot -(r-1))]^{\mathrm{T}}$, now it is straightforward to check that (2.18) holds for $n = 0$. Suppose that (2.18) holds for $n-1$. Now we show that (2.18) must hold for n. By the definition of f_n in (2.12), noting that a is finitely supported, by the induction hypothesis for $n-1$ and (2.17), we deduce that

$$\begin{aligned}
\int_{\mathbb{R}} f_n(x-j)\overline{f_n(x)}^{\mathrm{T}}\,\mathrm{d}x &= d^2 \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} a(k) f_{n-1}(dx-dj-k)\overline{\sum_{k'\in\mathbb{Z}} a(k')f_{n-1}(dx-k')}^{\mathrm{T}}\,\mathrm{d}x \\
&= d^2 \sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} a(k)\left[\int_{\mathbb{R}} f_{n-1}(dx-dj-k)\overline{f_{n-1}(dx-k')}^{\mathrm{T}}\,\mathrm{d}x\right]\overline{a(k')}^{\mathrm{T}} \\
&= d \sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} a(k)\left[\int_{\mathbb{R}} f_{n-1}(x-(dj+k-k'))\overline{f_{n-1}(x)}^{\mathrm{T}}\,\mathrm{d}x\right]\overline{a(k')}^{\mathrm{T}} \\
&= d \sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} a(k)\frac{1}{r}\delta_{dj+k-k'} I_r \overline{a(k')}^{\mathrm{T}} \\
&= \frac{d}{r}\sum_{k\in\mathbb{Z}} a(k)\overline{a(dj+k)}^{\mathrm{T}} \\
&= r^{-1}\delta_j I_r.
\end{aligned}$$

Hence, (2.18) has been verified for n. By induction, (2.18) holds for all $n \in \mathbb{N} \cup \{0\}$. Now it is easy to conclude from $\lim_{n\to\infty} \|f_n - \phi\|_{(L_2(\mathbb{R}))^{r\times 1}} = 0$ and (2.18) that (2.15) is true. ■

In the rest of this section, we shall investigate the structure of the vector $\hat{y}$ in the definition of the sum rules in (2.2) for the particular family of interpolatory masks of type (d, r) given in (2.8). Notice that the sum rule condition in (2.2) can be rewritten as

$$\hat{y}(d\xi)\hat{a}(\xi + 2\pi\ell/d) = \delta_\ell \hat{y}(\xi) + O(|\xi|^\kappa), \qquad \xi \to 0,\, \ell = 0, \ldots, d-1. \tag{2.19}$$

Now we shall express the sum rule condition above in terms of the cosets of $\hat{a}(\xi)$. Define the cosets $\hat{a}^m(\xi)$ of $\hat{a}(\xi)$ by

$$\hat{a}^m(\xi) := \sum_{k\in\mathbb{Z}} a(m+dk)\mathrm{e}^{-\mathrm{i}\xi(m+dk)}, \qquad m = 0, \ldots, d-1. \tag{2.20}$$

By a simple calculation, from (2.20) we have $\hat{a}^m(\xi + 2\pi\ell/d) = \mathrm{e}^{-\mathrm{i}2\pi\ell m/d}\hat{a}^m(\xi)$ for all $\ell \in \mathbb{Z}$. Since $\hat{a}(\xi) = \sum_{m=0}^{d-1}\hat{a}^m(\xi)$, now (2.19) becomes

$$\sum_{m=0}^{d-1} \mathrm{e}^{-\mathrm{i}2\pi\ell m/d}\hat{y}(d\xi)\hat{a}^m(\xi) = \delta_\ell \hat{y}(\xi), \qquad \ell = 0, \ldots, d-1.$$

Rewrite the above identities in the form of a matrix, we have

$$U_d \begin{bmatrix} \hat{y}(d\xi)\hat{a}^0(\xi) \\ \hat{y}(d\xi)\hat{a}^1(\xi) \\ \vdots \\ \hat{y}(d\xi)\hat{a}^{d-1}(\xi) \end{bmatrix} = \begin{bmatrix} \hat{y}(\xi) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{with } U_d := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{-\mathrm{i}\frac{2\pi}{d}} & \cdots & \mathrm{e}^{-\mathrm{i}\frac{2\pi(d-1)}{d}} \\ \vdots & \vdots & & \vdots \\ 1 & \mathrm{e}^{-\mathrm{i}\frac{2\pi(d-1)}{d}} & \cdots & \mathrm{e}^{-\mathrm{i}\frac{2\pi(d-1)^2}{d}} \end{bmatrix}.$$

Note that $U_d\overline{U_d}^{\mathrm{T}} = dI_d$. Now it follows from the above relation that (2.19) is equivalent to

$$\hat{y}(d\xi)\hat{a}^m(\xi) = d^{-1}\hat{y}(\xi) + O(|\xi|^\kappa), \qquad \xi \to 0,\, m = 0, \ldots, d-1. \tag{2.21}$$

The following result determines the $\hat{y}$ vector in the definition of sum rules in (2.2) for interpolatory masks of type (d, r); this result will play an important role in later sections for our construction of interpolatory masks of type (d, r) with high orders of sum rules.

Theorem 2.3. *Let d and r be positive integers such that $d > 1$. Let $a : \mathbb{Z} \mapsto \mathbb{C}^{r\times r}$ be a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$. Suppose that a is an interpolatory mask of type (d, r), that is, $[1, \ldots, 1]\hat{a}(0) = [1, \ldots, 1]$ and (2.8) holds. If a satisfies the sum rules of order κ in (2.2) with a sequence $y \in (\ell_0(\mathbb{Z}))^{1\times r}$ and $\hat{y}(0) = [1, \ldots, 1]$, then*

$$\hat{y}^{(j)}(0) = \mathrm{i}^j r^{-j}[\delta_j, 1^j, 2^j, \ldots, (r-1)^j], \qquad j = 0, \ldots, \kappa-1. \tag{2.22}$$

In other words, $\hat{y}(\xi) = \hat{Y}(\xi) + O(|\xi|^\kappa)$ as $\xi \to 0$, where $Y(\xi) := [1, \mathrm{e}^{\mathrm{i}\xi/r}, \ldots, \mathrm{e}^{\mathrm{i}(r-1)\xi/r}]$.

Proof. Using the cosets of the mask a, it is easy to see that (2.8) can be equivalently rewritten as

$$\hat{a}^{R_\ell}(\xi)E_{Q_\ell+1} = d^{-1}\mathrm{e}^{-\mathrm{i}R_\ell\xi}E_{\ell+1} \qquad \forall\, \ell = 0,\dots,r-1. \tag{2.23}$$

Since a satisfies the sum rules of order κ with the vector $\hat{y}$, we have (2.21). In particular, using (2.21) with $m = R_\ell$, we deduce from (2.23) that

$$d^{-1}\hat{y}(\xi)E_{Q_\ell+1} = \hat{y}(d\xi)\hat{a}^{R_\ell}(\xi)E_{Q_\ell+1} + O(|\xi|^\kappa) = d^{-1}\mathrm{e}^{-\mathrm{i}R_\ell\xi}\hat{y}(d\xi)E_{\ell+1} + O(|\xi|^\kappa).$$

Denote $[\hat{y}_1(\xi),\dots,\hat{y}_r(\xi)] := \hat{y}(\xi)$, that is, we denote $\hat{y}_j$ to be the jth component of the row vector $\hat{y}$. Then the above identity can be rewritten as

$$\hat{y}_{Q_\ell+1}(\xi) = \mathrm{e}^{-\mathrm{i}R_\ell\xi}\hat{y}_{\ell+1}(d\xi) + O(|\xi|^\kappa).$$

That is, by $\hat{y}(0) = [1,\dots,1]$, we must have

$$\hat{y}_{\ell+1}(0) = 1, \qquad \hat{y}_{\ell+1}(d\xi) = \mathrm{e}^{\mathrm{i}R_\ell\xi}\hat{y}_{Q_\ell+1}(\xi) + O(|\xi|^\kappa), \qquad \xi \to 0,\ \ell = 0,\dots,r-1. \tag{2.24}$$

Note that the above relation is just a system of linear equations on the unknowns $\hat{y}^{(j)}(0)$ for $j = 1,\dots,\kappa-1$. In the following, we shall argue that the above system of linear equations in (2.24) has a unique solution for the unknowns $\{\hat{y}^{(j)}(0) : j = 1,\dots,\kappa-1\}$. Moreover, we shall prove that the unique solution to (2.24) must be given in (2.22).

For simplicity of discussion, let us rewrite Q_ℓ in the form of an operator. Define an operator $Q : \{0,\dots,r-1\} \mapsto \{0,\dots,r-1\}$ by $Q(\ell) := Q_\ell = d\ell \bmod r$. For all $\ell \in \{0,\dots,r-1\}$ and $n \in \mathbb{N}$, employing (2.24) iteratively, we have

$$\begin{aligned}\hat{y}_{\ell+1}(\xi) &= \mathrm{e}^{\mathrm{i}d^{-1}R_\ell\xi}\hat{y}_{Q_\ell+1}(d^{-1}\xi) + O(|\xi|^\kappa) = \mathrm{e}^{\mathrm{i}\xi(d^{-2}R_{Q(\ell)}+d^{-1}R_\ell)}\hat{y}_{Q^2(\ell)+1}(d^{-2}\xi) + O(|\xi|^\kappa)\\ &= \mathrm{e}^{\mathrm{i}\xi(\sum_{k=1}^{n} d^{-k}R_{Q^{k-1}(\ell)})}\hat{y}_{Q^n(\ell)+1}(d^{-n}\xi) + O(|\xi|^\kappa).\end{aligned}$$

That is, we have

$$\hat{y}_{\ell+1}(\xi) = \mathrm{e}^{\mathrm{i}\xi(\sum_{k=1}^{n} d^{-k}R_{Q^{k-1}(\ell)})}\hat{y}_{Q^n(\ell)+1}(d^{-n}\xi) + O(|\xi|^\kappa) \qquad \forall\, \ell = 0,\dots,r-1,\ n \in \mathbb{N}. \tag{2.25}$$

Note that $\hat{y}(0) = [1,\dots,1]$ is equivalent to $\hat{y}_{\ell+1}(0) = 1$ for all $\ell = 0,\dots,r-1$. Let S denote the set of all $\ell \in \{0,\dots,r-1\}$ such that $\ell \in S$ means that there exists $n_\ell \in \mathbb{N}$ such that $Q^{n_\ell}(\ell) = \ell$. For every $\ell \in S$, since $\hat{y}_{\ell+1}(0) = 1$, by [8, Lemma 2.2], (2.25) with $n = n_\ell$ has a unique solution $\{\hat{y}^{(j)}_{\ell+1}(0) : j = 1,\dots,\kappa-1\}$, which can be obtained recursively. More precisely, since for $\ell \in S$, we have $Q^{n_\ell}(\ell) = \ell$ for some $n_\ell \in \mathbb{N}$. Therefore, (2.25) becomes

$$\hat{y}_{\ell+1}(\xi) := X_\ell(\xi)\hat{y}_{\ell+1}(d^{-n_\ell}\xi) + O(|\xi|^\kappa), \qquad \ell \in S \quad \text{with} \quad X_\ell(\xi) := \mathrm{e}^{\mathrm{i}\xi\left(\sum_{k=1}^{n_\ell} d^{-k}R_{Q^{k-1}(\ell)}\right)}.$$

Now applying the Leibniz differentiation formula to the above relation, we have

$$\begin{aligned}\hat{y}^{(j)}_{\ell+1}(0) &= \sum_{k=0}^{j}\frac{j!}{k!(j-k)!}X^{(j-k)}(0)d^{-n_\ell k}\hat{y}^{(k)}_{\ell+1}(0)\\ &= X(0)d^{-n_\ell j}\hat{y}^{(j)}_{\ell+1}(0) + \sum_{k=0}^{j-1}\frac{j!}{k!(j-k)!}X^{(j-k)}(0)d^{-n_\ell k}\hat{y}^{(k)}_{\ell+1}(0), \qquad j = 1,\dots,\kappa-1.\end{aligned}$$

Since $X(0) = 1$ and $n_\ell \geqslant 1$, we have $d^{-n_\ell j} < 1$ for $j \geqslant 1$. Now it follows from the above relation that the value $\hat{y}^{(j)}_{\ell+1}(0)$ is completely determined by the values $\hat{y}^{(k)}_{\ell+1}(0)$, $k = 0,\dots,j-1$ via the following recursive formula:

$$\hat{y}^{(j)}_{\ell+1}(0) = [1 - d^{-n_\ell j}]^{-1}\sum_{k=0}^{j-1}\frac{j!}{k!(j-k)!}X^{(j-k)}(0)d^{-n_\ell k}\hat{y}^{(k)}_{\ell+1}(0), \qquad j = 1,\dots,\kappa-1.$$

Therefore, for any $\ell \in S$, the values $\hat{y}^{(j)}_{\ell+1}(0)$, $j = 1,\dots,\kappa-1$ are completely determined by the relation (2.24).

For $\ell \in \{0,\dots,r-1\} \setminus S$, since $Q^N(\ell) \in \{0,\dots,r-1\}$ for all $N \in \mathbb{N}$, there must exist $N_\ell \in \mathbb{N}$ such that $Q^{N_\ell}(\ell) \in S$. Therefore, by (2.25) with $n = N_\ell$, we have

$$\hat{y}_{\ell+1}(\xi) = \mathrm{e}^{\mathrm{i}\xi(\sum_{k=1}^{N_\ell} d^{-k}R_{Q^{k-1}(\ell)})}\hat{y}_{Q^{N_\ell}(\ell)+1}(\xi) + O(|\xi|^\kappa), \qquad \xi \to 0 \tag{2.26}$$

with $Q^{N_\ell}(\ell) \in S$. By what has been proved, all $\hat{y}^{(j)}_{Q^{N_\ell}(\ell)+1}(0)$, $j = 0,\dots,\kappa-1$ are completely determined by (2.24). It follows from (2.26) that for every $\ell \in \{0,\dots,r-1\} \setminus S$, the values $\hat{y}^{(j)}_{\ell+1}(0)$, $j = 1,\dots,\kappa-1$ are completely determined by (2.26) and, therefore, are uniquely determined by the system of linear equations in (2.24).

That is, we proved that if (2.24) holds, then all $\hat{y}^{(j)}_{\ell+1}(0)$, $\ell = 0, \ldots, r-1$ and $j = 0, \ldots, \kappa-1$ are uniquely determined by (2.24). Therefore, if there is a solution to the system of linear equations in (2.24), then the solution must be unique according to the above argument.

In the following, we show that the system of linear equations in (2.24) indeed has a solution. Let $Y_{\ell+1}(\xi) = e^{i\ell\xi/r}$, $\ell = 0, \ldots, r-1$. By (2.11), $d\ell/r = R_\ell + Q_\ell/r$ and we have

$$Y(0) = [1, \ldots, 1] \quad \text{and} \quad Y_{\ell+1}(d\xi) = e^{id\ell\xi/r} = e^{iR_\ell\xi}e^{iQ_\ell\xi/r} = e^{iR_\ell\xi}Y_{Q_\ell+1}(\xi).$$

Therefore, if we take $\hat{y}^{(j)}_{\ell+1}(0) = \hat{Y}^{(j)}_{\ell+1}(0)$ for all $\ell = 0, \ldots, r-1$ and $j = 0, \ldots, \kappa-1$, then it is a solution to the system of linear equations in (2.24). By the uniqueness of the solution to (2.24), we must have (2.22), which completes the proof. ■

3. General construction of interpolatory masks of type (d, r)

Based on the results in Section 2, in this section, we shall present a family of interpolatory masks of type (d, r) with increasing orders of sum rules.

Before we present the construction of interpolatory masks of type (d, r) in this section, let us lay out the whole picture of our construction and the idea of the proof first. Our construction in this section largely follows the key idea of the proposed CBC (coset by coset) algorithm in [6]. Roughly speaking, a mask $a : \mathbb{Z} \mapsto \mathbb{C}^{r\times r}$ can be regarded as a disjoint union of its cosets: $\{a(k)\}_{k\in\mathbb{Z}} = \cup_{m=0}^{d-1}\{a(m+dk)\}_{k\in\mathbb{Z}}$. So, in order to obtain a mask a with multiplicity r with some desirable properties, it suffices to design its d cosets $\{a(m+dk)\}_{k\in\mathbb{Z}}$, $m = 0, \ldots, d-1$ appropriately. That is, a desired mask can be constructed *coset by coset*. Following the notation of matlab, for a matrix A, we denote $[A]_{:,n}$ the nth column of the matrix A and $[A]_{k,n}$ the (k, n)-entry of A. For each coset $m + d\mathbb{Z}$, we can further split the mask a on the coset $m + d\mathbb{Z}$ as a disjoint union of columns. More precisely, in order to design $\{a(m+dk)\}_{k\in\mathbb{Z}}$ on the coset $m + d\mathbb{Z}$, one needs to design its columns: $\{[a(m+dk)]_{:,n}\}_{k\in\mathbb{Z}}$ for each $n = 1, \ldots, r$. The condition (2.8) for an interpolatory mask of type (d, r) can be expressed as

$$[a(R_\ell + dk)]_{:,Q_\ell+1} = d^{-1}\delta_k E_{\ell+1}, \qquad k \in \mathbb{Z}, \ell = 0, \ldots, r-1. \tag{3.1}$$

In other words, the $Q_\ell + 1$ columns of the mask a on the coset $R_\ell + d\mathbb{Z}$, that is, $\{[a(R_\ell + dk)]_{:,Q_\ell+1}\}_{k\in\mathbb{Z}}$, $\ell = 0, \ldots, r-1$, are completely determined by the condition (2.8) for an interpolatory mask of type (d, r). Denote

$$\Gamma_{d,r} := \{(m, n) \ : \ m = 0, \ldots, d-1, n = 1, \ldots, r\} \setminus \{(R_\ell, Q_\ell + 1) \ : \ \ell = 0, \ldots, r-1\}. \tag{3.2}$$

Therefore, in order to construct an interpolatory mask a of type (d, r) with the sum rules of order κ, it suffices to construct $\{[a(m+dk)]_{:,n}\}_{k\in\mathbb{Z}}$ for every $(m, n) \in \Gamma_{d,r}$ such that the sum rule conditions in (2.21) are satisfied.

We have the following result on interpolatory masks of type (d, r) with increasing orders of sum rules.

Theorem 3.1. *Let d and r be positive integers such that $d > 1$. Let N be a positive integer. Suppose that for every $(m, n) \in \Gamma_{d,r}$, $S_{m,n}$ is a subset of $\mathbb{Z}$ such that $\#S_{m,n} = N$, where $\#S_{m,n}$ denotes the cardinality of the set $S_{m,n}$. Then there exists a unique finitely supported mask $a : \mathbb{Z} \mapsto \mathbb{C}^{r\times r}$ such that*

(1) *a is an interpolatory mask of type (d, r), that is, $[1, \ldots, 1]\hat{a}(0) = [1, \ldots, 1]$ and (2.8) holds.*
(2) *For every $(m, n) \in \Gamma_{d,r}$, $[a(m+dk)]_{:,n} = 0$ for all $k \in \mathbb{Z} \setminus S_{m,n}$, where $[a(m+dk)]_{:,n}$ denotes the n-th column of the $r \times r$ matrix $a(m+dk)$, that is, the nth column of the mask a on the coset $m + d\mathbb{Z}$ vanishes outside the set $m + dS_{m,n}$ for all $(m, n) \in \Gamma_{d,r}$.*
(3) *a satisfies the sum rules of order rN.*

In fact, the unique mask a must be real-valued, that is, $a : \mathbb{Z} \mapsto \mathbb{R}^{r\times r}$.

Proof. Note that (2.8) is equivalent to (3.1), that is, $[\hat{a}^{R_\ell}(\xi)]_{:,Q_\ell+1} = d^{-1}e^{-iR_\ell\xi}E_{\ell+1}$ for $\ell = 0, \ldots, r-1$. Let $\hat{y}(\xi) := [1, e^{i\xi/r}, \ldots, e^{i\xi(r-1)/r}]$. Then by $d\ell/r = R_\ell + Q_\ell/r$, it is straightforward to see that (3.1) implies

$$\begin{aligned}\hat{y}(d\xi)[\hat{a}^{R_\ell}(\xi)]_{:,Q_\ell+1} &= d^{-1}\hat{y}(d\xi)e^{-iR_\ell\xi}E_{\ell+1} = d^{-1}\hat{y}_{\ell+1}(d\xi)e^{-iR_\ell\xi} = d^{-1}e^{i\xi d\ell/r}e^{-i\xi R_\ell}\\ &= d^{-1}e^{i\xi Q_\ell/r} = d^{-1}\hat{y}_{Q_\ell+1}(\xi) \qquad \forall\, \xi \to 0, \ell = 0, \ldots, r-1.\end{aligned} \tag{3.3}$$

For an interpolatory mask a of type (d, r) such that a satisfies the sum rules of order rN in (2.21), by Theorem 2.3 and (3.3), it is necessary and sufficient to require

$$\hat{y}(d\xi)[\hat{a}^m(\xi)]_{:,n} = d^{-1}\hat{y}_n(\xi) + O(|\xi|^{rN}), \qquad \xi \to 0, (m, n) \in \Gamma_{d,r}. \tag{3.4}$$

Since $[a(m+dk)]_{:,n} = 0$ for all $k \in \mathbb{Z} \setminus S_{m,n}$, we have $[\hat{a}^m(\xi)]_{:,n} = \sum_{k\in S_{m,n}}[a(m+dk)]_{:,n}e^{-i(m+dk)\xi}$. Since $\hat{y}(\xi) = [1, e^{i\xi/r}, \ldots, e^{i\xi(r-1)/r}]$, we deduce that (3.4) is equivalent to

$$\sum_{\ell=0}^{r-1}\sum_{k\in S_{m,n}}[a(m+dk)]_{\ell+1,n}e^{i\xi(d\ell/r-m-dk)} = d^{-1}e^{i\xi(n-1)/r} + O(|\xi|^{rN}),\ (m, n) \in \Gamma_{d,r}. \tag{3.5}$$

Finding a mask a such that a satisfies all the three conditions in (1), (2) and (3) is now equivalent to solving the system of linear equations in (3.5) for each pair $(m, n) \in \Gamma_{d,r}$.

Taking jth derivatives on both sides of (3.5) and evaluating them at $\xi = 0$, we see that (3.5) is equivalent to: for each $(m, n) \in \Gamma_{d,r}$,

$$\sum_{\ell=0}^{r-1} \sum_{k\in S_{m,n}} [a(m+dk)]_{\ell+1,n}(d\ell/r - m - dk)^j = d^{-1}\left(\frac{n-1}{r}\right)^j, \qquad j = 0, \dots, rN-1. \tag{3.6}$$

Since $\#S_{m,n} = N$ and $S_{m,n} \subseteq \mathbb{Z}$, we see that for each $(m, n) \in \Gamma_{d,r}$, the set $\{d\ell/r - m - dk : k \in S_{m,n}, \ell = 0, \dots, r-1\}$ consists of rN distinct points on $\mathbb{R}$. So, the coefficient matrix $((d\ell/r - m - dk)^j)_{k\in S_{m,n},\ell=0,\dots,r-1;j=0,\dots,rN-1}$ is a Vandermonde matrix and therefore, it is invertible. Also note that for each $(m, n) \in \Gamma_{d,r}$, the number of unknowns in $\{[a(m+dk)]_{\ell+1,n} : k \in S_{m,n}, \ell = 0, \dots, r-1\}$ is also rN. So, the solution to the system of linear equations in (3.6) is unique. Moreover, it is straightforward to see that the unique solution $\{[a(m+dk)]_{\ell+1,n} : k \in S_{m,n}, \ell = 0, \dots, r-1\}$ to (3.6) must be real-valued. This completes the proof. ■

The following result is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let d and r be positive integers such that $d > 1$. Let S be any subset of $\mathbb{Z}$ such that $N = \#(S \cap (m + d\mathbb{Z}))$ for all $m \in \mathbb{Z}$ and $R_\ell \in S$ for all $\ell = 0, \dots, r-1$. Then there exists a unique mask $a : \mathbb{Z} \mapsto \mathbb{R}^{r\times r}$ such that*

(1) *a is an interpolatory mask of type (d, r).*
(2) *a is supported inside S, that is, $a(k) = 0$ for all $k \in \mathbb{Z} \setminus S$.*
(3) *a satisfies the sum rules of order rN.*

In particular, if $S = [-N_0, dN - N_0 - 1] \cap \mathbb{Z}$ for any $N_0 \in \mathbb{Z}$, then $\#(S \cap (m + d\mathbb{Z})) = N$ for all $m \in \mathbb{Z}$.

Proof. By calculation, for each $(m, n) \in \Gamma_{d,r}$, we have $S_{m,n} = \{k \in \mathbb{Z} : m + dk \in S\}$. By our assumption on S, it is not difficult to check that $\#S_{m,n} = N$. Now the claim follows directly from Theorem 3.1. ■

Let us consider the condition (2.8) for interpolatory masks with the special choice $r \mid d$, that is, $d = rr'$ for some $r' \in \mathbb{N}$. In this case, by the definition of Q_ℓ and R_ℓ in (2.9), we have $Q_\ell = 0$ and $R_\ell = r'\ell$ for all $\ell = 0, \dots, r-1$. Consequently, for $d = rr'$, (2.8) becomes

$$\hat{a}^{r'\ell}(\xi)E_1 = d^{-1}e^{-ir'\ell\xi}E_{\ell+1}, \qquad \ell = 0, \dots, d-1. \tag{3.7}$$

Note that $\hat{a}(\xi) = \sum_{m=0}^{d-1} \hat{a}^m(\xi)$. In particular, if $d = r$, that is, $r' = 1$, then the interpolatory condition in (2.8) is equivalent to

$$\hat{a}(\xi) = \frac{1}{d}\begin{bmatrix} 1 & * & \cdots & * \\ e^{-i\xi} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ e^{-i(d-1)\xi} & * & \cdots & * \end{bmatrix}, \tag{3.8}$$

where $*$ denotes some 2π-periodic trigonometric polynomial.

To understand better the definition of an interpolatory mask of type (d, r) in (2.8) (or equivalently in (3.1)), in the following we present another equivalent expression for an interpolatory mask of type (d, r). For a matrix mask $a : \mathbb{Z} \mapsto \mathbb{C}^{r\times r}$ with multiplicity r, there correspond r scalar sequences $A^1, \dots, A^r : \mathbb{Z} \mapsto \mathbb{C}$, which are as

$$A^\ell(k) := [a(\lfloor k/r \rfloor)]_{\ell,k-r\lfloor k/r\rfloor+1}, \qquad k \in \mathbb{Z}, \ell = 1, \dots, r,$$

where $\lfloor x \rfloor$ denotes the largest integer that is no greater than x. Or equivalently, the scalar sequences $A^1, \dots, A^r$ are uniquely determined by the following relation:

$$[a(k)]_{\ell,m} = A^\ell(rk + m - 1), \qquad k \in \mathbb{Z}, \ell, m = 1, \dots, r.$$

Informally speaking, each scalar sequence A^ℓ is just obtained from the ℓth row of the matrix mask a by regarding the ℓth row $\{[a(k)]_{\ell,:}\}_{k\in\mathbb{Z}}$ of the matrix mask a as one scalar sequence.

Then a is an interpolatory mask of type (d, r) if and only if

$$A^\ell(d(k + \ell - 1)) = d^{-1}\delta_k \qquad \forall\, k \in \mathbb{Z}, \ell = 1, \dots, r.$$

That is, each scalar sequence A^ℓ is an interpolatory mask with the dilation factor d and with the center $A^\ell(d(\ell - 1)) = 1/d$, $\ell = 1, \dots, r$.

For the case $d = r = 2$, we have the following result on interpolatory masks of type $(2, 2)$ with symmetry.

Corollary 3.3. *For any positive integer N, there exists a unique interpolatory mask a of type $(2, 2)$ such that*

(1) *a is supported inside $[1 - N, N]$.*
(2) *The mask a is real-valued and satisfies the sum rules of order $2N - 1$.*
(3) *The mask a is symmetric: $\hat{a}(\xi) = \mathrm{diag}(1, e^{i2\xi})\hat{a}(\xi)\mathrm{diag}(1, e^{-i\xi})$. In other words, $\phi_1(-x) = \phi_1(x)$ and $\phi_2(1-x) = \phi_2(x)$ for all $x \in \mathbb{R}$, where $[\phi_1, \phi_2]^T$ is the compactly supported 2-refinable function vector associated with mask a.*

Table 1

N	1	2	3	4	5	6	7	8
$\nu_2(a_{(2,2,N)}, 2)$	0.5	1.891641	2.310887	2.665375	2.864820	3.022116	3.148026	3.255143
$\nu_2(a^{sym}_{(2,2,N)}, 2)$	0.5	1.839036	2.159779	2.676161	2.850528	3.022694	3.147442	3.255166
$\nu_2(a_{(3,2,N)}, 3)$	0.5	1.699021	2.119868	2.384743	2.569875	2.705482	2.808354	2.889563
$\nu_2(a_{(3,3,N)}, 3)$	0.5	1.307524	2.239411	2.346999	2.685627	2.738300	2.942924	2.969225

The first row lists the quantities $\nu_2(a_{(2,2,N)}, 2)$ for the interpolatory masks $a_{(2,2,N)}$ constructed in Corollary 3.2 with $d = r = 2$ and $S := [1 - N, N]$. The second row lists the quantities $\nu_2(a^{sym}_{(2,2,N)}, 2)$ for the interpolatory masks $a^{sym}_{(2,2,N)}$ constructed in Corollary 3.3. The third and fourth rows list the quantities $\nu_2(a_{(3,2,N)}, 3)$ and $\nu_2(a_{(3,3,N)}, 3)$, respectively, for the interpolatory masks $a_{(3,r,N)}$ constructed in Corollary 3.2 with $d = 3$ and $S := [-N_0, 3N - N_0 - 1]$ with $N_0 := \lfloor 3(N-1)/2 \rfloor$. Note that we always have $\nu_\infty(a, d) \geqslant \nu_2(a, d) - 1/2$ for any mask a [8].

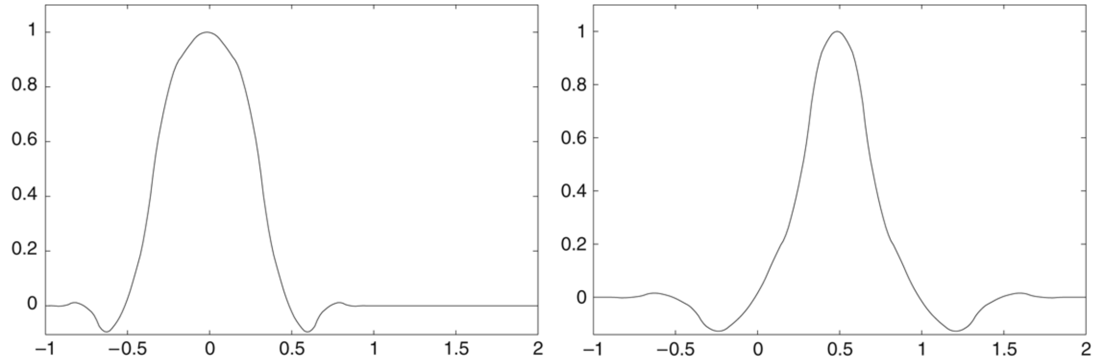

Fig. 1. The graphs of ϕ_1 (left) and ϕ_2 (right) in the symmetric interpolating 2-refinable function vector $\phi = [\phi_1, \phi_2]^T$ of Example 4.1. Moreover, $\nu_2(\phi) \approx 1.839036$ and $\phi_1(-x) = \phi_1(x)$ and $\phi_2(1-x) = \phi_2(x)$ for all $x \in \mathbb{R}$.

Proof. Since $d = r = 2$, by the interpolatory condition in (3.8), we see that $\{[a(k)]_{:,1}\}_{k\in\mathbb{Z}}$ is completely determined by (3.8). By the symmetry condition in (3), we see that $\{[a(2k+1)]_{:,2}\}_{k\in\mathbb{Z}}$ is completely determined by $\{[a(2k)]_{:,2}\}_{k\in\mathbb{Z}}$. Note that $\#([1-N, N] \cap (2\mathbb{Z})) = N$. The unknowns of the second column of the mask a on the coset $2\mathbb{Z}$ are $[a(k)]_{:,2}$, $k \in [1-N, N] \cap (2\mathbb{Z})$, plus one extra requirement $[a(N)]_{1,2} = 0$ if N is even, or $[a(1-N)]_{2,2} = 0$ if N is odd, due to the condition in (1). The proof is now completed by a similar proof as in Theorem 3.1. ■

To complete this section, let us present in Table 1 the smoothness of some families of the interpolatory masks constructed in Corollaries 3.2 and 3.3.

4. Some examples of interpolating refinable function vectors

In this section, we shall present several examples of interpolatory masks of type (d, r), as well as several examples of masks for orthogonal interpolating refinable function vectors.

Example 4.1. Let $d = r = 2$ and $N = 2$ in Corollary 3.3. Then we have a symmetric interpolatory mask a of type $(2, 2)$ satisfying the sum rules of order 3 given by

$$a(-1) = \frac{1}{16}\begin{bmatrix} 0 & 6 \\ 0 & -1 \end{bmatrix}, \quad a(0) = \frac{1}{16}\begin{bmatrix} 8 & 6 \\ 0 & 3 \end{bmatrix}, \quad a(1) = \frac{1}{16}\begin{bmatrix} 0 & 0 \\ 8 & 3 \end{bmatrix}, \quad a(2) = \frac{1}{16}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

with $a(k) = 0$ for all $k \in \mathbb{Z} \setminus \{-1, 0, 1, 2\}$. Then we have $\nu_2(a, 2) \approx 1.839036$. Therefore, $\nu_\infty(a, 2) \geqslant \nu_2(a, 2) - 1/2 \approx 1.339036 > 0$. By Theorem 2.1, its associated refinable function vector $\phi = [\phi_1, \phi_2]^T$ is interpolating. Moreover, $\phi_1(-x) = \phi_1(x)$ and $\phi_2(1-x) = \phi_2(x)$ for all $x \in \mathbb{R}$. See Fig. 1 for the graph of the interpolating 2-refinable function vector ϕ associated with the mask a.

Example 4.2. Let $d = 3$, $r = 2$, and $S = \{-2, -1, 0, 1, 2, 3\}$ in Corollary 3.2. Then we have an interpolatory mask a of type $(3, 2)$ satisfying the sum rules of order 4. The mask a is supported inside $[-2, 3]$ and is given by

$$a(-2) = \frac{1}{243}\begin{bmatrix} -21 & 0 \\ 4 & 0 \end{bmatrix}, \quad a(-1) = \frac{1}{243}\begin{bmatrix} 30 & 60 \\ -4 & -5 \end{bmatrix}, \quad a(0) = \frac{1}{243}\begin{bmatrix} 81 & 84 \\ 0 & 14 \end{bmatrix},$$
$$a(1) = \frac{1}{243}\begin{bmatrix} 14 & 0 \\ 84 & 81 \end{bmatrix}, \quad a(2) = \frac{1}{243}\begin{bmatrix} -5 & -4 \\ 60 & 30 \end{bmatrix}, \quad a(3) = \frac{1}{243}\begin{bmatrix} 0 & 4 \\ 0 & -21 \end{bmatrix},$$

with $a(k) = 0$ for all $k \in \mathbb{Z} \setminus \{-2, -1, 0, 1, 2, 3\}$. Then we have $\nu_2(a, 3) \approx 1.348473$. Therefore, $\nu_\infty(a, 3) \geqslant \nu_2(a, 3) - 1/2 \approx 0.848473 > 0$. By Theorem 2.1, its associated refinable function vector $\phi = [\phi_1, \phi_2]^T$ is interpolating. See Fig. 2 for the graph of the interpolating 3-refinable function vector ϕ associated with the mask a.

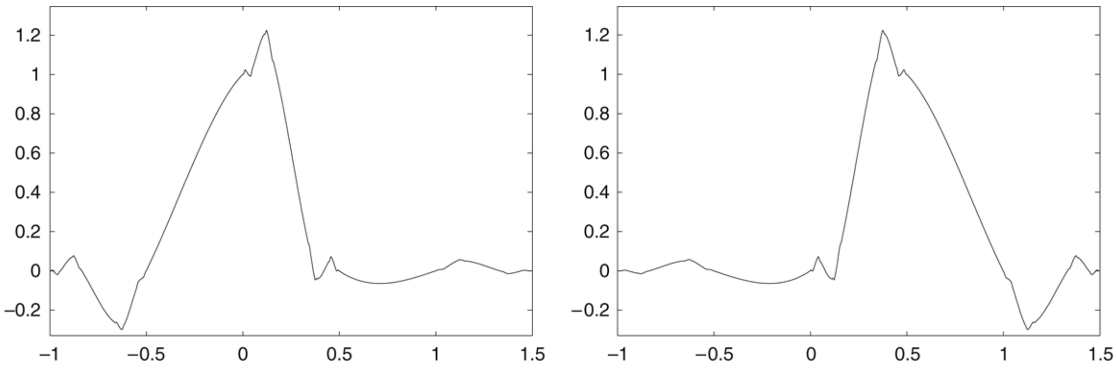

Fig. 2. The graphs of ϕ_1 (left) and ϕ_2 (right) in the interpolating 3-refinable function vector $\phi = [\phi_1, \phi_2]^{\mathrm{T}}$ of Example 4.2. Moreover, $\nu_2(\phi) \approx 1.348473$.

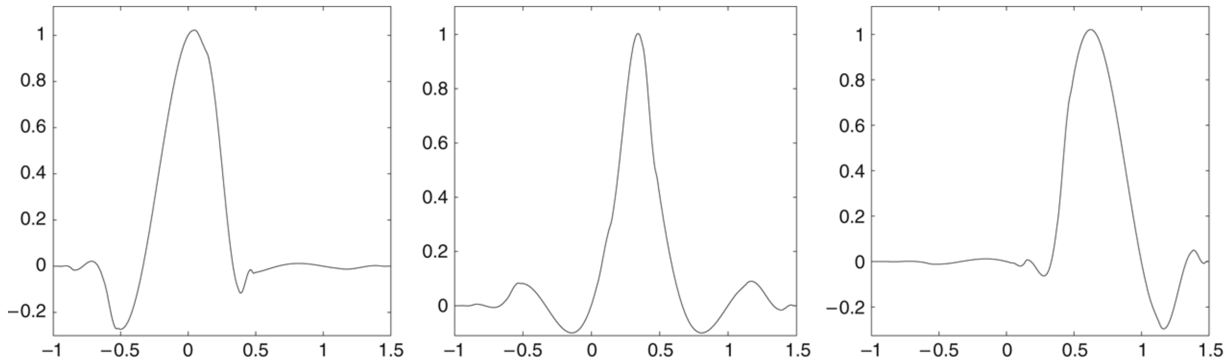

Fig. 3. The graphs of ϕ_1 (left), ϕ_2 (middle) and ϕ_3 (right) in the interpolating 3-refinable function vector $\phi = [\phi_1, \phi_2, \phi_3]^{\mathrm{T}}$ of Example 4.3. Moreover, $\nu_2(\phi) \approx 2.589443$.

Example 4.3. Let $d = r = 3$ and $S = \{-2, -1, 0, 1, 2, 3\}$ in Corollary 3.2. Then we have an interpolatory mask a of type $(3, 3)$ satisfying the sum rules of order 6. The mask a is supported inside $[-2, 3]$ and is given by

$$a(-2) = \frac{1}{2187}\begin{bmatrix} 0 & -176 & -175 \\ 0 & 55 & 50 \\ 0 & -8 & -7 \end{bmatrix}, \quad a(-1) = \frac{1}{2187}\begin{bmatrix} 0 & 280 & 560 \\ 0 & -56 & -70 \\ 0 & 7 & 8 \end{bmatrix},$$

$$a(0) = \frac{1}{2187}\begin{bmatrix} 729 & 700 & 440 \\ 0 & 175 & 440 \\ 0 & -14 & -22 \end{bmatrix}, \quad a(1) = \frac{1}{2187}\begin{bmatrix} 0 & -22 & -14 \\ 729 & 440 & 175 \\ 0 & 440 & 700 \end{bmatrix},$$

$$a(2) = \frac{1}{2187}\begin{bmatrix} 0 & 8 & 7 \\ 0 & -70 & -56 \\ 729 & 560 & 280 \end{bmatrix}, \quad a(3) = \frac{1}{2187}\begin{bmatrix} 0 & -7 & -8 \\ 0 & 50 & 55 \\ 0 & -175 & -176 \end{bmatrix},$$

with $a(k) = 0$ for all $k \in \mathbb{Z} \setminus \{-2, -1, 0, 1, 2, 3\}$. Then we have $\nu_2(a, 3) \approx 2.589443$. Therefore, $\nu_\infty(a, 3) \geqslant \nu_2(a, 3) - 1/2 \approx 2.089443 > 0$. By Theorem 2.1, its associated refinable function vector $\phi = [\phi_1, \phi_2, \phi_3]^{\mathrm{T}}$ is interpolating and belongs to $C^2(\mathbb{R})$. See Fig. 3 for the graph of the interpolating 3-refinable function vector ϕ.

Examples of orthogonal interpolating 2-refinable function vectors with multiplicity 2 have been given in [14–16]. Next, let us present some examples of orthogonal interpolating d-refinable function vectors.

Example 4.4. Let $d = 3$ and $r = 2$. The orthogonal and interpolatory mask a of type $(3, 2)$ is supported inside $[-2, 3]$ and is given by

$$a(-2) = \begin{bmatrix} -\frac{17}{702} - \frac{\sqrt{17}}{351} & 0 \\ -\frac{8}{351} + \frac{5\sqrt{17}}{702} & 0 \end{bmatrix}, \quad a(-1) = \begin{bmatrix} \frac{85}{702} - \frac{8\sqrt{17}}{351} & \frac{68}{351} + \frac{29\sqrt{17}}{702} \\ \frac{1}{351} + \frac{\sqrt{17}}{702} & \frac{11}{702} - \frac{7\sqrt{17}}{351} \end{bmatrix},$$

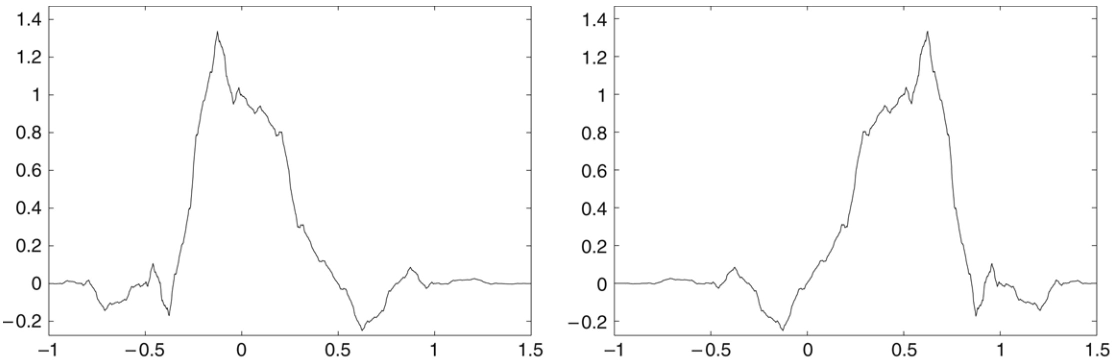

Fig. 4. The graphs of ϕ_1 (left) and ϕ_2 (right) in the orthogonal interpolating 3-refinable function vector $\phi = [\phi_1, \phi_2]^T$ of Example 4.4. Moreover, $\nu_2(\phi) \approx 1.046673$ and $\phi_2(1/2 - x) = \phi_1(x)$ for all $x \in \mathbb{R}$.

$$a(0) = \begin{bmatrix} \frac{1}{3} & \frac{119}{351} - \frac{11\sqrt{17}}{702} \\ 0 & \frac{29}{702} + \frac{4\sqrt{17}}{351} \end{bmatrix},$$

$$a(1) = \begin{bmatrix} \frac{29}{702} + \frac{4\sqrt{17}}{351} & 0 \\ \frac{119}{351} - \frac{11\sqrt{17}}{702} & \frac{1}{3} \end{bmatrix}, \quad a(2) = \begin{bmatrix} \frac{11}{702} - \frac{7\sqrt{17}}{351} & \frac{1}{351} + \frac{\sqrt{17}}{702} \\ \frac{68}{351} + \frac{29\sqrt{17}}{702} & \frac{85}{702} - \frac{8\sqrt{17}}{351} \end{bmatrix}, \quad a(3) = \begin{bmatrix} 0 & -\frac{8}{351} + \frac{5\sqrt{17}}{702} \\ 0 & -\frac{17}{702} - \frac{\sqrt{17}}{351} \end{bmatrix},$$

with $a(k) = 0$ for all $k \in \mathbb{Z} \setminus \{-2, -1, 0, 1, 2, 3\}$. The mask a satisfies the sum rules of order 2 and is an orthogonal interpolatory mask. Then we have $\nu_2(a, 3) \approx 1.046673$. Therefore, $\nu_\infty(a, 3) \geqslant \nu_2(a, 3) - 1/2 \approx 0.546673 > 0$. By Corollary 2.2, the associated 3-refinable function vector $\phi = [\phi_1, \phi_2]^T$ associated with the mask a is interpolating and orthogonal. See Fig. 4 for the graph of the orthogonal interpolating 3-refinable function vector ϕ.

Example 4.5. Let $d = 2$ and $r = 3$. The orthogonal and interpolatory masks a of type $(2, 3)$ is supported on $[-1, 2]$ and is given by

$$a(-1) = \begin{bmatrix} \frac{15}{482} - \frac{8\sqrt{15}}{241} & 0 & \frac{225}{482} + \frac{\sqrt{15}}{482} \\ 0 & 0 & -\frac{\sqrt{15}}{32} \\ 0 & 0 & \frac{1}{32} \end{bmatrix}, \quad a(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{482} + \frac{15\sqrt{15}}{482} & 0 \\ 0 & \frac{15}{32} & \frac{1}{2} \\ 0 & -\frac{\sqrt{15}}{32} & 0 \end{bmatrix},$$

$$a(1) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{482} + \frac{15\sqrt{15}}{482} & 0 & \frac{15}{482} - \frac{15\sqrt{15}}{7712} \\ \frac{225}{482} + \frac{\sqrt{15}}{482} & 1/2 & -\frac{225}{7712} + \frac{15\sqrt{15}}{482} \end{bmatrix}, \quad a(2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{15}{7712} + \frac{\sqrt{15}}{482} & 0 \\ 0 & \frac{15}{482} - \frac{15\sqrt{15}}{7712} & 0 \end{bmatrix},$$

with $a(k) = 0$ for $k \in \mathbb{Z} \setminus \{-1, 0, 1, 2\}$. The mask a satisfies the sum rules of order 1 and is an orthogonal interpolatory mask of type $(2, 3)$. Then we have $\nu_2(a, 2) \approx 0.892777$. Therefore, $\nu_\infty(a, 2) \geqslant \nu_2(a, 2) - 1/2 \approx 0.392777 > 0$. By Corollary 2.2, the associated 2-refinable function vector $\phi = [\phi_1, \phi_2, \phi_3]^T$ is interpolating and orthogonal. See Fig. 5 for the graph of the orthogonal interpolating 2-refinable function vector ϕ.

In passing, we mention that several examples of symmetric (interpolating) orthogonal scalar 4-refinable functions have been reported in [4].

5. Biorthogonal multiwavelets derived from interpolating refinable function vectors

It is of interest to construct biorthogonal multiwavelets from interpolating refinable function vectors, due to their interesting interpolation property. In this section, let us discuss how to derive biorthogonal multiwavelets from interpolating refinable function vectors that have been investigated and constructed in this paper. To do so, let us introduce some necessary concepts.

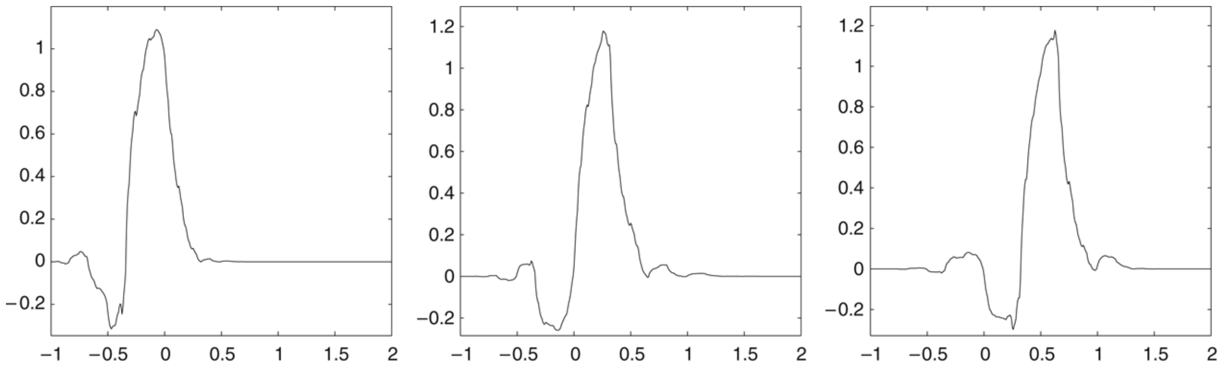

Fig. 5. The graphs of ϕ_1 (left), ϕ_2 (middle) and ϕ_3 (right) in the orthogonal interpolating 2-refinable function vector $\phi = [\phi_1, \phi_2, \phi_3]^{\mathrm{T}}$ of Example 4.5. Moreover, $\nu_2(\phi) \approx 0.892777$.

For two $r \times 1$ vectors ϕ and $\tilde{\phi}$ of compactly supported functions in $L_2(\mathbb{R})$, we say that $(\phi, \tilde{\phi})$ is *a pair of dual function vectors* (or $\tilde{\phi}$ is a dual function vector of ϕ) if

$$\int_{\mathbb{R}} \phi(x-j)\overline{\tilde{\phi}(x)}^{\mathrm{T}}\,\mathrm{d}x = \delta_j I_r, \qquad j \in \mathbb{Z}. \tag{5.1}$$

For two $r \times r$ matrices $\hat{a}$ and $\hat{\tilde{a}}$ of 2π-periodic trigonometric polynomials, we say that $(a, \tilde{a})$ is *a pair of dual masks* (or $\tilde{a}$ is a dual mask of a) with a dilation factor d if

$$\sum_{m=0}^{d-1} \hat{a}(\xi + 2\pi m/d)\overline{\hat{\tilde{a}}(\xi + 2\pi m/d)}^{\mathrm{T}} = I_r. \tag{5.2}$$

If a is a dual mask of itself, then (5.2) becomes (2.16) and a is an orthogonal mask. Let ϕ and $\tilde{\phi}$ be two compactly supported d-refinable function vectors with masks a and $\tilde{a}$, respectively. Assume that $\hat{\phi}(0)$ and $\hat{\tilde{\phi}}(0)$ are appropriately normalized so that $\overline{\hat{\phi}(0)}^{\mathrm{T}}\hat{\tilde{\phi}}(0) = 1$. Then it is known that $(\phi, \tilde{\phi})$ is a pair of dual d-refinable function vectors in $L_2(\mathbb{R})$, if and only if, $(a, \tilde{a})$ is a pair of dual masks, and both $\nu_2(a, d) > 0$ and $\nu_2(\tilde{a}, d) > 0$. In wavelet analysis, for a given mask a, it is of interest to construct a dual mask $\tilde{a}$ of a such that $\tilde{a}$ can attain the sum rules of any preassigned order $\tilde{\kappa}$ with a sequence $\tilde{y}$, that is, $\hat{\tilde{y}}(0) \neq 0$ and

$$\hat{\tilde{y}}(d\xi)\hat{\tilde{a}}(\xi + 2\pi m/d) = \delta_m \hat{\tilde{y}}(\xi) + O(|\xi|^{\tilde{\kappa}}), \qquad \xi \to 0, m = 0, \ldots, d-1. \tag{5.3}$$

A systematic way, called the CBC (coset by coset) algorithm, of constructing such desirable dual masks $\tilde{a}$ has been introduced in [5] and further developed in [1,6]. There are two key ingredients in the proposed CBC algorithm in [1,5,6]. In the following, let us outline the main ideas of the CBC algorithm and use it to construct biorthogonal multiwavelets for the interpolating refinable function vectors obtained in this paper.

The first key ingredient of the CBC algorithm in [1,5,6] is the following interesting fact, whose proof is given in [6], as well as [5] for the scalar case. For the purpose of completeness, we shall provide a self-contained proof here.

Proposition 5.1. *Let d be a dilation factor. Let $\hat{a}$ be an $r \times r$ matrix of 2π-periodic trigonometric polynomials such that 1 is a simple eigenvalue of $\hat{a}(0)$ and for every $j \in \mathbb{N}$, d^j is not an eigenvalue of $\hat{a}(0)$. Suppose that $\tilde{a}$ is a dual mask of a and $\tilde{a}$ satisfies the sum rules of order $\tilde{\kappa}$ in (5.3) with a sequence $\tilde{y}$. Then up to a multiplicative constant, the values $\hat{\tilde{y}}^{(j)}(0)$, $j = 0, \ldots, \tilde{\kappa}-1$ are uniquely determined by the mask $\hat{a}$ via the following recursive formula: $\hat{\tilde{y}}(0) = \hat{\tilde{y}}(0)\overline{\hat{a}(0)}^{\mathrm{T}}$ and*

$$\hat{\tilde{y}}^{(j)}(0) = \left[\sum_{k=0}^{j-1} \frac{j!}{k!(j-k)!}\hat{\tilde{y}}^{(k)}(0)\overline{\hat{a}^{(j-k)}(0)}^{\mathrm{T}}\right][d^j I_r - \overline{\hat{a}(0)}^{\mathrm{T}}]^{-1}, \qquad j = 1, \ldots, \tilde{\kappa}. \tag{5.4}$$

In other words, if ϕ is a compactly supported d-refinable function vector satisfying $\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ and $\hat{\phi}(0) \neq 0$, then $\hat{\tilde{y}}(\xi) = c\overline{\hat{\phi}(\xi)}^{\mathrm{T}} + O(|\xi|^{\tilde{\kappa}})$ as $\xi \to 0$ for some nonzero constant c.

Proof. By (5.2), we deduce that

$$\overline{\hat{\tilde{y}}(d\xi)}^{\mathrm{T}} = \sum_{m=0}^{d-1} \hat{a}(\xi + 2\pi m/d)\overline{\hat{\tilde{a}}(\xi + 2\pi m/d)}^{\mathrm{T}}\overline{\hat{\tilde{y}}(d\xi)}^{\mathrm{T}} = \sum_{m=0}^{d-1} \hat{a}(\xi + 2\pi m/d)\overline{\hat{\tilde{y}}(d\xi)\hat{\tilde{a}}(\xi + 2\pi m/d)}^{\mathrm{T}}.$$

Now by (5.3) we get

$$\overline{\hat{\tilde{y}}(d\xi)}^{\mathrm{T}} = \hat{a}(\xi)\overline{\hat{\tilde{y}}(\xi)}^{\mathrm{T}} + O(|\xi|^{\tilde{\kappa}}), \qquad \xi \to 0.$$

That is, the vector $\hat{\hat{y}}$ must satisfy

$$\hat{\hat{y}}(d\xi) = \hat{\hat{y}}(\xi)\overline{\hat{\tilde{a}}(\xi)}^{\mathrm{T}} + O(|\xi|^{\tilde{\kappa}}), \qquad \xi \to 0. \tag{5.5}$$

By Leibniz differentiation formula, it follows from (5.5) that

$$d^j \hat{\hat{y}}^{(j)}(0) = \hat{\hat{y}}^{(j)}(0)\overline{\hat{\tilde{a}}(0)}^{\mathrm{T}} + \sum_{k=0}^{j-1} \frac{j!}{k!(j-k)!} \hat{\hat{y}}^{(k)}(0)\overline{\hat{\tilde{a}}^{(j-k)}(0)}^{\mathrm{T}}, \qquad j = 1, \ldots, \tilde{\kappa}.$$

Since 1 is a simple eigenvalue of $\hat{a}(0)$ and d^j is not an eigenvalue of $\hat{a}(0)$ for all $j \in \mathbb{N}$, now the recursive formula in (5.4) can be easily deduced from the above relation. Moreover, the relation $\hat{\hat{y}}(\xi) = c\overline{\hat{\phi}(\xi)}^{\mathrm{T}} + O(|\xi|^{\tilde{\kappa}})$ follows directly from (5.5) and the identity $\hat{\phi}(d\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. ■

By obtaining the values $\hat{\hat{y}}^{(j)}(0), j \in \mathbb{N} \cup \{0\}$ from a given mask a via the recursive formula in (5.4) of Proposition 5.1, the CBC algorithm reduces the system of nonlinear equations (in terms of both $\tilde{a}(k)$, $k \in \mathbb{Z}$ and $\hat{\hat{y}}^{(j)}(0), j = 0, \ldots, \tilde{\kappa} - 1$) in (5.3) into a system of linear equations, since now $\hat{\hat{y}}^{(j)}(0), j = 0, \ldots, \tilde{\kappa} - 1$ are known. On the other hand, both conditions in Eqs. (5.2) and (5.3) can be equivalently rewritten in terms of the cosets of the masks a and $\tilde{a}$. More precisely, it is easy to verify that (5.2) is equivalent to

$$\sum_{m=0}^{d-1} \hat{a}^m(\xi)\overline{\hat{\tilde{a}}^m(\xi)}^{\mathrm{T}} = d^{-1} I_r, \tag{5.6}$$

where $\hat{\tilde{a}}^m(\xi) := \sum_{k\in\mathbb{Z}} \tilde{a}(m + dk)\mathrm{e}^{-\mathrm{i}\xi(m+dk)}$, and (5.3) is equivalent to

$$\hat{\hat{y}}(d\xi)\hat{\tilde{a}}^m(\xi) = d^{-1}\hat{\hat{y}}(\xi) + O(|\xi|^{\tilde{\kappa}}), \qquad \xi \to 0, m = 0, \ldots, d-1. \tag{5.7}$$

The second key ingredient of the CBC algorithm lies in that using Proposition 5.1, the CBC reduces the big system of linear equations in both Eqs. (5.3) and (5.2) into small systems of linear equations using the idea of coset by coset construction and the equations in Eqs. (5.6) and (5.7). Moreover, the CBC algorithm in [6] guarantees that as long as a possesses at least one finitely supported dual mask, for any given positive integer $\tilde{\kappa}$, there always exists a finitely supported dual mask $\tilde{a}$ of a such that $\tilde{a}$ satisfies the sum rules of order $\tilde{\kappa}$, see [6, Theorem 3.4] and [1,5] for more details on the CBC algorithm.

We also mention that due to Theorem 2.3, all biorthogonal multiwavelets derived from interpolating refinable function vectors in this paper have the highest possible balancing order, that is, its balancing order matches the order of sum rules. See [10] and references therein on balanced biorthogonal multiwavelets and balanced dual multiframelets.

In the following, let us present several examples of dual masks for some given interpolatory masks constructed in this paper.

Example 5.2. Let $d = r = 2$. Let a denote the mask given in Example 4.1. By (5.4) of Proposition 5.1 with $\tilde{\kappa} = 3$, we have

$$\hat{\hat{y}}(0) = [3/2, 1], \qquad \frac{-\mathrm{i}}{1!}\hat{\hat{y}}^{(1)}(0) = [0, 1/2], \qquad \frac{(-\mathrm{i})^2}{2!}\hat{\hat{y}}^{(2)}(0) = [3/136, 7/68],$$

where i here denotes the imaginary unit with $\mathrm{i}^2 = -1$. By the CBC algorithm in [6], we have a dual mask $\tilde{a}$ of a such that $\tilde{a}$ satisfies the sum rules of order 3. The dual mask $\tilde{a}$ is supported inside $[-1, 3]$ and is given by

$$\tilde{a}(-1) = \frac{1}{384}\begin{bmatrix} -28 & 112 \\ 21 & -36 \end{bmatrix}, \quad \tilde{a}(0) = \frac{1}{384}\begin{bmatrix} 216 & 112 \\ -18 & 60 \end{bmatrix}, \quad \tilde{a}(1) = \frac{1}{384}\begin{bmatrix} -28 & 0 \\ 330 & 60 \end{bmatrix},$$

$$\tilde{a}(2) = \frac{1}{384}\begin{bmatrix} 0 & 0 \\ -18 & -36 \end{bmatrix}, \quad \tilde{a}(3) = \frac{1}{384}\begin{bmatrix} 0 & 0 \\ 21 & 0 \end{bmatrix}$$

with $\tilde{a}(k) = 0$ for $k \in \mathbb{Z}\backslash\{-1, 0, 1, 2, 3\}$. By calculation, we have $\nu_2(\tilde{a}, 2) \approx 1.117992$. So, the associated 2-refinable function vectors ϕ and $\tilde{\phi}$ with masks a and $\tilde{a}$ indeed satisfy the biorthogonal relation in (5.1). See Fig. 6 for the graph of the dual 2-refinable function vector $\tilde{\phi} = [\tilde{\phi}_1, \tilde{\phi}_2]^{\mathrm{T}}$. Note that $\tilde{\phi}_1(-x) = \tilde{\phi}_1(x)$ and $\tilde{\phi}_2(1 - x) = \tilde{\phi}_2(x)$ for all $x \in \mathbb{R}$.

Example 5.3. Let $d = 3$ and $r = 2$. Let a denote the mask given in Example 4.2. By (5.4) of Proposition 5.1 with $\tilde{\kappa} = 2$, we have

$$\hat{\hat{y}}(0) = [1, 1], \qquad \frac{-\mathrm{i}}{1!}\hat{\hat{y}}^{(1)}(0) = [16/387, 355/774].$$

By the CBC algorithm in [6], we have a dual mask $\tilde{a}$ of a such that $\tilde{a}$ satisfies the sum rules of order 2. The dual mask $\tilde{a}$ is supported inside $[-2, 3]$ and is given by

$$\tilde{a}(-2) = \frac{1}{34884}\begin{bmatrix} 1292 & -4773 \\ -969 & 1866 \end{bmatrix}, \quad \tilde{a}(-1) = \frac{1}{34884}\begin{bmatrix} 2844 & 9682 \\ 386 & -1284 \end{bmatrix},$$

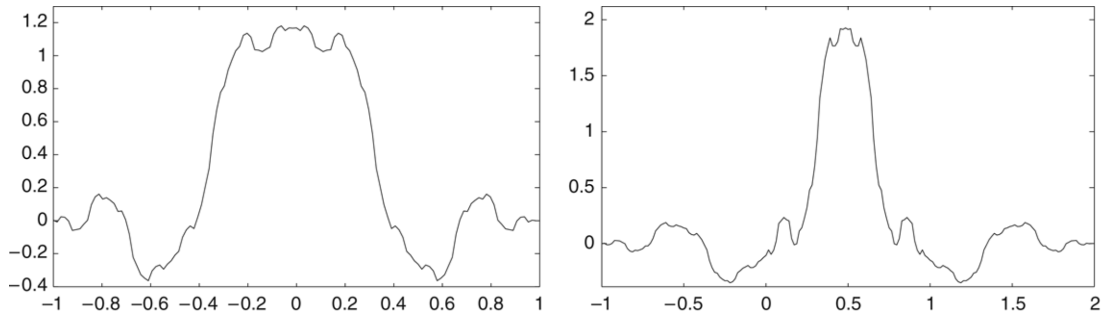

Fig. 6. The graphs of $\tilde{\phi}_1$ (left) and $\tilde{\phi}_2$ (right) in the symmetric dual 2-refinable function vector $\tilde{\phi} = [\tilde{\phi}_1, \tilde{\phi}_2]^{\mathrm{T}}$ constructed in Example 5.2 for the interpolating 2-refinable vector in Example 4.1. Moreover, $\nu_2(\tilde{\phi}) \approx 1.117992$ and $\tilde{\phi}_1(-x) = \tilde{\phi}_1(x)$ and $\tilde{\phi}_2(1-x) = \tilde{\phi}_2(x)$ for all $x \in \mathbb{R}$.

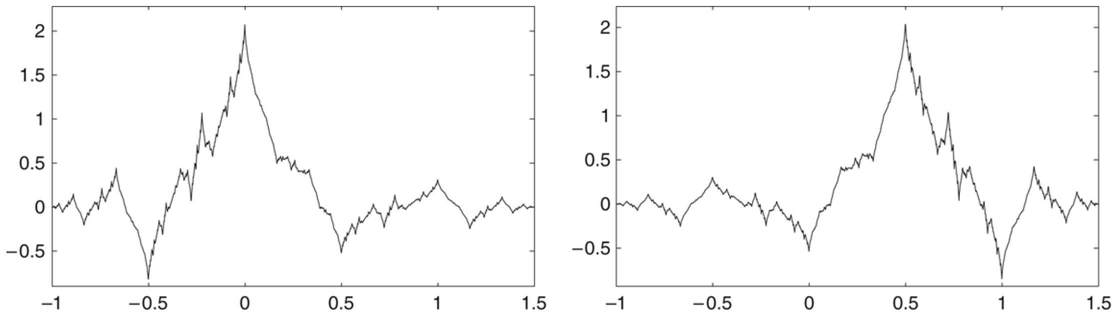

Fig. 7. The graphs of $\tilde{\phi}_1$ (left) and $\tilde{\phi}_2$ (right) in the dual 3-refinable function vector $\tilde{\phi} = [\tilde{\phi}_1, \tilde{\phi}_2]^{\mathrm{T}}$ constructed in Example 5.3 for the interpolating 3-refinable vector in Example 4.2. Moreover, $\nu_2(\tilde{\phi}) \approx 0.736519$.

$$\tilde{a}(0) = \frac{1}{34884}\begin{bmatrix} 17496 & 8715 \\ -2961 & 2590 \end{bmatrix}, \quad \tilde{a}(1) = \frac{1}{34884}\begin{bmatrix} 2590 & -2961 \\ 8715 & 17496 \end{bmatrix},$$
$$\tilde{a}(2) = \frac{1}{34884}\begin{bmatrix} -1284 & 386 \\ 9682 & 2844 \end{bmatrix}, \quad \tilde{a}(3) = \frac{1}{34884}\begin{bmatrix} 1866 & -969 \\ -4773 & 1292 \end{bmatrix}$$

with $\tilde{a}(k) = 0$ for $k \in \mathbb{Z} \setminus \{-2, -1, 0, 1, 2, 3\}$. By calculation, we have $\nu_2(\tilde{a}, 3) \approx 0.736519$. So, the associated 3-refinable function vectors ϕ and $\tilde{\phi}$ with masks a and $\tilde{a}$ indeed satisfy the biorthogonal relation in (5.1). See Fig. 7 for the graph of the dual 3-refinable function vector $\tilde{\phi} = [\tilde{\phi}_1, \tilde{\phi}_2]^{\mathrm{T}}$. Note that $\tilde{\phi}_1(x) = \tilde{\phi}_2(1/2 - x)$ for all $x \in \mathbb{R}$.

6. Conclusions and remarks

In this paper, we present in Theorem 2.1 a complete characterization of a generalized interpolating refinable function vector in terms of its mask. As a consequence, we have a criterion for orthogonal interpolating refinable function vectors in Corollary 2.2. We introduce the notion of an interpolatory mask of type (d, r) and study its sum rule structure in Theorem 2.3. We provide in Section 3 a family of interpolatory masks of type (d, r) with arbitrarily high orders of sum rules and address in Section 5 how to construct biorthogonal multiwavelets using the CBC algorithm in [6] from the interpolatory masks of type (d, r) obtained in this paper. Examples are given in Sections 4 and 5 to illustrate the theoretical results of this paper.

Symmetry property is one of the most important and desired properties in wavelet analysis (e.g. [2,4,7,12]). Though we provide a family of symmetric interpolatory masks of type $(2, 2)$ in Corollary 3.3, we did not address the symmetry properties of a general interpolatory mask of type (d, r) and its associated refinable function vector. When the dilation factor $d > 2$ and the multiplicity $r > 1$, except for some special cases (e.g. [12]), it seems that little is known in the literature about the connections between the symmetry property of a matrix mask and that of its associated refinable function vector. For an interpolating refinable function vector $\phi = [\phi_1, \ldots, \phi_r]^{\mathrm{T}}$ with an interpolatory mask a of type (d, r), it is natural that each function ϕ_ℓ, $\ell = 1, \ldots, r$, is symmetric about the point $(\ell-1)/r$, more precisely, $\phi_\ell(2(\ell-1)/r - \cdot) = \phi_\ell$. However, for $d > 2$ and $r > 1$, it is unclear to us so far under which kind of symmetry conditions on its interpolatory mask a, the interpolating refinable function vector ϕ is guaranteed to possess the desired symmetry property. That is, what is the right symmetry condition for an interpolatory mask of type (d, r) so that its associated interpolating refinable function vector possesses certain desired symmetry.

For the families of interpolatory masks of type (d, r) in Section 3, it is desirable that the smoothness quantity $\nu_2(a, d)$ could increase linearly with respect to the order of sum rules of the mask a (or equivalently, with respect to the length of the support of the mask a). Though Table 1 seems to suggest that this is true for our families of interpolatory masks, we are unable to prove this at this moment. We shall leave these questions as well as other related problems on generalized interpolating refinable function vectors for future study.

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References

[1] D.R. Chen, B. Han, S.D. Riemenschneider, Construction of multivariate biorthogonal wavelets with arbitrary vanishing moments, Adv. Comput. Math. 13 (2000) 131–165.
[2] I. Daubechies, Ten Lectures on Wavelets, in: CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia, 1992.
[3] G. Deslauriers, S. Dubuc, Symmetric iterative interpolation processes, Constr. Approx. 5 (1989) 49–68.
[4] B. Han, Symmetric orthonormal scaling functions and wavelets with dilation factor 4, Adv. Comput. Math. 8 (1998) 221–247.
[5] B. Han, Analysis and construction of optimal multivariate biorthogonal wavelets with compact support, SIAM J. Math. Anal. 31 (1999) 274–304.
[6] B. Han, Approximation properties and construction of Hermite interpolants and biorthogonal multiwavelets, J. Approx. Theory 110 (2001) 18–53.
[7] B. Han, Symmetry property and construction of wavelets with a general dilation matrix, Linear Algebra Appl. 353 (2002) 207–225.
[8] B. Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, J. Approx. Theory. 124 (2003) 44–88.
[9] B. Han, Solutions in Sobolev spaces of vector refinement equations with a general dilation matrix, Adv. Comput. Math. 24 (2006) 375–403.
[10] B. Han, The structure of balanced multivariate biorthogonal multiwavelets and dual multiframelets, 2007, preprint.
[11] B. Han, R.Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, SIAM J. Math. Anal. 29 (1998) 1177–1199.
[12] B. Han, Q. Mo, Multiwavelet frames from refinable function vectors, Adv. Comput. Math. 18 (2003) 211–245.
[13] R.Q. Jia, C.A. Micchelli, On linear independence of integer translates of a finite number of functions, Proc. Edinb. Math. Soc. (1992) 69–85.
[14] K. Koch, Interpolating scaling vectors, Int. J. Wavelets Multiresolut. Info. Process. 3 (2005) 389–416.
[15] I.W. Selesnick, Interpolating multiwavelet bases and the sampling theorem, IEEE Trans. Signal Process. 47 (1999) 1615–1621.
[16] D.X. Zhou, Interpolatory orthogonal multiwavelets and refinable functions, IEEE Trans. Signal Process. 50 (2002) 520–527.