Acta Appl Math (2009) 107: 143–171
DOI 10.1007/s10440-008-9399-8

Analysis and Construction of Multivariate Interpolating Refinable Function Vectors

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Received: 28 September 2008 / Accepted: 28 November 2008 / Published online: 10 December 2008

Abstract In this paper, we shall introduce and study a family of multivariate interpolating refinable function vectors with some prescribed interpolation property. Such interpolating refinable function vectors are of interest in approximation theory, sampling theorems, and wavelet analysis. In this paper, we characterize a multivariate interpolating refinable function vector in terms of its mask and analyze the underlying sum rule structure of its generalized interpolatory matrix mask. We also discuss the symmetry property of multivariate interpolating refinable function vectors. Based on these results, we construct a family of univariate generalized interpolatory matrix masks with increasing orders of sum rules and with symmetry for interpolating refinable function vectors. Such a family includes several known important families of univariate refinable function vectors as special cases. Several examples of bivariate interpolating refinable function vectors with symmetry will also be presented.

Keywords Interpolating refinable function vectors · Interpolatory masks · Interpolation property · Sum rules · Smoothness · Symmetry · Symmetry groups · Orthogonality

Mathematics Subject Classification (2000) 42C40 · 41A05 · 41A15 · 42B05 · 65T60

1 Introduction and Motivation

Refinable functions and refinable function vectors play a fundamental role in wavelet analysis and its applications such as signal processing, sampling theorems, numerical algorithms, and computer graphics. Let us recall the definition of a refinable function vector first.

Research was supported in part by NSERC Canada under Grant RGP 228051.

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We say that a $d \times d$ integer matrix M is *a dilation matrix* if $\lim_{n\to\infty} M^{-n} = 0$, that is, all the eigenvalues of M are greater than 1 in modulus. An *M-refinable function (or distribution) vector* $\phi = (\phi_1, \dots, \phi_L)^T$ satisfies the vector refinement equation

$$\phi = |\det M| \sum_{k\in\mathbb{Z}^d} a(k)\phi(M\cdot -k), \tag{1.1}$$

where $a : \mathbb{Z}^d \to \mathbb{C}^{L\times L}$ is called *a (matrix) mask* with multiplicity L for ϕ. When the multiplicity $L = 1$, ϕ and a are called a (scalar) refinable function and a scalar mask, respectively.

The refinement equation in (1.1) can be also stated in the frequency domain. For a sequence $u : \mathbb{Z}^d \to \mathbb{C}^{m\times n}$, its *(formal) Fourier series* $\hat{u}$ is defined to be

$$\hat{u}(\xi) := \sum_{k\in\mathbb{Z}^d} u(k)e^{-ik\cdot\xi}, \quad \xi \in \mathbb{R}^d, \tag{1.2}$$

where $k \cdot \xi$ denotes the inner product of the vectors k and ξ in $\mathbb{R}^d$. For a finitely supported sequence u, $\hat{u}$ is a matrix of 2π-periodic trigonometric polynomials in d-variables. Now the refinement equation in (1.1) can be rewritten in the frequency domain as

$$\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi), \quad \text{a.e. } \xi \in \mathbb{R}^d, \tag{1.3}$$

where M^T denotes the transpose of the matrix M and for $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\hat{f}$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx$, $\xi \in \mathbb{R}^d$, which can be naturally extended to tempered distributions and $L_2(\mathbb{R}^d)$.

Refinable function vectors with some interpolation property are of particular interest in wavelet analysis [1, 3, 5, 6, 10, 11, 15, 17, 22, 25, 26]. In the following, let us mention some known examples that motivate this paper.

Throughout this paper, δ denotes the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$. The sinc function is defined to be $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ for $x \in \mathbb{R}\backslash\{0\}$ and $\operatorname{sinc}(0) = 1$. The sinc function is continuous and symmetric about the origin. Moreover, sinc is interpolating: $\operatorname{sinc}|_{\mathbb{Z}} = \delta$. The sinc function is used in the Shannon sampling theorem saying that for $f \in C(\mathbb{R}) \cap L_1(\mathbb{R})$ such that $\hat{f}$ is supported inside $[-\pi, \pi]$ (that is, f is bandlimited with band π), $f(x) = \lim_{n\to\infty} \sum_{k=-n}^{n} f(k)\operatorname{sinc}(x - k)$, where the series converges uniformly for any $x \in \mathbb{R}$. The Shannon sampling theorem can be restated using shift-invariant spaces. For a function vector $f = (f_1, \dots, f_L)^T$ in $L_2(\mathbb{R}^d)$, we denote by $V(f)$ the smallest closed subspace in $L_2(\mathbb{R}^d)$ containing $f_1(\cdot - k), \dots, f_L(\cdot - k)$ for all $k \in \mathbb{Z}^d$. Due to the interpolation property of the sinc function, for any continuous function $f \in V(\operatorname{sinc}) \cap L_1(\mathbb{R}^d)$, one always has $f = \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}(\cdot - k)$. Note that $\widehat{\operatorname{sinc}} = \chi_{(-\pi,\pi]}$ is the characteristic function of the interval $(-\pi, \pi]$. It is well known in the literature ([1, 25, 26] and references therein) that sinc is a 2-refinable function satisfying $\widehat{\operatorname{sinc}}(2\xi) = \hat{a}(\xi)\widehat{\operatorname{sinc}}(\xi)$, where $\hat{a}$ is a 2π-periodic function defined by $\hat{a}(\xi) := \chi_{(-\pi/2,\pi/2]}(\xi)$ for all $\xi \in (-\pi, \pi]$. So, sinc is an interpolating 2-refinable function. For a function f that is not bandlimited with band π, one often considers its projection onto $V(\operatorname{sinc})$, for example, $f \approx \tilde{f} = \sum_{k\in\mathbb{Z}} \langle f, \operatorname{sinc}(\cdot - k)\rangle \operatorname{sinc}(\cdot - k)$. Note that $\langle \operatorname{sinc}, \operatorname{sinc}(\cdot - k)\rangle = \delta(k)$ for all $k \in \mathbb{Z}$ and $\tilde{f}$ is the orthogonal projection of f onto $V(\operatorname{sinc})$. However, sinc is not compactly supported and has a slow decay rate near ∞. In the literature, shift-invariant spaces generated by compactly supported (interpolating) refinable function vectors are often useful in sampling theorems in signal processing, see [1, 25, 26] and many references therein on applications of (interpolating) refinable function vectors in sampling theorems.

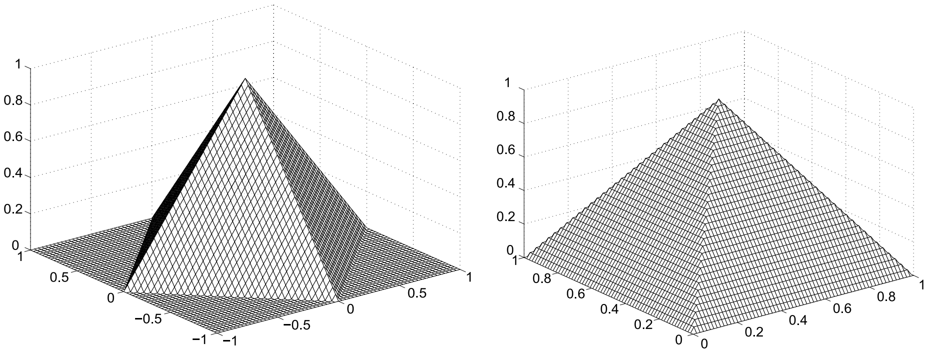


Fig. 1 The two components ϕ_0 (*left*) and ϕ_1 (*right*) of the $M_{\sqrt{2}}$-refinable function vector in [8]

However, there is no compactly supported 2-refinable function that is both interpolating and orthogonal [23, 26, 27]. In order to achieve both interpolation and orthogonality, Selesnick [23] and Zhou [27] presented several examples of compactly supported 2-refinable function vectors $\phi = (\phi_0, \phi_1)^T$ such that ϕ is orthogonal: $\langle \phi_\ell, \phi_n(\cdot - k)\rangle = \frac{1}{2}\delta(k)\delta(\ell - n)$ for all $k \in \mathbb{Z}$ and $\ell, n \in \{0, 1\}$, and ϕ is a continuous function vector satisfying the following interpolation property:

$$\phi_0(k/2) = \delta(k) \quad \text{and} \quad \phi_1(1/2 + k/2) = \delta(k) \quad \forall k \in \mathbb{Z}. \tag{1.4}$$

Unfortunately, as showed in [23, 27], such 2-refinable function vectors cannot have symmetry. Refinable function vectors with the interpolation property in (1.4) and with symmetry have been further developed in [16] for dimension one which extends [23, 27] to any dilation factor and any refinable function vector $\phi = (\phi_1, \ldots, \phi_L)^T$ with $L \geq 1$. In this paper, we shall generalize [16] from the univariate case to the multivariate case in a more general setting. This paper is greatly motivated by the following interesting bivariate example of Goodman [8], which obtains an $M_{\sqrt{2}}$-refinable function vector $\phi = (\phi_0, \phi_1)^T$ (see Fig. 1), where

$$M_{\sqrt{2}} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1.5}$$

From Fig. 1, it is evident that ϕ is a piecewise linear function vector satisfying the following interpolation property:

$$\phi_0(k) = \delta(k) \quad \text{and} \quad \phi_1((1/2, 1/2)^T + k) = \delta(k), \quad k \in M_{\sqrt{2}}^{-1}\mathbb{Z}^2. \tag{1.6}$$

But ϕ in Fig. 1 is not a function vector in $C^1(\mathbb{R}^2)$. It is of interest to have compactly supported $M_{\sqrt{2}}$-refinable function vectors satisfying the interpolation property in (1.6) and having higher order of smoothness, such as C^1 smoothness.

Motivated by the above examples, now we introduce the concept of interpolating refinable function vectors. Let N be a $d \times d$ invertible integer matrix. Noting that $\mathbb{Z}^d \subseteq N^{-1}\mathbb{Z}^d$, we denote by Γ_N an ordered complete set of representatives of the cosets of $[N^{-1}\mathbb{Z}^d]/\mathbb{Z}^d$ with the first element of Γ_N being $\mathbf{0}$; in dimension one, we simply take $\Gamma_N = \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\}$. Note that $\Gamma_N + \mathbb{Z}^d = N^{-1}\mathbb{Z}^d$. We say that a column vector $\phi = (\phi_\gamma)_{\gamma \in \Gamma_N}$ of functions on $\mathbb{R}^d$ is *an interpolating function vector* of type $(\Gamma_N, 0)$ if ϕ is continuous and satisfies the following interpolation property:

$$\phi_\gamma(\beta+k)=\delta(k)\delta(\beta-\gamma), \quad \forall \beta,\gamma\in\Gamma_N, k\in\mathbb{Z}^d. \tag{1.7}$$

Now for any continuous function $f\in C(\mathbb{R}^d)$, defining

$$\tilde{f}(x):=\sum_{\gamma\in\Gamma_N}\sum_{k\in\mathbb{Z}^d} f(k+\gamma)\phi_\gamma(x-k), \quad x\in\mathbb{R}^d, \tag{1.8}$$

we see that $f(x)=\tilde{f}(x)$ for all $x\in N^{-1}\mathbb{Z}^d$. That is, the function value of $\tilde{f}$ agrees with that of f on the lattice $N^{-1}\mathbb{Z}^d$.

Obviously, the sinc function is an interpolating 2-refinable function of type $(\Gamma_1,0)$. More generally, a continuous function ϕ satisfying $\phi|_{\mathbb{Z}^d}=\delta$ is simply an interpolating function of type $(\Gamma_{I_d},0)$, where I_d is the $d\times d$ identity matrix. The examples in [23, 27] are just interpolating 2-refinable function vectors of type $(\Gamma_2,0)$ and the one-dimensional examples in [16] correspond to interpolating M-refinable function vectors of type $(\Gamma_N,0)$ for any positive integers M ($M>1$) and N. The example in [8] shown in Fig. 1 is an example of interpolating $M_{\sqrt{2}}$-refinable function vectors of type $(\Gamma_{M_{\sqrt{2}}},0)$.

Denote $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. An interpolating function vector of type $(\Gamma_N,0)$ is a special case of interpolating function vectors of type (Γ_N,h) for $h\in\mathbb{N}_0$ that we will introduce in Sect. 3. For the more general interpolating refinable function vectors of type (Γ_N,h), all the derivatives up to degree h of f and $\tilde{f}$ (see (3.3)) agree on the lattice $N^{-1}\mathbb{Z}^d$. To improve the readability of this paper and to avoid introducing complicated notions at the very beginning, in this paper we shall breifly discuss interpolating refinable function vectors of type $(\Gamma_N,0)$ first in Sect. 2, and deal with the more general case in Sect. 3.

The structure of this paper is as follows. In Sect. 2, we present a criterion for an interpolating M-refinable function vector of type $(\Gamma_N,0)$ in terms of its interpolatory mask, as well as a criterion for an orthogonal and interpolating M-refinable function vector of type $(\Gamma_N,0)$. Several examples will be presented to illustrate the results. In Sect. 3, we shall introduce a more general notion of an interpolating function vector of type (Γ_N,h) for $h\in\mathbb{N}_0$ by interpolating all derivatives up to degree h of a function as well as a notion of an interpolatory mask of type (M,Γ_N,h). The Hermite interpolatory masks considered in [10, 12] correspond to interpolatory masks of type (M,Γ_{I_d},h). A $d\times d$ dilation matrix M is *isotropic* if it is similar to a diagonal matrix $\mathrm{diag}(\sigma_1,\dots,\sigma_d)$ such that $|\sigma_1|=\cdots=|\sigma_d|=|\det M|^{1/d}$. In Sect. 3, for an isotropic dilation matrix M, we shall characterize an interpolating M-refinable function vector of type (Γ_N,h) in terms of its mask and study the underlying sum rule structure of its interpolatory mask of type (M,Γ_N,h). Due to the importance of the symmetry property in applications, we shall also discuss the symmetry property of an interpolating M-refinable function vector and its interpolatory mask. For the design of interpolatory masks of type (M,Γ_N,h) with any preassigned orders of sum rules, these results will reduce the problem of solving a system of nonlinear equations into a problem of solving a system of linear equations. In Sect. 4, using the results in Sect. 3, for any dilation factor M and integers $N\in\mathbb{N}$ and $h\in\mathbb{N}_0$, we construct a family of univariate interpolatory masks of type (M,Γ_N,h) with increasing orders of sum rules. Such a family includes the family of the famous Deslauriers-Dubuc interpolatory masks in [5] and the family of Hermite interpolatory masks in [10] as special cases. The proofs to the main results in Sect. 3 will be given in Sect. 5.

2 Interpolating M-Refinable Function Vectors of Type $(\Gamma_N, 0)$

Before introducing interpolating M-refinable function vectors of type (Γ_N, h), in this section, we briefly study interpolating M-refinable function vectors of type $(\Gamma_N, 0)$ and present several examples to illustrate the results.

Before proceeding further, we recall a quantity $\nu_p(a, M)$, which plays an important role in wavelet analysis. Let ∂_j denote the differentiation operator with respect to the j-th coordinate. For $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$, ∂^μ is the differentiation operator $\partial_1^{\mu_1} \cdots \partial_d^{\mu_d}$. By $(\ell_0(\mathbb{Z}^d))^{m\times n}$ we denote the linear space of all finitely supported sequences of $m \times n$ matrices on $\mathbb{Z}^d$. Similarly, $u \in (\ell_p(\mathbb{Z}^d))^{m\times n}$ for $1 \le p \le \infty$ means that u is a sequence of $m \times n$ matrices on $\mathbb{Z}^d$ and $\|u\|_{(\ell_p(\mathbb{Z}^d))^{m\times n}} := (\sum_{k\in\mathbb{Z}^d} \|u(k)\|^p)^{1/p} < \infty$ for $1 \le p < \infty$ and $\|u\|_{(\ell_\infty(\mathbb{Z}^d))^{m\times n}} := \sup_{k\in\mathbb{Z}^d} \|u(k)\|$, where $\|\cdot\|$ denotes a matrix norm on $m \times n$ matrices. Throughout this paper, the notation $f(\xi) = g(\xi) + O(\|\xi\|^\kappa)$ as $\xi \to 0$ just means $\partial^\mu f(0) = \partial^\mu g(0)$ for all $|\mu| < \kappa$ and $\mu \in \mathbb{N}_0^d$, where $|\mu| := |\mu_1| + \cdots + |\mu_d|$ and $\mathbb{N}_0^d := (\mathbb{N} \cup \{0\})^d$.

For a matrix mask a with multiplicity L, we say that a satisfies the *sum rules* of order κ with a dilation matrix M if there exists a sequence $y \in (\ell_0(\mathbb{Z}^d))^{1\times L}$ such that $\hat{y}(0) \ne 0$ and

$$\begin{aligned} \partial^\mu[\hat{y}(M^T\cdot)\hat{a}(\cdot)](0) &= \partial^\mu \hat{y}(0) \quad \forall |\mu| < \kappa,\ \mu \in \mathbb{N}_0^d, \\ \partial^\mu[\hat{y}(M^T\cdot)\hat{a}(\cdot)](2\pi\gamma) &= 0 \quad \forall |\mu| < \kappa,\ \gamma \in [(M^T)^{-1}\mathbb{Z}^d] \setminus \mathbb{Z}^d. \end{aligned} \tag{2.1}$$

Let Ω_{M^T} be a complete set of representatives of the cosets $\mathbb{Z}^d/[M^T\mathbb{Z}^d]$ such that $\mathbf{0} \in \Omega_{M^T}$. Then, similar to [16], one can show that (2.1) is equivalent to

$$\hat{y}(M^T\xi)\hat{a}^\omega(\xi) = |\det M|^{-1}\hat{y}(\xi) + O(\|\xi\|^\kappa), \quad \xi \to 0,\ \omega \in \Omega_{M^T}, \tag{2.2}$$

where $\hat{a}^\omega(\xi) := \sum_{k\in\mathbb{Z}^d} a(\omega + Mk)e^{-i\xi\cdot(\omega+Mk)}$ is called the *coset* of $\hat{a}(\xi)$.

The convolution of two sequences u and v is defined to be

$$[u * v](j) := \sum_{k\in\mathbb{Z}^d} u(k)v(j-k), \quad u \in (\ell_0(\mathbb{Z}^d))^{r\times m},\ v \in (\ell_0(\mathbb{Z}^d))^{m\times n}.$$

Clearly, $\widehat{u * v} = \hat{u}\hat{v}$. For $y \in (\ell_0(\mathbb{Z}^d))^{1\times L}$ and a positive integer κ, as in [12], we define the space $\mathcal{V}_{\kappa,y}$ by

$$\mathcal{V}_{\kappa,y} := \{v \in (\ell_0(\mathbb{Z}^d))^{L\times 1} : \partial^\mu[\hat{y}(\cdot)\hat{v}(\cdot)](0) = 0\ \forall |\mu| < \kappa,\ \mu \in \mathbb{N}_0^d\}. \tag{2.3}$$

By convention, $\mathcal{V}_{0,y} := (\ell_0(\mathbb{Z}^d))^{L\times 1}$. Note that the above equations in (2.1), (2.2), and (2.3) depend only on the values $\partial^\mu \hat{y}(0)$, $|\mu| < \kappa$. For a mask a with multiplicity L, a sequence $y \in (\ell_0(\mathbb{Z}^d))^{1\times L}$ and a dilation matrix M, we define

$$\rho_\kappa(a, M, y, p) := \sup\Big\{\limsup_{n\to\infty} \|a_n * v\|^{1/n}_{(\ell_p(\mathbb{Z}^d))^{L\times 1}} : v \in \mathcal{V}_{\kappa,y}\Big\}, \quad \kappa \in \mathbb{N}_0, \tag{2.4}$$

where $\widehat{a_n}(\xi) := \hat{a}((M^T)^{n-1}\xi) \cdots \hat{a}(M^T\xi)\hat{a}(\xi)$. For $1 \le p \le \infty$, define

$$\begin{aligned} \rho(a, M, p) := \inf\{\rho_\kappa(a, M, y, p) &: (2.1) \text{ holds for some } \kappa \in \mathbb{N}_0 \\ &\text{and some } y \in (\ell_0(\mathbb{Z}^d))^{1\times L} \text{ with } \hat{y}(0) \ne 0\}. \end{aligned} \tag{2.5}$$

As in [12, p. 61], we define the following quantity:

$$\nu_p(a, M) := -\log_{\rho(M)}\big[|\det M|^{1-1/p}\rho(a, M, p)\big], \quad 1 \le p \le \infty, \tag{2.6}$$

where $\rho(M)$ denotes the spectral radius of the matrix M. In the above definition of $\rho(a, M, p)$, it seems that the sequences y (more precisely, the vectors $\partial^\mu \hat{y}(0)$, $|\mu| < \kappa$) are not uniquely determined. Up to a scalar multiplicative constant, we point out that the vectors $\partial^\mu \hat{y}(0)$, $\mu \in \mathbb{N}_0^d$ are quite often uniquely determined ([12] and Theorem 3.2 of this paper).

The above quantity $\nu_p(a, M)$ plays a very important role in characterizing the convergence of a vector cascade algorithm in a Sobolev space and in characterizing the L_p smoothness of a refinable function vector. It was showed in [12, Theorem 4.3] that the vector cascade algorithm associated with mask a and an isotropic dilation matrix M converges in the Sobolev space $W_p^h(\mathbb{R}^d) := \{f \in L_p(\mathbb{R}^d) : \partial^\mu f \in L_p(\mathbb{R}^d)\ \forall |\mu| \le h\}$ if and only if $\nu_p(a, M) > h$. In general, $\nu_p(a, M)$ provides a lower bound for the L_p smoothness exponent of a refinable function vector ϕ with a mask a and a dilation matrix M, that is, $\nu_p(a, M) \le \nu_p(\phi)$ always holds, where $\nu_p((f_1, \dots, f_L)^T) := \min_{1 \le \ell \le L} \nu_p(f_\ell)$ and for $f \in L_p(\mathbb{R}^d)$,

$$\nu_p(f) := \sup\{n + \nu : \|\partial^\mu f - \partial^\mu f(\cdot - t)\|_{L_p(\mathbb{R}^d)} \le C_f |t|^\nu\ \forall |\mu| = n; t \in \mathbb{R}^d\}. \tag{2.7}$$

Moreover, if the shifts of the refinable function vector ϕ associated with a mask a and an isotropic dilation matrix M are stable in $L_p(\mathbb{R}^d)$ (see (5.1)), then $\nu_p(\phi) = \nu_p(a, M)$. That is, $\nu_p(a, M)$ indeed characterizes the L_p smoothness exponent of a refinable function vector ϕ with a mask a and an isotropic dilation matrix M. Furthermore, we also have $\nu_p(a, M) \ge \nu_q(a, M) \ge \nu_p(a, M) + (1/q - 1/p) \log_{\rho(M)} |\det M|$ for $1 \le p \le q \le \infty$. In particular, we have $\nu_2(a, M) \ge \nu_\infty(a, M) \ge \nu_2(a, M) - s/2$ when M is isotropic. For a finitely supported matrix mask a, the quantity $\nu_2(a, M)$ can be numerically computed by finding the spectral radius of certain finite matrix using an interesting algorithm in [20] (also see [13] for computing $\nu_2(a, M)$ using the symmetry of the mask a). Interested readers should consult [2, 6, 9, 10, 12, 14, 15, 18, 24, 28] and many references therein for more details on the convergence of vector cascade algorithms and smoothness of refinable function vectors.

For an ordered set Γ, we denote by $\#\Gamma$ the cardinality of Γ. For a $(\#\Gamma) \times (\#\Gamma)$ matrix A and $\alpha, \beta \in \Gamma$, we use $[A]_{\alpha,\beta}$ to denote the (α, β)-entry of A.

The following result generalizes [16, Theorem 2.1] from dimension one to high dimensions for interpolating M-refinable function vectors of type $(\Gamma_N, 0)$ and is a special case of Theorem 3.1 in Sect 3.

Theorem 2.1 *Let M be a $d \times d$ dilation matrix and N be a $d \times d$ integer matrix such that*

$$N \text{ is invertible} \quad \text{and} \quad NMN^{-1} \text{ is an integer matrix.} \tag{2.8}$$

Let Γ_N be a given ordered complete set of representatives of $[N^{-1}\mathbb{Z}^d]/\mathbb{Z}^d$ with the first element of Γ_N being $\mathbf{0}$. Let $\phi = (\phi_\gamma)_{\gamma \in \Gamma_N}$ be a $(\#\Gamma_N) \times 1$ column vector of compactly supported distributions such that $\hat{\phi}(M^T \xi) = \hat{a}(\xi)\hat{\phi}(\xi)$, where $a : \mathbb{Z}^d \mapsto \mathbb{C}^{(\#\Gamma_N) \times (\#\Gamma_N)}$ is a finitely supported matrix mask for ϕ. Then ϕ is an interpolating M-refinable function vector of type $(\Gamma_N, 0)$ (that is, ϕ is continuous and (1.7) holds) if and only if the following statements hold:

(i) $(1, 1, \dots, 1)\hat{\phi}(0) = 1$ (*this is a normalization condition on ϕ*);
(ii) $\nu_\infty(a, M) > 0$;
(iii) *The mask a is an interpolatory mask of type $(M, \Gamma_N, 0)$; that is, the mask a satisfies*

$$(1, 1, \dots, 1)\hat{a}(0) = (1, 1, \dots, 1), \tag{2.9}$$

$$[a(Mk + [M\alpha]_{\Gamma_N})]_{\gamma,\langle M\alpha\rangle_{\Gamma_N}} = |\det M|^{-1}\delta(k)\delta(\alpha-\gamma),$$
$$\forall \alpha,\gamma\in\Gamma_N,\ k\in\mathbb{Z}^d, \tag{2.10}$$

where $[M\alpha]_{\Gamma_N}\in\mathbb{Z}^d$ *and* $\langle M\alpha\rangle_{\Gamma_N}\in\Gamma_N$ *are uniquely determined by the relation* $M\alpha = [M\alpha]_{\Gamma_N} + \langle M\alpha\rangle_{\Gamma_N}$.

The condition in (2.8) is quite natural since it is equivalent to the requirement that $MN^{-1}\mathbb{Z}^d \subseteq N^{-1}\mathbb{Z}^d$. As proved in [12, Theorem 4.3], a mask a must satisfy the sum rules of order κ, where κ is the largest integer such that $\kappa < \nu_p(a, M)$. In order to have a smooth refinable function vector, one often needs to design a mask a with a large quantity $\nu_p(a, M)$. Therefore, it is of practical interest to have matrix masks with high orders of sum rules. In order to obtain an interpolatory mask a of type $(M,\Gamma_N,0)$, though the equations induced by (2.9) and (2.10) are indeed linear equations, the equations induced by the sum rule condition in (2.1) are nonlinear equations, since the values $\partial^\mu\hat{y}(0), |\mu|<\kappa$ are not known in advance in (2.1). We shall discuss the sum rule structure of an interpolatory mask in Sect. 3 and we shall see in Theorem 3.2 that under a mild condition, (2.9) and (2.10) together with (2.1) will uniquely determine the values $\partial^\mu\hat{y}(0), |\mu|<\kappa$. Consequently, to design an interpolatory mask with any preassigned order of sum rules, one only needs to solve a system of linear equations. For more details, see Sect. 3.

The example in Goodman [8] (also cf. [4]) is an interpolating $M_{\sqrt{2}}$-refinable function vector of type $(\Gamma_{M_{\sqrt{2}}}, 0)$, but it is not in $C^1(\mathbb{R}^2)$. In the following, we present a C^1 interpolating $M_{\sqrt{2}}$-refinable function vector of type $(\Gamma_{M_{\sqrt{2}}}, 0)$.

Example 2.2 Let $M = N = M_{\sqrt{2}}$, where $M_{\sqrt{2}}$ is defined in (1.5). Let a be an interpolatory mask of type $(M,\Gamma_N,0)$ with support $[-2,1]\times[-2,1]$ and of D_4-symmetry (see (3.9) in Sect. 3). Solving a system of linear equations, we obtain a parametric expression of the mask a which satisfies the sum rules of order 2 and is given by:

$$a(-2,-2)=\begin{bmatrix}0 & t_3\\ 0 & 0\end{bmatrix},\qquad a(-2,-1)=\begin{bmatrix}0 & t_5\\ 0 & 0\end{bmatrix},$$
$$a(-2,0)=\begin{bmatrix}0 & t_5\\ 0 & 0\end{bmatrix},\qquad a(-2,1)=\begin{bmatrix}0 & t_3\\ 0 & 0\end{bmatrix},$$
$$a(-1,-2)=\begin{bmatrix}0 & t_5\\ 0 & t_4\end{bmatrix},\qquad a(-1,1)=\begin{bmatrix}0 & t_5\\ 0 & t_4\end{bmatrix},$$
$$a(1,-2)=\begin{bmatrix}0 & t_3\\ 0 & t_2\end{bmatrix},\qquad a(1,-1)=\begin{bmatrix}0 & t_5\\ 0 & t_1\end{bmatrix},$$
$$a(1,0)=\begin{bmatrix}0 & t_5\\ \frac{1}{2} & t_1\end{bmatrix},\qquad a(1,1)=\begin{bmatrix}0 & t_3\\ 0 & t_2\end{bmatrix},$$
$$a(0,-2)=\begin{bmatrix}0 & t_5\\ 0 & t_2\end{bmatrix},\qquad a(0,1)=\begin{bmatrix}0 & t_5\\ 0 & t_2\end{bmatrix},$$
$$a(-1,0)=\begin{bmatrix}0 & \frac{1}{4}-t_3-2t_5-2t_2-t_4-t_1\\ 0 & t_2\end{bmatrix},$$
$$a(-1,-1)=\begin{bmatrix}0 & \frac{1}{4}-t_3-2t_5-2t_2-t_4-t_1\\ 0 & t_2\end{bmatrix},$$

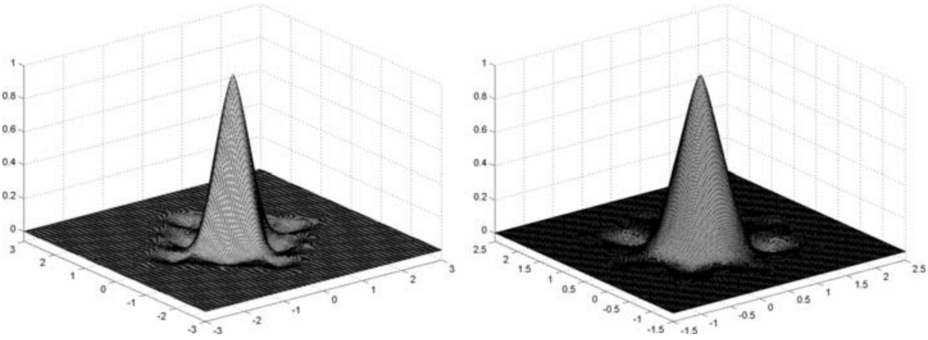


Fig. 2 The graphs of $\phi_{(0,0)}$ (*left*) and $\phi_{(\frac{1}{2},\frac{1}{2})}$ (*right*) of the D_4-symmetric interpolating $M_{\sqrt{2}}$-refinable function vector of type $(\Gamma_{M_{\sqrt{2}}}, 0)$ in Example 2.2 for $t_1 = \frac{3}{32}, t_2 = t_4 = 0, t_3 = -\frac{5}{256}$, and $t_5 = -\frac{3}{256}$

$$a(0,-1) = \begin{bmatrix} 0 & \frac{1}{4} - t_3 - 2t_5 - 2t_2 - t_4 - t_1 \\ 0 & t_1 \end{bmatrix},$$

$$a(0,0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} - t_3 - 2t_5 - 2t_2 - t_4 - t_1 \\ 0 & t_1 \end{bmatrix},$$

where $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}$ are free parameters. When $t_2 = t_3 = t_4 = t_5 = 0$, the mask a is supported inside $[-1, 1] \times [-1, 0]$. If in addition $t_1 = 0$, then we have $\nu_2(a, M) = 1.5$ (and therefore, $\nu_\infty(a, M) \geq \nu_2(a, M) - 1 \geq 0.5$) and this is the mask for the interpolating $M_{\sqrt{2}}$-refinable function vector given in Goodman [8] (see Fig. 1).

When $t_1 = \frac{1}{4} + 8t_3, t_2 = t_4 = 0, t_5 = -\frac{1}{32} - t_3$, the mask a satisfies the sum rules of order 4. If in addition $t_3 = -\frac{5}{256}$, we have $\nu_2(a, M_{\sqrt{2}}) \approx 2.535219$. Therefore, $\nu_\infty(a, M_{\sqrt{2}}) \geq \nu_2(a, M_{\sqrt{2}}) - 1 = 1.535219 > 1$. By Theorem 2.1, its associated $M_{\sqrt{2}}$-refinable function vector $\phi = (\phi_{(0,0)}, \phi_{(\frac{1}{2},\frac{1}{2})})^T$ is an interpolating $M_{\sqrt{2}}$-refinable function vector in $(C^1(\mathbb{R}^2))^{2\times 1}$ of type $(\Gamma_{M_{\sqrt{2}}}, 0)$ (see Fig. 2).

By Theorem 2.1, using the same proof of [16, Corollary 2.2], we have the following result on orthogonal and interpolating M-refinable function vectors of type $(\Gamma_N, 0)$.

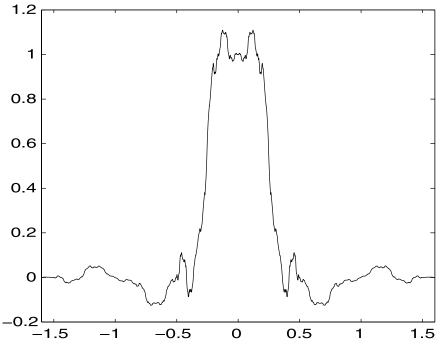
Corollary 2.3 *Let* M, N, Γ_N, ϕ, a *be as in Theorem* 2.1. *Then* ϕ *is an interpolating* M*-refinable function vector of type* $(\Gamma_N, 0)$ *and satisfies the following orthogonality condition*:

$$\int_{\mathbb{R}^d} \phi(x-j)\overline{\phi(x)}^T dx = \frac{1}{\#\Gamma_N}\delta(j) I_{\#\Gamma_N}, \quad \forall j \in \mathbb{Z}^d, \tag{2.11}$$

if and only if, (i)–(iii) *of Theorem* 2.1 *hold and* a *is an orthogonal mask*:

$$\sum_{k\in\mathbb{Z}^d} a(k)\overline{a(Mj+k)}^T = \frac{1}{|\det M|}\delta(j) I_{\#\Gamma_N}, \quad \forall j \in \mathbb{Z}^d. \tag{2.12}$$

As shown in [23, 27] for dimension one, one cannot obtain symmetric, orthogonal and interpolating 2-refinable function vectors of type $(\Gamma_2, 0)$. Some orthogonal and interpolating M-refinable function vectors of type $(\Gamma_N, 0)$ have been given in [16] but without symmetry.



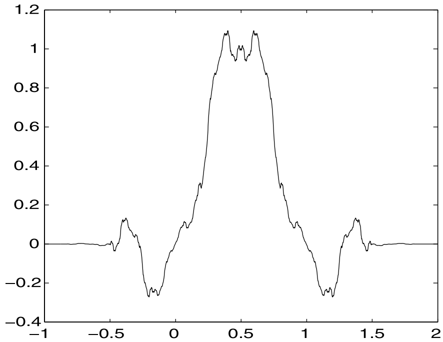


Fig. 3 The graphs of ϕ_0 and $\phi_{\frac{1}{2}}$ of the symmetric, orthogonal, and interpolating 3-refinable function vector ϕ of type $(\Gamma_2, 0)$ in Example 2.4. $\nu_2(\phi) \approx 0.976503$. $\phi_0(-x) = \phi_0(x)$ and $\phi_{\frac{1}{2}}(1-x) = \phi_{\frac{1}{2}}(x)$

Here, we provide a symmetric, orthogonal and interpolating 3-refinable function vector of type $(\Gamma_2, 0)$.

Example 2.4 Let $M = 3$ and $N = 2$. An orthogonal and interpolatory mask a of type $(3, \Gamma_2, 0)$ satisfying the sum rules of order 2 is supported inside $[-4, 4]$ and is given by

$$a(0) = \begin{bmatrix} \frac{1}{3} & \frac{29}{108} + \frac{\sqrt{41}}{108} \\ 0 & \frac{7}{60} - \frac{\sqrt{41}}{180} \end{bmatrix}, \qquad a(1) = \begin{bmatrix} \frac{11}{108} - \frac{\sqrt{41}}{108} & 0 \\ \frac{37}{180} + \frac{\sqrt{41}}{60} & \frac{1}{3} \end{bmatrix},$$

$$a(2) = \begin{bmatrix} -\frac{2}{135} - \frac{\sqrt{41}}{270} & \frac{1}{540} - \frac{\sqrt{41}}{540} \\ \frac{37}{180} + \frac{\sqrt{41}}{60} & \frac{7}{60} - \frac{\sqrt{41}}{180} \end{bmatrix}, \qquad a(3) = \begin{bmatrix} 0 & \frac{17}{270} - \frac{\sqrt{41}}{135} \\ 0 & -\frac{7}{60} + \frac{\sqrt{41}}{180} \end{bmatrix},$$

$$a(4) = \begin{bmatrix} -\frac{47}{540} + \frac{7\sqrt{41}}{540} & 0 \\ \frac{23}{180} - \frac{\sqrt{41}}{60} & 0 \end{bmatrix},$$

while $a(-4), a(-3), a(-2), a(-1)$ can be obtained by the symmetry condition in (3.12). Then we have $\nu_2(a, 2) \approx 0.976503$. Therefore, $\nu_\infty(a, 2) \geq \nu_2(a, 2) - 1/2 \approx 0.476503 > 0$. By Corollary 2.3 and Theorem 3.3, its associated 3-refinable function vector $\phi = (\phi_0, \phi_{\frac{1}{2}})^T$ is a symmetric, orthogonal and interpolating 3-refinable function vector of type $(\Gamma_2, 0)$. Moreover, $\phi_0(-x) = \phi_0(x)$ and $\phi_{\frac{1}{2}}(1-x) = \phi_{\frac{1}{2}}(x)$. See Fig. 3 for the graphs of ϕ_0 and $\phi_{\frac{1}{2}}$.

3 Interpolating Refinable Function Vectors of Type (Γ_N, h)

In this section, we shall introduce a more general class of interpolating function vectors by interpolating not only the function values at the lattice $N^{-1}\mathbb{Z}^d$ but also all its derivatives up to degree h on $N^{-1}\mathbb{Z}^d$. This includes interpolating function vectors discussed in Sect. 2 as a special case with $h = 0$ and the Hermite interpolants in [10, 12, 18, 28] as a special case with $N = I_d$.

To introduce the notion of interpolating function vectors of type (Γ_N, h), we need a little bit more complicated notations. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}_0^d$, we denote $\mu! := \mu_1! \cdots \mu_d!$, $|x| := |x_1| + \cdots + |x_d|$ and $x^\mu := x_1^{\mu_1} \cdots x_d^{\mu_d}$. For a given order of derivative $h \in \mathbb{N}_0$, we denote $O_h := \{\mu : |\mu| = h, \mu \in \mathbb{N}_0^d\}$ and $\Lambda_h := \{\mu : |\mu| \leq h, \mu \in \mathbb{N}_0^d\}$. It is easy to see that $\#\Lambda_h = \binom{h+d}{d}$. Throughout this paper, the elements in O_h and Λ_h will be always ordered in such a way that $\nu = (\nu_1, \ldots, \nu_d)$ is less than $\mu = (\mu_1, \ldots, \mu_d)$ if either $|\nu| < |\mu|$ or if $|\nu| = |\mu|$, $\nu_j = \mu_j$ for $j = 1, \ldots, \ell - 1$ and $\nu_\ell < \mu_\ell$ for some $1 \le \ell \le d$.

For simplicity, we denote $\mathcal{D}^{\Lambda_h} := (\partial^\mu)_{\mu\in\Lambda_h}$ which is a $1 \times (\#\Lambda_h)$ row vector of differentiation. For a $d \times d$ matrix N, $S(N, O_h)$ is defined to be the following $(\#O_h) \times (\#O_h)$ matrix [10, 13], uniquely determined by

$$\frac{(Nx)^\mu}{\mu!} = \sum_{\nu\in O_h} S(N, O_h)_{\mu,\nu} \frac{x^\nu}{\nu!}, \quad \mu \in O_h. \tag{3.1}$$

Clearly, $S(N, \Lambda_h) := \operatorname{diag}(S(N, O_0), S(N, O_1), \ldots, S(N, O_h))$, which is a $(\#\Lambda_h) \times (\#\Lambda_h)$ matrix. It is obvious that $S(A, O_h)S(B, O_h) = S(AB, O_h)$.

For matrices $A = (a_{i,j})_{1\le i\le I, 1\le j\le J}$ and $B = (b_{\ell,k})_{1\le\ell\le L, 1\le k\le K}$, the (right) *Kronecker product* $A\otimes B$ is defined to be $A\otimes B := (a_{i,j}B)_{1\le i\le I,1\le j\le J}$; its $((i-1)L+\ell, (j-1)K+k)$-entry is $a_{i,j}b_{\ell,k}$ and can be conveniently denoted by $[A \otimes B]_{i,j;\ell,k}$. Throughout this paper, for an $I \times J$ block matrix A with each block of size $L \times K$, we will use $[A]_{i,j}$ to denote the (i, j)-block of A and $[A]_{i,j;\ell,k}$ to denote the (ℓ, k)-entry of the block $[A]_{i,j}$. It is well known that $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$, $C \otimes (A + B) = (C \otimes A) + (C \otimes B)$, $(A \otimes B)(C \otimes E) = (AC) \otimes (BE)$ and $(A \otimes B)^T = A^T \otimes B^T$.

Now we are ready to introduce *an interpolating function vector of type* (Γ_N, h). Let N and Γ_N be defined as before. Let $\phi = (\phi_\gamma)_{\gamma\in\Gamma_N}$ be a $(\#\Gamma_N)(\#\Lambda_h) \times 1$ column vector of compactly supported distributions with each $\phi_\gamma = (\phi_{\gamma,\mu})_{\mu\in\Lambda_h}$ being a $(\#\Lambda_h) \times 1$ column vector. For $h \in \mathbb{N}_0$, we say that ϕ is *an interpolating function vector of type* (Γ_N, h) if $\phi \in (C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}$ and satisfies the following interpolation property:

$$[\mathcal{D}^{\Lambda_h} \otimes \phi_\gamma](\beta + k) = \delta(k)\delta(\beta - \gamma)I_{\#\Lambda_h}, \quad \forall \beta, \gamma \in \Gamma_N, k \in \mathbb{Z}^d. \tag{3.2}$$

The above notation seems a little bit complicated, but it essentially says that the components of the function vector ϕ interpolate all the derivatives up to degree h on the lattice $N^{-1}\mathbb{Z}^d$. Let ϕ be an interpolating function vector of type (Γ_N, h). For a function $f \in C^h(\mathbb{R}^d)$, defining

$$\tilde{f}(x) := \sum_{\gamma\in\Gamma_N} \sum_{\mu\in\Lambda_h} \sum_{k\in\mathbb{Z}^d} [\partial^\mu f](k + \gamma)\phi_{\gamma,\mu}(x - k), \quad x \in \mathbb{R}^d, \tag{3.3}$$

then $\partial^\mu \tilde{f}(x) = \partial^\mu f(x)$ for all $\mu \in \Lambda_h$ and $x \in N^{-1}\mathbb{Z}^d$; that is, $\tilde{f}$ agrees with f on the lattice $N^{-1}\mathbb{Z}^d$ with all derivatives up to degree h. When $h = 0$, the above definition reduces to interpolating function vectors of type $(\Gamma_N, 0)$. When $N = I_d$ and $\Gamma_N = \{0\}$, it is called the Hermite interpolants of order h in [10, 12, 18, 28].

As a generalization of [12, Corollary 5.2] on refinable Hermite interpolants (also cf. [6, 21, 28]) and [16, Theorem 2.1] on generalized interpolating refinable function vectors, the following result characterizes a compactly supported interpolating M-refinable function vector of type (Γ_N, h) in terms of its mask.

Theorem 3.1 *Let $h \in \mathbb{N}_0$ and M be a $d \times d$ dilation matrix. When $h > 0$, we further assume that M is isotropic. Let N be a $d \times d$ integer matrix such that (2.8) holds. Let Γ_N be a given ordered complete set of representatives of $[N^{-1}\mathbb{Z}^d]/\mathbb{Z}^d$ with the first element of Γ_N being $\mathbf{0}$. Let $\phi = (\phi_\gamma)_{\gamma\in\Gamma_N}$ be a $(\#\Gamma_N)(\#\Lambda_h) \times 1$ column vector of compactly supported distributions such that $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$, where $a : \mathbb{Z}^d \mapsto \mathbb{C}^{(\#\Gamma_N)(\#\Lambda_h)\times(\#\Gamma_N)(\#\Lambda_h)}$ is a $(\#\Gamma_N) \times (\#\Gamma_N)$ block matrix mask for ϕ. Then ϕ is an interpolating M-refinable function vector of type*

(Γ_N, h) (*that is*, ϕ *is a vector of functions in* $C^h(\mathbb{R}^d)$ *and* (3.2) *holds*) *if and only if the following statements hold*:

(1) $[(1,1,\ldots,1)\otimes(1,0,\ldots,0)]\hat{\phi}(0)=1$ (*this is a normalization condition on* ϕ);
(2) $\nu_\infty(a,M)>h$;
(3) *The mask a is an interpolatory mask of type* (M,Γ_N,h); *that is*,
  (i) *The mask a satisfies the following condition*:

$$[a(Mk+[M\alpha]_{\Gamma_N})]_{\gamma,\langle M\alpha\rangle_{\Gamma_N}}=|\det M|^{-1}S(M^{-1},\Lambda_h)\delta(k)\delta(\alpha-\gamma),$$
$$\forall\alpha,\gamma\in\Gamma_N, k\in\mathbb{Z}^d, \tag{3.4}$$

  where $[M\alpha]_{\Gamma_N}\in\mathbb{Z}^d$ *and* $\langle M\alpha\rangle_{\Gamma_N}\in\Gamma_N$ *are uniquely determined by the relation* $M\alpha=[M\alpha]_{\Gamma_N}+\langle M\alpha\rangle_{\Gamma_N}$.
  (ii) *The mask a satisfies the sum rules of order* $h+1$ *with a* $1\times(\#\Gamma_N)(\#\Lambda_h)$ *row vector* $y=(y_\gamma)_{\gamma\in\Gamma_N}$ *in* $(\ell_0(\mathbb{Z}^d))^{1\times(\#\Gamma_N)(\#\Lambda_h)}$ *such that*

$$\widehat{y_\gamma}(\xi)=e^{i\gamma\cdot\xi}((i\xi)^\nu)_{\nu\in\Lambda_h}+O(\|\xi\|^{h+1}),\quad \xi\to 0,\ \gamma\in\Gamma_N, \tag{3.5}$$

  or equivalently,

$$\hat{y}(\xi)=(e^{i\gamma\cdot\xi})_{\gamma\in\Gamma_N}\otimes((i\xi)^\nu)_{\nu\in\Lambda_h}+O(\|\xi\|^{h+1}),\quad \xi\to 0. \tag{3.6}$$

To improve the readability of the paper, we shall present the proof of Theorem 3.1 in Sect 5. We mention that the sufficiency part of Theorem 3.1 still holds without assuming that M is isotropic. In general, the conditions in (3.4) and (2.1) with $\kappa=h+1$ cannot guarantee that up to a scalar multiplicative constant, the vector $\hat{y}$ in (2.1) must satisfy (3.6). However, if in addition $\nu_\infty(a,M)>h$, then up to a scalar multiplicative constant, the vector $\hat{y}$ in (2.1) must be unique and satisfy (3.6).

As we discussed before, to design an interpolatory mask of type (M,Γ_N,h) with a preassigned order of sum rules, it is of importance to investigate its sum rule structure; in particular, it is of interest to know the values $\partial^\mu\hat{y}(0)$, $|\mu|<\kappa$ in advance so that the nonlinear equations in (2.1) will become linear equations. We have the following result, whose proof will also be given in Sect. 5.

Theorem 3.2 *Let* M *be a* $d\times d$ *dilation matrix and* N,Γ_N,h *as in Theorem* 3.1. *Suppose that a is an interpolatory mask of type* (M,Γ_N,h) *satisfying the sum rules of order* κ *with* $\kappa>h$ *in* (2.1) *with a sequence* $y\in(\ell_0(\mathbb{Z}^d))^{1\times(\#\Gamma_N)(\#\Lambda_h)}$ *satisfying* (3.6). *Let* $\sigma=(\sigma_1,\ldots,\sigma_d)^T$, *where* $\sigma_1,\ldots,\sigma_d$ *are all the eigenvalues of* M. *If*

$$\sigma^\mu\notin\{\sigma^\nu:\nu\in\Lambda_h\},\quad \forall h<|\mu|<\kappa \tag{3.7}$$

((3.7) *clearly holds when* M *is an isotropic dilation matrix*), *then we must have*

$$\hat{y}(\xi)=\left(e^{i\gamma\cdot\xi}\right)_{\gamma\in\Gamma_N}\otimes\left((i\xi)^\nu\right)_{\nu\in\Lambda_h}+O(\|\xi\|^\kappa),\quad \xi\to 0. \tag{3.8}$$

The above result reveals the structure of the vector $\hat{y}$ in the definition of sum rules in (2.1) for interpolatory masks of type (M,Γ_N,h). This result is very important for the construction of interpolatory masks since it allows us to reduce the system of nonlinear equations in (2.1), in terms of free parameters in the mask a and the unknown values $\partial^\mu\hat{y}(0)$, $|\mu|<\kappa$, into a system of linear equations by knowing the values $\partial^\mu \hat{y}(0)$, $|\mu| < \kappa$ in advance. For a related result on Hermite interpolatory masks, see [12, Proposition 5.2].

In high dimensions, symmetry becomes important for at least two reasons. First of all, wavelets and refinable function vectors with symmetry generally provide better results in applications. Secondly, when designing a matrix mask, symmetry significantly reduces the number of free parameters in the system of linear equations. In the following, we will discuss symmetry in high dimensions and characterize an interpolating refinable function vectors with symmetry in terms of its mask. With the symmetry condition on the masks and the vector y determined in Theorem 3.2, obtaining interpolatory masks of type (M, Γ_N, h) with high orders of sum rules becomes far more easy (see examples in Sects. 2 and 4).

Let G be a finite set of $d \times d$ integer matrices. We say that G is a *symmetry group with respect to a dilation matrix M* [13] if G forms a group under matrix multiplication and

$$|\det E| = 1 \quad \text{and} \quad MEM^{-1} \in G \quad \forall E \in G.$$

For dimension $d = 1$, there is only one nontrivial symmetry group $G = \{-1, 1\}$ with respect to any dilation factor $M > 1$. In dimension two, two commonly used symmetry groups are D_4 and D_6 for the quadrilateral and triangular meshes, respectively:

$$\begin{aligned}
D_4 &:= \left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}, \\
D_6 &:= \left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pm\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right\}.
\end{aligned} \tag{3.9}$$

Let G be a symmetry group with respect to a dilation matrix M. Let $\phi = (\phi_\gamma)_{\gamma \in \Gamma_N}$ be an interpolating M-refinable function vector of type (Γ_N, h). We say that ϕ is *G-symmetric* if

$$\phi_\beta(E(\cdot - \beta) + \beta) = S(E, \Lambda_h)\phi_\beta \quad \forall E \in G,\ \beta \in \Gamma_N. \tag{3.10}$$

For a G-symmetric interpolating M-refinable function vector of type (Γ_N, h), we have the following result, whose proof will be given in Sect. 5 as well.

Theorem 3.3 *Let M, N, Γ_N, h and Λ_h be as in Theorem 3.1. Let $\phi = (\phi_\gamma)_{\gamma \in \Gamma_N}$ be an interpolating M-refinable function vector of type (Γ_N, h) with a matrix mask a. Let G be a symmetry group with respect to M. If*

$$E\Gamma_N \subset \Gamma_N + \mathbb{Z}^d \quad \forall E \in G \tag{3.11}$$

and ϕ is G-symmetric, then the mask a is (G, M)-symmetric:

$$\begin{aligned}
&[a(j)]_{\beta,\alpha} = S(E^{-1}, \Lambda_h)[a(MEM^{-1}j + [J_{E,\alpha,\beta}]_{\Gamma_N})]_{\beta,\langle J_{E,\alpha,\beta}\rangle_{\Gamma_N}} S(MEM^{-1}, \Lambda_h), \\
&\quad \forall j \in \mathbb{Z}^d, \alpha, \beta \in \Gamma_N;\ E \in G,
\end{aligned} \tag{3.12}$$

where

$$J_{E,\alpha,\beta} := MEM^{-1}\alpha + M(I_d - E)\beta. \tag{3.13}$$

Conversely, if (3.12) *holds and*

$$(I_d - E)\Gamma_N \subset \mathbb{Z}^d \quad \forall E \in G, \tag{3.14}$$

then ϕ is G-symmetric.

Note that (3.14) implies (3.11). So, if (3.14) is satisfied, then an interpolating M-refinable function vector ϕ of type (Γ_N, h) with a mask a and a dilation matrix M is G-symmetric if and only if the mask a is (G, M)-symmetric. In dimension one, it is evident that (3.14) is satisfied if $N = 2$ and $G = \{-1, 1\}$ (see Example 2.4 and Corollary 4.3).

To illustrate the results of this section, we present an example. Note that D_4 is a symmetry group with respect to $M_{\sqrt{2}}$ and (3.14) is satisfied for $G = D_4$ and $N = M_{\sqrt{2}}$.

Example 3.4 Let $M = N = M_{\sqrt{2}}$ and $h = 1$. Then $\Lambda_h = \{(0,0), (0,1), (1,0)\}$, $\Gamma_N = \{(0,0)^T, (\frac{1}{2}, \frac{1}{2})^T\}$. Let a (with multiplicity 6) be an interpolatory mask of type $(M_{\sqrt{2}}, \Gamma_N, h)$. Suppose that a is $(M_{\sqrt{2}}, D_4)$-symmetric and supported inside $[-1, 1] \times [-1, 0]$. We obtain an $(M_{\sqrt{2}}, D_4)$-symmetric interpolatory mask a of type $(M_{\sqrt{2}}, \Gamma_N, h)$ which satisfies the sum rules of order 4 and is given by:

$$a(-1,-1) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 64t & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -16t & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a(-1,0) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 64t & -6 & 6 \\ 0 & 0 & 0 & -16t & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a(0,-1) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 64t & 6 & -6 \\ 0 & 0 & 0 & 16t & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4-64t & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16t-1 & -1 & -1 \end{bmatrix},$$

$$a(0,0) = \frac{1}{16}\begin{bmatrix} 8 & 0 & 0 & 64t & -6 & -6 \\ 0 & -4 & 4 & 0 & 0 & 0 \\ 0 & 4 & 4 & 16t & -1 & -1 \\ 0 & 0 & 0 & 4-64t & -6 & 6 \\ 0 & 0 & 0 & 16t-1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a(1,-1) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4-64t & 6 & -6 \\ 0 & 0 & 0 & 1-16t & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a(1,0) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 4-64t & -6 & -6 \\ 0 & -4 & 4 & 0 & 0 & 0 \\ 0 & 4 & 4 & 1-16t & -1 & -1 \end{bmatrix},$$

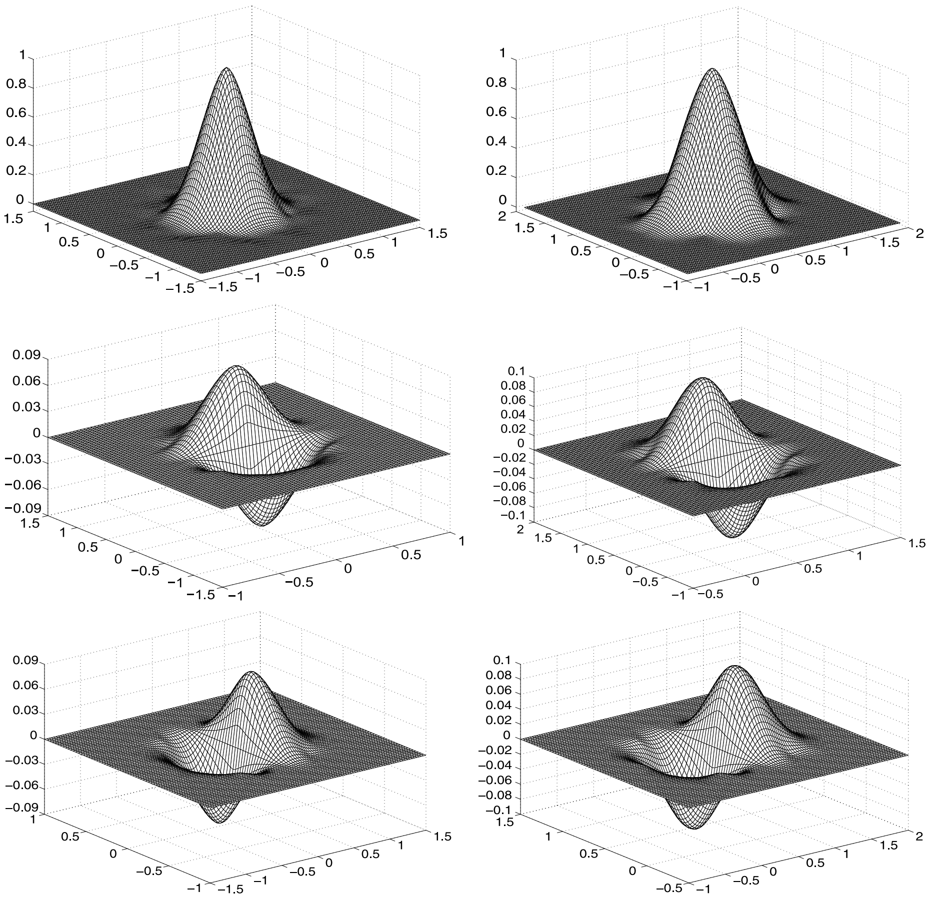


Fig. 4 The graphs of $\phi_{(0,0),\mu}$ (*left*) and $\phi_{(\frac{1}{2},\frac{1}{2}),\mu}$ (*right*), $\mu \in \Lambda_h$ in the D_4-symmetric interpolating $M_{\sqrt{2}}$-refinable function vector of type $(\Gamma_{M_{\sqrt{2}}}, 1)$ in Example *3.4* with $t = \frac{3}{128}$

where $t \in \mathbb{R}$ is a free parameter. For $t = \frac{3}{128}$, we have $\nu_2(a, M_{\sqrt{2}}) = 2.5$. Therefore, $\nu_\infty(a, M_{\sqrt{2}}) \geq \nu_2(a, M_{\sqrt{2}}) - 1 = 1.5 > 1$. By Theorem 3.1, its associated $M_{\sqrt{2}}$-refinable function vector ϕ is an interpolating function vector of type $(\Gamma_{M_{\sqrt{2}}}, 1)$. See Fig. 4 for the graph of ϕ with $t = \frac{3}{128}$.

4 Construction of Univariate Interpolatory Masks of Type (M, Γ_N, h)

Based on the results in Sect. 3, in this section we shall present a family of interpolatory masks of type (M, Γ_N, h) (more precisely, of type $(M, \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\}, h)$ with increasing orders of sum rules in dimension one. Here $M > 1$ is the dilation factor and $h \in \mathbb{N}_0$ is the degree of derivative and $N \geq 1$ is an integer.

Before we present the construction of interpolatory masks of type (M, Γ_N, h) in this section, let us lay out the whole picture of our construction and the idea of the proof first. Our construction in this section largely follows the key idea of the CBC (coset by coset) algorithm in [9, 10]. Roughly speaking, a mask $a : \mathbb{Z} \mapsto \mathbb{C}^{L\times L}$ can be regarded as a disjoint union of its cosets: $\{a(k)\}_{k\in\mathbb{Z}} = \bigcup_{m=0}^{M-1}\{a(m+Mk)\}_{k\in\mathbb{Z}}$. So, in order to obtain a mask a with multiplicity L with some desirable properties, it suffices to design its M cosets $\{a(m+Mk)\}_{k\in\mathbb{Z}}$, or equivalently, $\hat{a}^m(\xi) := \sum_{k\in\mathbb{Z}} a(m+Mk)e^{-i\xi(m+Mk)}$, for $m = 0,\dots,M-1$ separately and appropriately. That is, a desired mask can be constructed *coset by coset*.

Recall that $[A]_{i,j}$ is the (i, j)-block of a block matrix A and $[A]_{i,j;\ell,k}$ is the (ℓ,k)-entry of the block $[A]_{i,j}$. Let a be an interpolatory mask of type (M,Γ_N,h) in dimension one. a can be viewed as an $N\times N$ block matrix with each block of size $(h+1)\times(h+1)$. For $\ell = 0,1,\dots,N-1$, let $E_{\ell+1} := [\mathbf{0},\dots,\mathbf{0}, I_{h+1},\mathbf{0},\dots,\mathbf{0}]^T$ be an $N\times 1$ block matrix with each block of size $(h+1)\times(h+1)$ and its nonzero block is located at the $(\ell+1)$-th position. Then (3.4) and (3.6) in dimension one become

(1) a satisfies the following condition:

$$[a(Mk+R_\ell)]_{:,Q_\ell+1} = M^{-1}\delta_k E_{\ell+1}D, \tag{4.1}$$

where $D := \mathrm{diag}(1, M^{-1},\dots,M^{-h})$, $R_\ell := \lfloor\frac{M\ell}{N}\rfloor$ and $Q_\ell := N(\frac{M\ell}{N} - R_\ell) = M\ell \bmod N$ for $\ell = 0,\dots,N-1$;

(2) a satisfies the sum rules of order $h+1$ with a vector y such that

$$\hat{y}(\xi) = (1, e^{i\frac{1}{N}\xi},\dots,e^{i\frac{N-1}{N}\xi}) \otimes (1, i\xi,\dots,(i\xi)^h) + O(|\xi|^{h+1}), \quad \xi\to 0. \tag{4.2}$$

In other words, the $(j, Q_\ell+1)$-block of the mask a for all $j = 1,\dots,N$ on the coset $R_\ell + M\mathbb{Z}$, that is, $\{[a(R_\ell+Mk)]_{:,Q_\ell+1}\}_{k\in\mathbb{Z}}$, $\ell = 0,\dots,N-1$, are completely determined by the condition (4.1) for an interpolatory mask of type (M,Γ_N,h). Denote

$$\Gamma_{M,N} := \{(m,n) : m = 0,\dots,M-1, n = 1,\dots,N\}\backslash\{(R_\ell, Q_\ell+1) : \ell = 0,\dots,N-1\}. \tag{4.3}$$

Then, in order to construct an interpolatory mask a of type (M,Γ_N,h) with sum rules of order κ, it suffices to construct $\{[a(m+Mk)]_{:,n}\}_{k\in\mathbb{Z}}$ for every $(m,n)\in\Gamma_{M,N}$ such that the sum rule conditions in (2.2) are satisfied.

We have the following result on interpolatory masks of type (M,Γ_N,h) with increasing orders of sum rules, which generalizes [10, Theorem 4.3] and [16, Theorem 3.1].

Theorem 4.1 *Let M, N, K be positive integers with $M > 1$. Let h be a nonnegative integer and $L = N(h+1)$. Suppose that for every $(m,n)\in\Gamma_{M,N}$, $S_{m,n}$ is a subset of $\mathbb{Z}$ such that $\#S_{m,n} = K$. Then there exists a unique finitely supported mask $a : \mathbb{Z}\to\mathbb{C}^{L\times L}$ satisfying the following conditions*:

(1) *a is an interpolatory mask of type (M,Γ_N,h)*;
(2) *For every $(m,n)\in\Gamma_{M,N}$, $[a(m+Mk)]_{:,n} = 0$ for all $k\in\mathbb{Z}\backslash S_{m,n}$*;
(3) *a satisfies the sum rules of order LK.*

In fact, the unique mask a must be real-valued, that is, $a : \mathbb{Z}\to\mathbb{R}^{L\times L}$.

Proof Note that (4.1) is equivalent to

$$[\hat{a}^{R_\ell}(\xi)]_{:,Q_\ell+1} = M^{-1}e^{-iR_\ell\xi}E_{\ell+1}D, \quad \ell = 0,\dots,N-1.$$

Let $\hat{y}(\xi) = (1, e^{i\frac{1}{N}\xi}, \dots, e^{i\frac{N-1}{N}\xi}) \otimes (1, i\xi, \dots, (i\xi)^h)$. It is easy to see that

$$\begin{aligned}\hat{y}(M\xi)[\hat{a}^{R_\ell}(\xi)]_{:,Q_\ell+1} &= M^{-1}\hat{y}(M\xi)e^{-iR_\ell\xi}E_{\ell+1}D \\ &= M^{-1}e^{i(\frac{M\ell}{N}-R_\ell)\xi}(1, iM\xi, \dots, (iM\xi)^h)D \\ &= M^{-1}e^{i\frac{Q_\ell}{N}\xi}(1, i\xi, \dots, (i\xi)^h) \\ &= M^{-1}\widehat{y_{Q_\ell+1}}(\xi), \quad \xi \to 0, \ell = 0, \dots, N-1,\end{aligned}$$

where $\widehat{y_n}(\xi) = e^{i\frac{n-1}{N}\xi}(1, i\xi, \dots, (i\xi)^h)$, $n = 1, 2, \dots, N$ and $\hat{a}^m(\xi) := \sum_{k\in\mathbb{Z}} a(m+Mk)e^{-i(m+Mk)\xi}$, $m = 0, \dots, M-1$ are the cosets of a. To require that a should satisfy the sum rules of order LK, by Theorem 3.2 and (2.2), it is necessary and sufficient to require

$$\hat{y}(M\xi)[\hat{a}^m(\xi)]_{:,n} = M^{-1}\widehat{y_n}(\xi) + O(|\xi|^{LK}), \quad \xi \to 0, \ \forall (m,n) \in \Gamma_{M,N}.$$

That is, as $\xi \to 0$,

$$\begin{aligned}&\sum_{\ell=0}^{N-1}\sum_{k\in S_{m,n}} \widehat{y_{\ell+1}}(M\xi)[a(m+Mk)]_{\ell+1,n}e^{-i(m+Mk)\xi} \\ &\quad = \frac{1}{M}e^{i\frac{n-1}{N}\xi}(1, i\xi, \dots, (i\xi)^h) + O(|\xi|^{LK}).\end{aligned} \tag{4.4}$$

Now we need to show that for every $(m,n) \in \Gamma_{M,N}$ the above system of linear equations on $\{[a(m+Mk)]_{\ell+1,n} : \ell = 0, \dots, N-1, k \in S_{m,n}\}$ has a unique solution. The case for $h = 0$ has been proved in [16]. Here we will prove the general case for any $h \in \mathbb{N}_0$.

For $x \in \mathbb{R}$ and $j, s \in \mathbb{N}_0$, denote

$$v_{j,s}(x) = \begin{cases} 0, & j < s; \\ \frac{j!}{(j-s)!}x^{j-s}, & j \geq s. \end{cases}$$

Note that (4.4) is equivalent to: for $t = 0, 1, \dots, h$,

$$\begin{aligned}&\sum_{\ell=0}^{N-1}\sum_{k\in S_{m,n}}\sum_{s=0}^{h}[a(m+Mk)]_{\ell+1,n;\,s+1,t+1}e^{i(M\ell/N-m-Mk)\xi}(iM\xi)^s \\ &\quad = M^{-1}e^{i\frac{n-1}{N}\xi}(i\xi)^t + O(|\xi|^{LK}), \quad \xi \to 0.\end{aligned} \tag{4.5}$$

For each $t = 0, 1, \dots, h$, taking j-th derivative on both side of (4.5) and evaluating them at $\xi = 0$, we obtain

$$\begin{aligned}&\sum_{\ell=0}^{N-1}\sum_{k\in S_{m,n}}\sum_{s=0}^{h}[a(m+Mk)]_{\ell+1,n;\,s+1,t+1}M^s v_{j,s}(M\ell/N-m-Mk) \\ &\quad = M^{-1}v_{j,t}\Big(\frac{n-1}{N}\Big), \quad j = 0, 1, \dots, LK-1.\end{aligned} \tag{4.6}$$

Since $\#S_{m,n} = K$ for all $(m,n) \in \Gamma_{M,N}$, we see that for each $(m,n) \in \Gamma_{M,N}$, the set $\{M\ell/N - m - Mk : k \in S_{m,n}, \ell = 0, 1, \dots, N-1\}$ consists of NK distinct points in $\mathbb{R}$. The coefficient matrix of the above linear system (4.6) is

$$C = (M^s\, v_{j,s}(M\ell/N - m - Mk))_{j=0,1,\dots,LN-1;\, s=0,1,\dots,h,\ k\in S_{m,n},\ \ell=0,1,\dots,N-1},$$

which is an $LK \times LK$ matrix. Notice that

$$V := (v_{j,s}(M\ell/N - m - Mk))_{j=0,1,\ldots,LK-1;\, s=0,1,\ldots,h,\ k\in S_{m,n},\ \ell=0,1,\ldots,N-1}$$

is a confluent Vandermonde matrix [7] of size $LK \times LK$, which is invertible, and

$$C = V \cdot \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{NK \text{ times}}, \underbrace{M, M, \ldots, M}_{NK \text{ times}}, \ldots, \underbrace{M^h, M^h, \ldots, M^h}_{NK \text{ times}}).$$

Hence C is an invertible matrix of size $LK \times LK$. Moreover, the number of unknowns $\{[a(m + Mk)]_{\ell+1,n;\, s+1,t+1} : s = 0, 1, \ldots, h,\ k \in S_{m,n},\ \ell = 0, 1, \ldots, N - 1\}$ in (4.6) is also LK. Consequently, the solution to the system of linear equations in (4.6) is unique. Furthermore, it is evident that the solution is real-valued. □

The following result is a direct consequence of Theorem 4.1.

Corollary 4.2 *Let M, N, K be positive integers such that $M > 1$. Let h be a nonnegative integer and $L = N(h + 1)$. Let S be any subset of $\mathbb{Z}$ such that $\#(S \cap (m + M\mathbb{Z})) = K$ for all $m \in \mathbb{Z}$ and $\{R_\ell\}_{\ell=0}^{N-1} \subset S$, where $R_\ell := \lfloor \frac{M\ell}{N} \rfloor$. Then there exists a unique finitely supported mask $a : \mathbb{Z} \to \mathbb{R}^{L\times L}$ satisfying the following conditions*:

(1) *a is an interpolatory mask of type (M, Γ_N, h)*;
(2) *a is supported inside S*;
(3) *a satisfies the sum rules of order LK.*

In particular, if $S = [-N_0, MK - N_0 - 1] \cap \mathbb{Z}$ for $N_0 \in \mathbb{Z}$, then $\#(S \cap (m + M\mathbb{Z})) = K$ for all $m \in \mathbb{Z}$.

For the case $M = Nr'$ for some $r' \in \mathbb{N}$, we have $Q_\ell = 0$, $R_\ell = r'\ell$ for all $\ell = 0, \ldots, N - 1$. Equation (4.1) is equivalent to

$$[\hat{a}^{r'\ell}(\xi)]_{:,1} = M^{-1} e^{-ir'\ell\cdot\xi} E_{\ell+1} D, \quad \ell = 0, \ldots, N - 1. \tag{4.7}$$

In particular, if $M = N$, i.e., $r' = 1$, then an interpolatory mask of type (M, Γ_M, h) is of the form

$$\hat{a}(\xi) = \frac{1}{M} \begin{bmatrix} D & * & \cdots & * \\ De^{-i\xi} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ De^{-i(M-1)\xi} & * & \cdots & * \end{bmatrix}, \tag{4.8}$$

where $D = \mathrm{diag}(1, M^{-1}, \ldots, M^{-h})$.

For the case $M = N = 2$, we have the following result on interpolatory masks of type $(2, \{0, \frac{1}{2}\}, h)$ with symmetry.

Corollary 4.3 *For any positive integer K and any nonnegative integer h, there exists a unique interpolatory mask a of type $(2, \{0, \frac{1}{2}\}, h)$ such that*

(1) *a is supported inside $[1 - K, K]$*;
(2) *a is real-valued and satisfies the sum rules of order $(h + 1)(2K - 1)$*;

Table 1 These two rows list the quantities $\nu_2(a_{(3,\{0,\frac{1}{2}\},1)}, 3)$ and $\nu_2(a_{(3,\{0,\frac{1}{2}\},2)}, 3)$, respectively, for the interpolatory masks $a_{(M,\Gamma_N,h)}$ constructed in Corollary 4.2 with $M=3$, $N=2$ and $h=1$ for the first row, $h=2$ for the second row. Here $S := [-N_0, 3K-N_0-1]$ with $N_0 := \lfloor 3(K-1)/2 \rfloor$

K	1	2	3	4	5	6
$a_{(3,\{0,\frac{1}{2}\},1)}$	0.5	2.557920	2.952713	3.223482	3.425445	3.583893
$a_{(3,\{0,\frac{1}{2}\},2)}$	0.5	3.286249	3.767089	4.065856	4.234592	4.311367

Table 2 These three rows list the quantities $\nu_2(a^{\text{sym}}_{(2,\{0,\frac{1}{2}\},h)}, 2)$ for the symmetric interpolatory masks $a^{\text{sym}}_{(2,\{0,\frac{1}{2}\},h)}$ constructed in Corollary 4.3 for $h=1,2,3$, respectively

K	1	2	3	4	5	6
$a^{\text{sym}}_{(2,\{0,\frac{1}{2}\},1)}$	0.5	2.494509	3.051766	3.646481	3.791163	4.000000
$a^{\text{sym}}_{(2,\{0,\frac{1}{2}\},2)}$	0.5	2.958569	3.931713	4.471009	4.421853	4.999996
$a^{\text{sym}}_{(2,\{0,\frac{1}{2}\},3)}$	0.5	3.351721	4.343120	4.650265	4.890424	5.498152

(3) *The mask a is* $(\{-1,1\},2)$*-symmetric*:

$$\overline{\hat{a}(\xi)} = \operatorname{diag}(P, Pe^{2i\xi})\hat{a}(\xi)\operatorname{diag}(P, Pe^{-i\xi}), \tag{4.9}$$

where $P := \operatorname{diag}((-1)^0, (-1)^1, \dots, (-1)^h)$ *is a diagonal matrix of size* $(h+1)\times(h+1)$.

Moreover, if $\nu_\infty(a,2) > h$, *then* $\phi = (\phi_{0,0}, \dots, \phi_{0,h}, \phi_{\frac{1}{2},0}, \dots, \phi_{\frac{1}{2},h})^T$ *is* $\{1,-1\}$*-symmetric, where ϕ is the* 2*-refinable function vector associated with mask a. More precisely,* $\phi_{0,j}(-\cdot) = (-1)^j\phi_{0,j}$ *and* $\phi_{\frac{1}{2},j}(1-\cdot) = (-1)^j\phi_{\frac{1}{2},j}$ *for* $j = 0, 1, \dots, h$.

Proof Since $M = N = 2$, we see that $\{[a(k)]_{:,1}\}_{k\in\mathbb{Z}}$ are completely determined by (4.8). By the symmetry condition (4.9), $\{[a(2k+1)]_{:,2}\}_{k\in\mathbb{Z}}$ are completely determined by $\{[a(2k)]_{:,2}\}_{k\in\mathbb{Z}}$. Moreover, $[a(k)]_{1,2} = \mathbf{0}$ if k is even, or $[a(1-k)]_{2,2} = \mathbf{0}$ if k is odd due to the symmetry condition (4.9). Now the proof is completed by a similar proof of Theorem 4.1. □

In the rest of this section, let us present in Tables 1 and 2 the smoothness exponents of some families of the interpolatory masks constructed in Corollaries 4.2 and 4.3. An example of an interpolatory mask of type $(2, \{0, \frac{1}{2}\}, 1)$ will also be given.

Example 4.4 Let $M = N = 2$, $h = 1$ and $K = 2$ in Corollary 4.3. Then we have a symmetric interpolatory mask a of type $(2, \{0, \frac{1}{2}\}, 1)$ satisfying the sum rules of order 6. The mask a is supported inside $[-1, 2]$ and $a(0)$, $a(2)$ are given by

$$a(0) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{9}{32} & \frac{-3}{4} \\ 0 & \frac{1}{4} & \frac{9}{128} & \frac{-3}{64} \\ 0 & 0 & \frac{45}{256} & \frac{93}{128} \\ 0 & 0 & \frac{-9}{512} & \frac{-15}{256} \end{bmatrix}, \qquad a(2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{11}{256} & \frac{3}{128} \\ 0 & 0 & \frac{3}{512} & \frac{1}{256} \end{bmatrix},$$

while $a(-1)$, $a(1)$ can be obtained by symmetry in (4.9). Then we have $\nu_2(a,2) \approx 2.494509$. Therefore, $\nu_\infty(a,2) \geq \nu_2(a,2) - 1/2 \approx 1.994509 > 1$. By Theorem 3.1, its as-

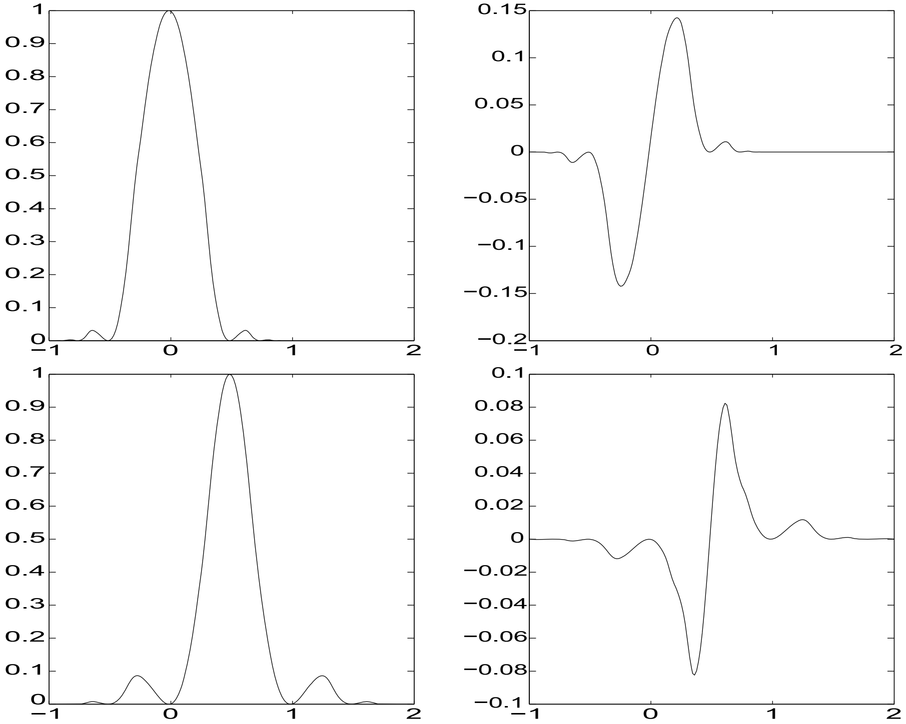


Fig. 5 The graphs of $\phi_{0,j}$, $j=0,1$ (*top*) and $\phi_{\frac{1}{2},j}$, $j=0,1$ (*bottom*) of the symmetric interpolating 2-refinable function vector ϕ in Example 4.4. $\nu_2(\phi)\approx 2.494509$. $\phi_{0,j}(-\cdot)=(-1)^j\phi_{0,j}$ and $\phi_{\frac{1}{2},j}(1-\cdot)=(-1)^j\phi_{\frac{1}{2},j}$ for all $j=0,1$

sociated 2-refinable function vector $\phi=(\phi_{0,0},\phi_{0,1},\phi_{\frac{1}{2},0},\phi_{\frac{1}{2},1})^T$ is a symmetric C^1 interpolating function vector of type $(\Gamma_2,1)$. Moreover, $\phi_{0,j}(-\cdot)=(-1)^j\phi_{0,j}$ and $\phi_{\frac{1}{2},j}(1-\cdot)=(-1)^j\phi_{\frac{1}{2},j}$ for all $j=0,1$. See Fig. 5 for the graph of ϕ.

5 Proofs of Theorems 3.1, 3.2, and 3.3

In this section, we prove Theorems 3.1, 3.2, and 3.3 in Sect. 3. Since stability and linear independence of a refinable function vector will be needed in our proofs, let us recall their definitions here. For an $L\times 1$ vector $\phi=(\phi_1,\dots,\phi_L)^T$ of compactly supported functions in $L_p(\mathbb{R}^d)$ for $1\le p\le\infty$, we say that the shifts of ϕ are *stable* in $L_p(\mathbb{R}^d)$ if there exist two positive constants C_1 and C_2 such that

$$C_1\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}|c_\ell(k)|^p\le\left\|\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}c_\ell(k)\phi_\ell(\cdot-k)\right\|^p_{L_p(\mathbb{R}^d)}\le C_2\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}|c_\ell(k)|^p \tag{5.1}$$

for all finitely supported sequences $c_1,\dots,c_L$ in $\ell_0(\mathbb{Z}^d)$. For a compactly supported function vector $\phi=(\phi_1,\dots,\phi_L)^T$, we say that the shifts of ϕ are *linearly independent* if for any sequences $c_1, \dots, c_L : \mathbb{Z}^d \mapsto \mathbb{C}$ such that

$$\sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^d} c_\ell(k)\phi_\ell(x-k) = 0, \quad \text{a.e. } x \in \mathbb{R}^d, \tag{5.2}$$

then one must have $c_\ell(k) = 0$ for all $\ell = 1, \dots, L$ and $k \in \mathbb{Z}^d$. Note that since ϕ is compactly supported, for any fixed $x \in \mathbb{R}^d$, the summation in $\sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^d} c_\ell(k)\phi_\ell(x-k)$ is in fact finite. For a compactly supported function vector $\phi = (\phi_1, \dots, \phi_L)^T$ in $L_p(\mathbb{R}^d)$, it is known in [19] that the shifts of ϕ are stable in $L_p(\mathbb{R}^d)$ (or linearly independent) if and only if $\text{span}\{\hat{\phi}(\xi + 2\pi k) : k \in \mathbb{Z}^d\} = \mathbb{C}^{L\times 1}$ for all $\xi \in \mathbb{R}^d$ (or for all $\xi \in \mathbb{C}^d$). Therefore, if the shifts of a compactly supported function vector ϕ in $L_p(\mathbb{R}^d)$ are linearly independent, then the shifts of ϕ must be stable in $L_p(\mathbb{R}^d)$.

The following lemma is needed in our proof of Theorem 3.1.

Lemma 5.1 *Let N be a $d \times d$ matrix. Then*

$$\mu! S(N, \Lambda_h)_{\mu,\nu} = \nu! S(N^T, \Lambda_h)_{\nu,\mu} \quad \forall \mu, \ \nu \in \Lambda_h, h \in \mathbb{N}_0. \tag{5.3}$$

Consequently, for a row vector $((i\xi)^\nu)_{\nu \in \Lambda_h}$, we have

$$((i\xi)^\nu)_{\nu\in\Lambda_h} S(N, \Lambda_h) = ((iN^T\xi)^\nu)_{\nu\in\Lambda_h}. \tag{5.4}$$

Proof Let $x, y \in \mathbb{R}^d$. Note that $(x \cdot y)^h = \sum_{\mu \in O_h} \frac{h!}{\mu!} x^\mu y^\mu$. Expanding $e^{x\cdot(Ny)}$ at the origin, we deduce that

$$\begin{aligned} e^{x\cdot(Ny)} &= \sum_{h=0}^{\infty} \frac{(x\cdot(Ny))^h}{h!} = \sum_{h=0}^{\infty} \sum_{\mu\in O_h} x^\mu \frac{(Ny)^\mu}{\mu!} \\ &= \sum_{h=0}^{\infty} \sum_{\mu\in O_h} \sum_{\nu\in O_h} \frac{1}{\nu!} S(N, O_h)_{\mu,\nu} x^\mu y^\nu. \end{aligned}$$

Similarly, we have

$$e^{y\cdot(N^T x)} = \sum_{h=0}^{\infty} \sum_{\nu\in O_h} \sum_{\mu\in O_h} \frac{1}{\mu!} S(N^T, O_h)_{\nu,\mu} x^\mu y^\nu.$$

Since $e^{x\cdot(Ny)} = e^{y\cdot(N^T x)}$, comparing the coefficients of $x^\mu y^\nu$ in both expressions, we conclude that $\frac{1}{\nu!} S(N, O_h)_{\mu,\nu} = \frac{1}{\mu!} S(N^T, O_h)_{\nu,\mu}$ for all $\mu, \nu \in O_h$. That is, (5.3) holds.

To prove (5.4), we have

$$((i\xi)^\nu)_{\nu\in\Lambda_h} S(N, \Lambda_h) = \left(\sum_{\nu\in\Lambda_h} (i\xi)^\nu S(N, \Lambda_h)_{\nu,\mu} \right)_{\mu\in\Lambda_h}.$$

By (5.3), we deduce that

$$\begin{aligned} \sum_{\nu\in\Lambda_h} (i\xi)^\nu S(N, \Lambda_h)_{\nu,\mu} &= \sum_{\nu\in\Lambda_h} \nu! S(N, \Lambda_h)_{\nu,\mu} \frac{(i\xi)^\nu}{\nu!} = \mu! \sum_{\nu\in\Lambda_h} S(N^T, \Lambda_h)_{\mu,\nu} \frac{(i\xi)^\nu}{\nu!} \\ &= \mu! \frac{(iN^T\xi)^\mu}{\mu!} = (iN^T\xi)^\mu. \end{aligned}$$

So, (5.4) is verified. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1 Necessity. By $\phi(M^{-1}\cdot) = |\det M| \sum_{k\in\mathbb{Z}^d} a(k)\phi(\cdot - k)$ and [12, Proposition 2.1], we have

$$[\mathcal{D}^{\Lambda_h} \otimes \phi](M^{-1}\cdot)S(M^{-1}, \Lambda_h) = \mathcal{D}^{\Lambda_h} \otimes [\phi(M^{-1}\cdot)] = |\det M| \sum_{j\in\mathbb{Z}^d} a(j)[\mathcal{D}^{\Lambda_h} \otimes \phi](\cdot - j).$$

Hence,

$$[\mathcal{D}^{\Lambda_h} \otimes \phi_\gamma](\cdot)S(M^{-1}, \Lambda_h) = |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\beta\in\Gamma_N} [a(j)]_{\gamma,\beta}[\mathcal{D}^{\Lambda_h} \otimes \phi_\beta](M\cdot - j).$$

That is, for $\alpha \in \Gamma_N$ and $k \in \mathbb{Z}^d$, we have

$$[\mathcal{D}^{\Lambda_h} \otimes \phi_\gamma](\alpha + k)S(M^{-1}, \Lambda_h) = |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\beta\in\Gamma_N} [a(j)]_{\gamma,\beta}[\mathcal{D}^{\Lambda_h} \otimes \phi_\beta](M\alpha + Mk - j).$$

Since NMN^{-1} is an integer matrix, we have $MN^{-1}\mathbb{Z}^d \subseteq N^{-1}\mathbb{Z}^d$, that is, $M[\Gamma_N + \mathbb{Z}^d] \subseteq \Gamma_N + \mathbb{Z}^d$. Thus, for each $\alpha \in \Gamma_N$, we can uniquely write $M\alpha = [M\alpha]_{\Gamma_N} + \langle M\alpha\rangle_{\Gamma_N}$ with $[M\alpha]_{\Gamma_N} \in \mathbb{Z}^d$ and $\langle M\alpha\rangle_{\Gamma_N} \in \Gamma_N$. Since ϕ is an interpolating function vector of type (Γ_N, h), applying (3.2) to the above equation, we obtain

$$\begin{aligned}
&\delta(k)\delta(\alpha - \gamma)S(M^{-1}, \Lambda_h)\\
&\quad = |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\beta\in\Gamma_N} [a(j)]_{\gamma,\beta}[\mathcal{D}^{\Lambda_h} \otimes \phi_\beta](\langle M\alpha\rangle_{\Gamma_N} + [M\alpha]_{\Gamma_N} + Mk - j)\\
&\quad = |\det M| \sum_{\beta\in\Gamma_N} [a(Mk + [M\alpha]_{\Gamma_N})]_{\gamma,\beta}\delta(\beta - \langle M\alpha\rangle_{\Gamma_N})\\
&\quad = |\det M| [a(Mk + [M\alpha]_{\Gamma_N})]_{\gamma,\langle M\alpha\rangle_{\Gamma_N}}.
\end{aligned}$$

That is, $[a(Mk + [M\alpha]_{\Gamma_N})]_{\gamma,\langle M\alpha\rangle_{\Gamma_N}} = |\det M|^{-1}S(M^{-1}, \Lambda_h)\delta(k)\delta(\alpha - \gamma)$ for all $\alpha, \gamma \in \Gamma_N$ and $k \in \mathbb{Z}^d$. So, (3.4) holds.

By the interpolation property of ϕ in (3.2), it is easy to see that the shifts of ϕ are linearly independent. In fact, suppose

$$0 = \sum_{\gamma\in\Gamma_N} \sum_{\mu\in\Lambda_h} \sum_{k\in\mathbb{Z}^d} c_{\gamma,\mu}(k)\phi_{\gamma,\mu}(x - k), \quad \text{a.e. } x \in \mathbb{R}^d.$$

Taking the differentiation operator ∂^ν on both sides and setting $x = \beta + j$, we obtain $c_{\beta,\nu}(j) = 0$ for all $\beta \in \Gamma_N$, $\nu \in \Lambda_h$ and $j \in \mathbb{Z}^d$. So the shifts of ϕ are linearly independent and therefore stable. Consequently, by [12, Corollary 5.1] and $\phi \in (C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}$, we must have $\nu_\infty(a, M) > h$. That is, Item (2) holds.

Since $\nu_\infty(a, M) > h$, by [12, Theorem 4.3], the mask a must satisfy the sum rules of order $h + 1$ with a vector $y \in (\ell_0(\mathbb{Z}^d))^{1\times(\#\Gamma_N)(\#\Lambda_h)}$ and $\hat{y}(0)\hat{\phi}(0) = 1$. But this implies [12] that $\partial^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](0) = \delta(\mu)$ and $\partial^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](2\pi k) = 0$ for all $|\mu| \le h$ and $k \in \mathbb{Z}^d\backslash\{0\}$. By the remark after [12, Proposition 3.2], this is equivalent to $(p * y) * \phi = p$ for all $p \in \Pi_h$, where Π_h denotes the linear space of all polynomials with total degree no greater than h.

More precisely, by [12, (2.13)], we have

$$\sum_{j\in\mathbb{Z}^d}\sum_{\mu\in\Lambda_h}\partial^\mu p(j)\frac{(-i\partial)^\mu}{\mu!}\hat{y}(0)\phi(\cdot-j)=p,\quad p\in\Pi_h.$$

That is,

$$\sum_{j\in\mathbb{Z}^d}\sum_{\mu\in\Lambda_h}\sum_{\gamma\in\Gamma_N}\partial^\mu p(j)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\gamma}(0)\phi_\gamma(\cdot-j)=p,\quad p\in\Pi_h.$$

Hence, we have

$$\sum_{j\in\mathbb{Z}^d}\sum_{\mu\in\Lambda_h}\sum_{\gamma\in\Gamma_N}\partial^\mu p(j)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\gamma}(0)[\mathcal{D}^{\Lambda_h}\otimes\phi_\gamma](\cdot-j)=\mathcal{D}^{\Lambda_h}\otimes p,\quad p\in\Pi_h.$$

So, for $x=\beta+k,\ \beta\in\Gamma_N$ and $k\in\mathbb{Z}^d$, for any $p\in\Pi_h$, we have

$$\sum_{j\in\mathbb{Z}^d}\sum_{\mu\in\Lambda_h}\sum_{\gamma\in\Gamma_N}\partial^\mu p(j)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\gamma}(0)[\mathcal{D}^{\Lambda_h}\otimes\phi_\gamma](\beta+k-j)=[\mathcal{D}^{\Lambda_h}\otimes p](\beta+k).$$

By (3.2) and the above identity, we obtain

$$\sum_{\mu\in\Lambda_h}\partial^\mu p(k)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\beta}(0)=[\mathcal{D}^{\Lambda_h}\otimes p](\beta+k),\quad p\in\Pi_h,k\in\mathbb{Z}^d,\beta\in\Gamma_N.$$

Set $p_\nu(x):=\frac{x^\nu}{\nu!}$. Taking $k=0$ in the above identity, we get

$$\frac{(-i\partial)^\nu}{\nu!}\widehat{y_\beta}(0)=[\mathcal{D}^{\Lambda_h}\otimes p_\nu](\beta)=([\partial^\mu p_\nu](\beta))_{\mu\in\Lambda_h}.\tag{5.5}$$

For $\mu=(\mu_1,\dots,\mu_d)$ and $\nu=(\nu_1,\dots,\nu_d)$, we say that $\nu\le\mu$ if $\nu_j\le\mu_j$ for all $j=1,\dots,d$. Denote $\operatorname{sgn}(\mu)=1$ if $\mu\ge0$ and 0, otherwise. Now it is easy to see that (5.5) is equivalent to

$$\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\beta}(0)=\left(\frac{\beta^{\mu-\nu}}{(\mu-\nu)!}\operatorname{sgn}(\mu-\nu)\right)_{\nu\in\Lambda_h}.\tag{5.6}$$

The relation in (5.6) is satisfied by the choice $\widehat{y_\beta}(\xi):=e^{i\beta\cdot\xi}[(i\xi)^\eta]_{\eta\in\Lambda_h}$, since

$$\begin{aligned}\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\beta}(0)&=\sum_{o\le\nu\le\mu}\frac{(-i\partial)^{\mu-\nu}}{(\mu-\nu)!}e^{i\beta\cdot\xi}\Big|_{\xi=0}\frac{(-i\partial)^\nu}{\nu!}[(i\xi)^\eta]_{\eta\in\Lambda_h}\Big|_{\xi=0}\\&=\sum_{0\le\nu\le\mu}\frac{\beta^{\mu-\nu}}{(\mu-\nu)!}[\delta_{\eta-\nu}]_{\eta\in\Lambda_h}=\left(\frac{\beta^{\mu-\nu}}{(\mu-\nu)!}\operatorname{sgn}(\mu-\nu)\right)_{\nu\in\Lambda_h}.\end{aligned}$$

Hence, a is an interpolatory mask of type (M,Γ_N,h). So, Item (3) holds.

Since Item (3) holds, by (3.6), we have $\hat{y}(0)=(1,1,\dots,1)\otimes(1,0,\dots,0)$. By $(p*y)*\phi=p$ with $p=1$, we must have $\hat{y}(0)*\phi=1$. Consequently, we have $\hat{y}(0)\hat{\phi}(0)=1$. Thus, Item (1) holds.

Sufficiency. Let $g \in (C^h(\mathbb{R}^d))^{(\#\Lambda_h)\times 1)}$ be an interpolating function vector of type (Γ_{I_d}, h) (see [10] and [12, Corollary 5.2] for the construction of such function vectors) such that

$$(1,0,\dots,0)\hat{g}(0) = 1 \quad \text{and} \quad ((i\xi)^\nu)_{\nu\in\Lambda_h}\hat{g}(\xi + 2\pi k) = O(\|\xi\|^{h+1}), \quad \xi \to 0, k \in \mathbb{Z}^d\backslash\{0\}. \tag{5.7}$$

Define a $(\#\Gamma_N)(\#\Lambda_h) \times 1$ column vector by

$$f := \big(S(N^{-1}, \Lambda_h)g(N(\cdot - \gamma))\big)_{\gamma\in\Gamma_N}. \tag{5.8}$$

Then we have

$$\hat{f}(\xi) = \big(|\det N|^{-1} e^{-i\gamma\cdot\xi} S(N^{-1}, \Lambda_h)\hat{g}((N^T)^{-1}\xi)\big)_{\gamma\in\Gamma_N}.$$

This can be rewritten as

$$\hat{f}(\xi) = |\det N|^{-1}[(e^{-i\gamma\cdot\xi})_{\gamma\in\Gamma_N}]^T \otimes [S(N^{-1}, \Lambda_h)\hat{g}((N^T)^{-1}\xi)]. \tag{5.9}$$

Note that by (5.7), the first component of $\hat{g}(0)$ is 1. Also, we observe that the first row of $S(N^{-1}, \Lambda_h)$ is $(1,0,\dots,0)$. Consequently, the first component of $S(N^{-1}, \Lambda_h)\hat{g}(0)$ is 1. Now by $\hat{y}(0) = (1,1,\dots,1) \otimes (1,0,\dots,0)$, we conclude from (5.9) that

$$\begin{aligned}
\hat{y}(0)\hat{f}(0) &= |\det N|^{-1}[(1,1,\dots,1)\otimes(1,0,\dots,0)] \times [(1,1,\dots,1)^T \otimes (S(N^{-1}, \Lambda_h)\hat{g}(0))] \\
&= |\det N|^{-1}[(1,1,\dots,1)\times(1,1,\dots,1)^T] \otimes [(1,0,\dots,0)\times(1,*,\dots,*)^T] \\
&= 1,
\end{aligned}$$

where $*$ denotes some number and we used the fact $S(N^{-1}, \Lambda_h)\hat{g}(0) = (1,*,\dots,*)^T$ in the last second identity.

On the other hand, we deduce from (5.9) that as $\xi \to 0$,

$$\begin{aligned}
&|\det N|\hat{y}(\xi)\hat{f}(\xi + 2\pi k) \\
&\quad = \big((e^{i\beta\cdot\xi})_{\beta\in\Gamma_N} \otimes ((i\xi)^\nu)_{\nu\in\Lambda_h}\big) \\
&\qquad \times \big([(e^{-i\gamma\cdot(\xi+2\pi k)})_{\gamma\in\Gamma_N}]^T \otimes [S(N^{-1}, \Lambda_h)\hat{g}((N^T)^{-1}(\xi + 2\pi k))]\big) + O(\|\xi\|^{h+1}) \\
&\quad = \left(\sum_{\gamma\in\Gamma_N} e^{-i2\pi k\cdot\gamma}\right)\big(((i\xi)^\nu)_{\nu\in\Lambda_h} S(N^{-1}, \Lambda_h)\hat{g}((N^T)^{-1}\xi + 2\pi(N^T)^{-1}k)\big) + O(\|\xi\|^{h+1}).
\end{aligned}$$

By Lemma 5.1, we see that $((i\xi)^\nu)_{\nu\in\Lambda_h} S(N^{-1}, \Lambda_h) = ((i(N^T)^{-1}\xi)^\nu)_{\nu\in\Lambda_h}$. Consequently, we have

$$\begin{aligned}
&|\det N|\hat{y}(\xi)\hat{f}(\xi + 2\pi k) \\
&\quad = \left(\sum_{\gamma\in\Gamma_N} e^{-i2\pi k\cdot\gamma}\right)\Big(((i(N^T)^{-1}\xi)^\nu)_{\nu\in\Lambda_h}\hat{g}((N^T)^{-1}\xi + 2\pi(N^T)^{-1}k)\Big) \\
&\qquad + O(\|\xi\|^{h+1}), \quad \xi \to 0.
\end{aligned} \tag{5.10}$$

For $k \in \mathbb{Z}^d\backslash[N^T\mathbb{Z}^d]$, we have $\sum_{\gamma\in\Gamma_N} e^{-i2\pi k\cdot\gamma} = 0$ and consequently it follows from the above identity that $\hat{y}(\xi)\hat{f}(\xi + 2\pi k) = O(\|\xi\|^{h+1})$ as $\xi \to 0$ for all $k \in \mathbb{Z}^d\backslash[N^T\mathbb{Z}^d]$. For $k \in [N^T\mathbb{Z}^d]\backslash\{0\}$, we have $k = N^T k'$ for some $k' \in \mathbb{Z}^d\backslash\{0\}$. Therefore, by (5.7), we have

$$((i(N^T)^{-1}\xi)^\nu)_{\nu\in\Lambda_h}\hat{g}((N^T)^{-1}\xi + 2\pi(N^T)^{-1}k) + O(\|\xi\|^{h+1}) = O(\|\xi\|^{h+1}),$$

as $\xi \to 0$. Therefore, we conclude that $\hat{y}(\xi)\hat{f}(\xi + 2\pi k) = O(\|\xi\|^{h+1})$, $\xi \to 0$ for all $k \in \mathbb{Z}^d \backslash \{0\}$. So, f is a suitable initial function vector (see [12]) with respect to y.

Let $f_0 := f$. Define $Q_{a,M} f := |\det M| \sum_{k\in\mathbb{Z}^d} a(k) f(M \cdot -k)$ and $f_n := Q_{a,M} f_{n-1}$, $n \in \mathbb{N}$. Now we prove by induction that all f_n are interpolating function vectors of type (Γ_N, h). When $n = 0$, we have $f_0 = f$. By the choice of the initial function f in (5.8) and by [12, Proposition 2.1], for $\gamma \in \Gamma_N$, we have

$$\begin{aligned}\mathcal{D}^{\Lambda_h} \otimes f_\gamma &= \mathcal{D}^{\Lambda_h} \otimes [S(N^{-1}, \Lambda_h) g(N(\cdot - \gamma))] \\ &= S(N^{-1}, \Lambda_h)[\mathcal{D}^{\Lambda_h} \otimes g](N(\cdot - \gamma)) S(N, \Lambda_h).\end{aligned}$$

Since g is an interpolating function vector of type (Γ_{I_d}, h), we deduce that for all $\beta \in \Gamma_N$ and $k \in \mathbb{Z}^d$,

$$\begin{aligned}[\mathcal{D}^{\Lambda_h} \otimes f_\gamma](\beta + k) &= S(N^{-1}, \Lambda_h)[\mathcal{D}^{\Lambda_h} \otimes g](Nk + N(\beta - \gamma)) S(N, \Lambda_h) \\ &= S(N^{-1}, \Lambda_h)\delta(k)\delta(\beta - \gamma) I_{\#\Lambda_h} S(N, \Lambda_h) \\ &= \delta(k)\delta(\beta - \gamma) I_{\#\Lambda_h}.\end{aligned}$$

So, f is an interpolating function vector of type (Γ_N, h). Suppose that f_{n-1} is an interpolating function vector of type (Γ_N, h). Then by

$$f_n = Q_{a,M} f_{n-1} = |\det M| \sum_{j\in\mathbb{Z}^d} a(j) f_{n-1}(M \cdot -j),$$

for $\gamma \in \Gamma_N$, we have

$$[f_n]_\gamma = |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} [f_{n-1}(M \cdot -j)]_\alpha.$$

Hence, by [12, Proposition 2.1], we have

$$\begin{aligned}\mathcal{D}^{\Lambda_h} \otimes [f_n]_\gamma &= |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} \mathcal{D}^{\Lambda_h} \otimes [f_{n-1}(M \cdot -j)]_\alpha \\ &= |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} [\mathcal{D}^{\Lambda_h} \otimes [f_{n-1}]_\alpha](M \cdot -j) S(M, \Lambda_h).\end{aligned}$$

So, for $\beta \in \Gamma_N$ and $k \in \mathbb{Z}^d$, we deduce that

$$\begin{aligned}[\mathcal{D}^{\Lambda_h} \otimes [f_n]_\gamma](\beta + k) = |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} [\mathcal{D}^{\Lambda_h} \\ \otimes [f_{n-1}]_\alpha](M\beta + Mk - j) S(M, \Lambda_h).\end{aligned}$$

Now by induction hypothesis, we have

$$\begin{aligned}[\mathcal{D}^{\Lambda_h} \otimes [f_{n-1}]_\alpha](M\beta + Mk - j) &= [\mathcal{D}^{\Lambda_h} \otimes [f_{n-1}]_\alpha](\langle M\beta\rangle_{\Gamma_N} + [M\beta]_{\Gamma_N} + Mk - j) \\ &= \delta(\langle M\beta\rangle_{\Gamma_N} - \alpha)\delta([M\beta]_{\Gamma_N} + Mk - j) I_{\#\Lambda_h}.\end{aligned}$$

Therefore, by (3.4), we get

$$
\begin{aligned}
&[\mathcal{D}^{\Lambda_h} \otimes [f_n]_\gamma](\beta + k)\\
&\quad = |\det M| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} \delta(\langle M\beta\rangle_{\Gamma_N} - \alpha)\delta([M\beta]_{\Gamma_N} + Mk - j)[a(j)]_{\gamma,\alpha} S(M, \Lambda_h)\\
&\quad = |\det M| [a(Mk + [M\beta]_{\Gamma_N})]_{\gamma,\langle M\beta\rangle_{\Gamma_N}} S(M, \Lambda_h)\\
&\quad = \delta(k)\delta(\beta - \gamma) S(M^{-1}, \Lambda_h) S(M, \Lambda_h)\\
&\quad = \delta(k)\delta(\beta - \gamma) I_{\#\Lambda_h}.
\end{aligned}
$$

Hence, f_n is an interpolating function vector of type (Γ_N, h). Now by induction, all f_n, $n = 0, 1, \dots,$ are interpolating function vectors of type (Γ_N, h).

Since $\nu_\infty(a, M) > h$, the cascade algorithm f_n converges in $(C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}$ [12, Theorem 4.3]. By (ii) of Item (3), we have $\hat{y}(0) = (1, 1, \dots, 1) \otimes (1, 0, \dots, 0)$. Now by Item (1), we see that $\hat{y}(0)\hat{\phi}(0) = 1$. Since $\hat{y}(0)\hat{\phi}(0) = \hat{y}(0)\hat{f}(0) = 1$, we see that $f_n \to \phi$ in $(C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}$ as $n \to \infty$. Consequently, since all f_n are interpolating function vectors of type (Γ_N, h), ϕ is also an interpolating function vector of type (Γ_N, h). □

Next, we prove Theorem 3.2.

Proof of Theorem 3.2 For simplicity, let us define two operators $R : \Gamma_N + \mathbb{Z}^d \to \mathbb{Z}^d$ and $Q : \Gamma_N + \mathbb{Z}^d \to \Gamma_N$ by $R(\alpha) := [M\alpha]_{\Gamma_N}$ and $Q(\alpha) = \langle M\alpha\rangle_{\Gamma_N}$. Let $E_\alpha := [\mathbf{0}, \dots, \mathbf{0}, I_{\#\Lambda_h}, \mathbf{0}, \dots, \mathbf{0}]^T$, $\alpha \in \Gamma_N$, be a $(\#\Gamma_N) \times 1$ block matrix with each block of size $(\#\Lambda_h) \times (\#\Lambda_h)$, whose nonzero block is located at the α-th position.

Using the cosets of the mask a, we see that (3.4) can be equivalently rewritten as

$$
\hat{a}^{R(\alpha)}(\xi) E_{Q(\alpha)} = |\det M|^{-1} e^{-iR(\alpha)\cdot\xi} E_\alpha S(M^{-1}, \Lambda_h) \quad \forall \alpha \in \Gamma_N. \tag{5.11}
$$

Since a satisfies the sum rules of order κ with the vector $\hat{y}$, we have (2.2). In particular, using (2.2) with $\omega = R(\alpha)$, we deduce from (5.11) that as $\xi \to 0$,

$$
\begin{aligned}
|\det M|^{-1} \hat{y}(\xi) E_{Q(\alpha)} &= \hat{y}(M^T\xi) \hat{a}^{R(\alpha)}(\xi) E_{Q(\alpha)} + O(\|\xi\|^\kappa)\\
&= |\det M|^{-1} e^{-iR(\alpha)\cdot\xi} \hat{y}(M^T\xi) E_\alpha S(M^{-1}, \Lambda_h) + O(\|\xi\|^\kappa).
\end{aligned}
$$

Denote $\hat{y}(\xi) := (\widehat{y_\alpha}(\xi))_{\alpha\in\Gamma_N}$ with each $\widehat{y_\alpha}$ being a $1 \times (\#\Lambda_h)$ row vector. Then the above identity can be rewritten as

$$
\widehat{y_{Q(\alpha)}}(\xi) = e^{-iR(\alpha)\cdot\xi} \widehat{y_\alpha}(M^T\xi) S(M^{-1}, \Lambda_h) + O(\|\xi\|^\kappa), \quad \xi \to 0.
$$

That is, since (3.5) is satisfied, we must have

$$
\begin{aligned}
&\widehat{y_\alpha}(\xi) = e^{i\alpha\cdot\xi}((i\xi)^\nu)_{\nu\in\Lambda_h} + O(\|\xi\|^{h+1}),\\
&\widehat{y_\alpha}(M^T\xi) = e^{iR(\alpha)\cdot\xi} \widehat{y_{Q(\alpha)}}(\xi) S(M, \Lambda_h) + O(\|\xi\|^\kappa), \quad \xi \to 0,\ \forall \alpha \in \Gamma_N.
\end{aligned} \tag{5.12}
$$

Note that the above relation is just a system of linear equations on the unknowns $\{\partial^\mu \hat{y}(0) : h < |\mu| < \kappa\}$. In the following, we shall argue that the above system of linear equations in (5.12) has a unique solution for the unknowns $\{\partial^\mu \hat{y}(0) : h < |\mu| < \kappa\}$. Moreover, we shall prove that the unique solution to (5.12) must be given in (3.8).

For all $\alpha \in \Gamma_N$ and $n \in \mathbb{N}$, employing (5.12) iteratively, as $\xi \to 0$, we have

$$\begin{aligned}\widehat{y_\alpha}(\xi) &= e^{i\xi\cdot M^{-1}R(\alpha)}\widehat{y_{Q(\alpha)}}((M^T)^{-1}\xi)S(M,\Lambda_h) + O(\|\xi\|^\kappa)\\ &= e^{i\xi\cdot(M^{-2}R(Q(\alpha))+M^{-1}R(\alpha))}\widehat{y_{Q^2(\alpha)}}((M^T)^{-2}\xi)S(M^2,\Lambda_h) + O(\|\xi\|^\kappa)\\ &\ \vdots\\ &= e^{i\xi\cdot(\sum_{k=1}^n M^{-k}R(Q^{k-1}(\alpha)))}\widehat{y_{Q^n(\alpha)}}((M^T)^{-n}\xi)S(M^n,\Lambda_h) + O(\|\xi\|^\kappa).\end{aligned}$$

That is, as $\xi \to 0$, we have

$$\widehat{y_\alpha}(\xi) = e^{i\xi\cdot(\sum_{k=1}^n M^{-k}R(Q^{k-1}(\alpha)))}\widehat{y_{Q^n(\alpha)}}((M^T)^{-n}\xi)S(M^n,\Lambda_h) + O(\|\xi\|^\kappa) \quad \forall \alpha \in \Gamma_N. \tag{5.13}$$

Let S denote the set of all $\alpha \in \Gamma_N$ such that $\alpha \in S$ means that there exists $n_\alpha \in \mathbb{N}$ satisfying $Q^{n_\alpha}(\alpha) = \alpha$. For every $\alpha \in S$, since $\{\partial^\mu \widehat{y_\alpha}(0) : |\mu| \le h\}$ is uniquely determined by (3.5), by [12, Lemma 2.2], (5.13) with $n = n_\alpha$ has a unique solution $\{\partial^\mu \widehat{y_\alpha}(0) : h < |\mu| < \kappa\}$, which can be obtained recursively. More precisely, since for $\alpha \in S$, we have $Q^{n_\alpha}(\alpha) = \alpha$ for some $n_\alpha \in \mathbb{N}$. Therefore, (5.13) becomes

$$\widehat{y_\alpha}(\xi) = X_\alpha((M^T)^{-n_\alpha}\xi)\widehat{y_\alpha}((M^T)^{-n_\alpha}\xi)S(M^{n_\alpha},\Lambda_h) + O(\|\xi\|^\kappa), \quad \xi \to 0,$$

where $X_\alpha((M^T)^{-n_\alpha}\xi) := e^{i\xi\cdot(\sum_{k=1}^{n_\alpha} M^{-k}R(Q^{k-1}(\alpha)))}$, or equivalently,

$$\widehat{y_\alpha}((M^T)^{n_\alpha}\xi)I_{\#\Lambda_h} = X_\alpha(\xi)\widehat{y_\alpha}(\xi)S(M^{n_\alpha},\Lambda_h) + O(\|\xi\|^\kappa), \quad \xi \to 0.$$

Note that $\sigma^{n_\alpha \nu}, \nu \in O_j$ and $\sigma^{-n_\alpha \mu}, \mu \in \Lambda_h$ are all the eigenvalues of $S(M^{n_\alpha}, O_j)$ and $S(M^{-n_\alpha}, \Lambda_h)$, respectively. By our assumption on M, we see that

$$\begin{aligned}&S(M^{n_\alpha},O_j)\otimes I_{\#\Lambda_h} - I_{\#O_j}\otimes S(M^{n_\alpha},\Lambda_h)^T\\ &\quad= [S(M^{n_\alpha},O_j)\otimes S(M^{-n_\alpha},\Lambda_h)^T - I_{\#O_j}\otimes I_{\#\Lambda_h}][I_{\#O_j}\otimes S(M^{n_\alpha},\Lambda_h)^T]\end{aligned}$$

is invertible for all $j = h+1,\dots,\kappa-1$. Therefore, by [12, Lemma 2.2],

$$\partial^\mu[\widehat{y_\alpha}((M^T)^{n_\alpha}\xi)I_{\#\Lambda_h}](0) = \partial^\mu[X_\alpha(\xi)\widehat{y_\alpha}(\xi)S(M^{n_\alpha},\Lambda_h)](0), \quad h < |\mu| < \kappa$$

has a unique solution $\{\partial^\mu \widehat{y_\alpha}(0) : h < |\mu| < \kappa\}$ for every $\alpha \in S$. Consequently, for every $\alpha \in S$, $\{\partial^\mu \widehat{y_\alpha}(0) : |\mu| < \kappa\}$ is completely determined by the relation (5.12).

For $\alpha \in \Gamma_N \backslash S$, since $Q^n(\alpha) \in \Gamma_N$ for all $n \in \mathbb{N}$, there must exist $N_\alpha \in \mathbb{N}$ such that $Q^{N_\alpha}(\alpha) \in S$. Hence, by (5.13) with $n = N_\alpha$, we have

$$\widehat{y_\alpha}(\xi) = e^{i\xi\cdot(\sum_{k=1}^{N_\alpha} M^{-k}R(Q^{k-1}(\alpha)))}\widehat{y_{Q^{N_\alpha}(\alpha)}}((M^T)^{-N_\alpha}\xi)S(M^{N_\alpha},\Lambda_h) + O(\|\xi\|^\kappa), \quad \xi \to 0. \tag{5.14}$$

By what has been proved, all $\{\partial^\mu \widehat{y_{Q^{N_\alpha}(\alpha)}}(0) : |\mu| < \kappa\}$ is completely determined by (5.12). Thus, it follows from (5.14) that for every $\alpha \in \Gamma_N \backslash S$, the values $\{\partial^\mu \widehat{y_\alpha}(0) : |\mu| < \kappa\}$ is completely determined by (5.14) and therefore, is uniquely determined by the system of linear equations in (5.12).

That is, we proved that if (5.12) holds, then all $\partial^\mu \widehat{y_\alpha}(0), h < |\mu| < \kappa, \alpha \in \Gamma_N$ are uniquely determined by (5.12). Therefore, if there is a solution to the system of linear equations in (5.12), then the solution must be unique according to the above argument.

In the following, let us show that the system of linear equations in (5.12) indeed has a solution. Let $Y(\xi) := (Y_\alpha(\xi))_{\alpha\in\Gamma_N}$ with $Y_\alpha(\xi) := e^{i\alpha\cdot\xi}\,((i\xi)^\nu)_{\nu\in\Lambda_h}$. Since

$$M\alpha = [M\alpha]_{\Gamma_N} + \langle M\alpha\rangle_{\Gamma_N} = R(\alpha) + Q(\alpha),$$

we have $Y_\alpha(\xi) = e^{i\alpha\cdot\xi}((i\xi)^\nu)_{\nu\in\Lambda_h} + O(\|\xi\|^{h+1})$ as $\xi \to 0$ and $\alpha \in \Gamma_N$, and by Lemma 5.1,

$$\begin{aligned} Y_\alpha(M^T\xi) &= e^{iM\alpha\cdot\xi}\big((iM^T\xi)^\nu\big)_{\nu\in\Lambda_h} \\ &= e^{iR(\alpha)\cdot\xi}\big[e^{iQ(\alpha)\cdot\xi}\big((i\xi)^\nu\big)_{\nu\in\Lambda_h}\big]S(M,\Lambda_h) \\ &= e^{iR(\alpha)\cdot\xi}Y_{Q(\alpha)}(\xi)S(M,\Lambda_h). \end{aligned}$$

Therefore, if we take $\partial^\mu\widehat{y_\alpha}(0) = \partial^\mu Y_\alpha(0)$ for all $\alpha \in \Gamma_N$ and $|\mu| < \kappa$, then it is a solution to the system of linear equations in (5.12). By the uniqueness of the solution to (5.12), we must have (3.8), which completes the proof. □

Finally, we prove Theorem 3.3.

Proof of Theorem 3.3 Suppose that ϕ is G-symmetric and (3.11) holds. By (3.10) and the refinement equation (1.1), for $\beta \in \Gamma_N$, we deduce that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(x-k) \\ &\quad= |\det M|^{-1}\phi_\beta(M^{-1}x) \\ &\quad= |\det M|^{-1}S(E^{-1},\Lambda_h)\phi_\beta(E(M^{-1}x-\beta)+\beta) \\ &\quad= S(E^{-1},\Lambda_h)\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(MEM^{-1}x - M(E-I_d)\beta - k) \\ &\quad= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} S(E^{-1},\Lambda_h)[a(k)]_{\beta,\gamma}S(MEM^{-1},\Lambda_h)\phi_\gamma(x - ME^{-1}M^{-1}k - J_{E^{-1},\gamma,\beta} + \gamma). \end{aligned}$$

Therefore, for $x = \alpha + j$ with $\alpha \in \Gamma_N$ and $j \in \mathbb{Z}^d$, we deduce that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}[\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\alpha+j-k) \\ &\quad= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} S(E^{-1},\Lambda_h)[a(k)]_{\beta,\gamma}S(MEM^{-1},\Lambda_h) \\ &\qquad\times[\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\alpha+j-ME^{-1}M^{-1}k - J_{E^{-1},\gamma,\beta}+\gamma). \end{aligned} \tag{5.15}$$

By (3.11) and the interpolation property of ϕ in (3.2), it is easy to verify that (5.15) implies (3.12).

Conversely, suppose that (3.12) and (3.14) are satisfied. By induction on n, we first prove that

$$\phi_\beta(E(x-\beta)+\beta) = S(E,\Lambda_h)\phi_\beta(x) \quad \forall x \in M^{-n}(\mathbb{Z}^d+\Gamma_N), n\in\mathbb{N}_0, E\in G, \beta\in\Gamma_N. \tag{5.16}$$

By $\phi_\gamma(\alpha+j)=\delta(\alpha-\gamma)\delta(j)(1,0,\ldots,0)^T$ for all $\alpha,\gamma\in\Gamma_N$ and $j\in\mathbb{Z}^d$, it is evident that (5.16) holds for $n=0$.

Suppose that (5.16) holds for $n-1$. Then for any $x\in M^{-n}(\mathbb{Z}^d+\Gamma_N)$, we have $x=M^{-1}y$ with $y:=Mx\in M^{-(n-1)}(\mathbb{Z}^d+\Gamma_N)$. Therefore,

$$\begin{aligned}|\det M|^{-1}S(E,\Lambda_h)\phi_\beta(x)&=S(E,\Lambda_h)\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(Mx-k)\\&=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}S(E,\Lambda_h)[a(k)]_{\beta,\gamma}\phi_\gamma(y-k).\end{aligned}$$

Since $y-k\in M^{-(n-1)}(\mathbb{Z}^d+\Gamma_N)$, by our induction hypothesis in (5.16), we have

$$\begin{aligned}S(MEM^{-1},\Lambda_h)\phi_\gamma(y-k)&=\phi_\gamma(MEM^{-1}(y-k-\gamma)+\gamma)\\&=\phi_\gamma(MEx-MEM^{-1}(k+\gamma)+\gamma).\end{aligned}$$

Note that

$$J_{E,\gamma,\beta}=MEM^{-1}\gamma+M(I_d-E)\beta=\gamma-(I_d-MEM^{-1})\gamma+M(I_d-E)\beta.$$

By our assumption in (3.14), it is not difficult to verify that $\langle J_{E,\gamma,\beta}\rangle_{\Gamma_N}=\gamma$ for all $\gamma,\beta\in\Gamma_N$. Now by (3.12) and the above identities, we deduce that for any $x\in M^{-n}(\mathbb{Z}^d+\Gamma_N)$,

$$\begin{aligned}&|\det M|^{-1}S(E,\Lambda_h)\phi_\beta(x)\\&\quad=\sum_{k\in\mathbb{Z}^d}\sum_{\beta\in\Gamma_N}S(E,\Lambda_h)[a(k)]_{\beta,\gamma}\phi_\gamma(y-k)\\&\quad=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}S(E,\Lambda_h)[a(k)]_{\beta,\gamma}S(ME^{-1}M^{-1},\Lambda_h)\phi_\gamma(MEx-MEM^{-1}(k+\gamma)+\gamma)\\&\quad=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(MEM^{-1}k+[J_{E,\gamma,\beta}]_{\Gamma_N})]_{\beta,\langle J_{E,\gamma,\beta}\rangle_{\Gamma_N}}\phi_\gamma(MEx-MEM^{-1}(k+\gamma)+\gamma)\\&\quad=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(MEM^{-1}k+[J_{E,\gamma,\beta}]_{\Gamma_N})]_{\beta,\gamma}\phi_\gamma(MEx-MEM^{-1}(k+\gamma)+\gamma)\\&\quad=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(MEx-k+\gamma-MEM^{-1}\gamma+[J_{E,\gamma,\beta}]_{\Gamma_N})\\&\quad=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(MEx+M(I_d-E)\beta-k)\\&\quad=|\det M|^{-1}\phi_\beta(Ex+\beta-E\beta)=|\det M|^{-1}\phi_\beta(E(x-\beta)+\beta).\end{aligned}$$

Hence, (5.16) holds for n. By induction, (5.16) holds for all $n\in\mathbb{N}_0$.

Since ϕ is continuous and $\{M^{-n}(\mathbb{Z}^d+\Gamma_N)\ :\ n\in\mathbb{N}_0\}$ is dense in $\mathbb{R}^d$, we conclude that (3.10) holds. So, ϕ is G-symmetric. □

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