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MATRIX EXTENSION WITH SYMMETRY AND ITS APPLICATION TO SYMMETRIC ORTHONORMAL MULTIWAVELETS*

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Abstract. Let P be an $r\times s$ matrix of Laurent polynomials with symmetry such that $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible. The matrix extension problem with symmetry is to find an $s\times s$ square matrix P_e of Laurent polynomials with symmetry such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$ (that is, the submatrix of the first r rows of P_e is the given matrix P), P_e is paraunitary satisfying $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$, and the symmetry of P_e is compatible. Moreover, it is highly desirable in many applications that the support of the coefficient sequence of P_e can be controlled by that of P. In this paper, we completely solve the matrix extension problem with symmetry by constructing such a desired matrix P_e from a given matrix P. Furthermore, using a cascade structure, we obtain a complete representation of any $r \times s$ paraunitary matrix P having compatible symmetry, which in turn leads to a construction of a desired matrix P_e from a given matrix P. Matrix extension plays an important role in many areas such as wavelet analysis, electronic engineering, system sciences, and so on. As an application of our general results on matrix extension with symmetry, we obtain a satisfactory algorithm for constructing symmetric orthonormal multiwavelets by deriving high-pass filters with symmetry from any given orthogonal low-pass filters with symmetry. Several examples of symmetric orthonormal multiwavelets are provided to illustrate the results in this paper.

Key words. orthonormal multiwavelets, matrix extension, symmetry, Laurent polynomials, paraunitary filter banks

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1. Introduction and main results.

It is well known in wavelet analysis that the construction of orthonormal wavelets and multiwavelets can be formulated as a matrix extension problem; see [1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 16, 18, 17]. The matrix extension problem also plays a fundamental role in many areas such as electronic engineering, system sciences, mathematics, etc. We mention only a few references here on this topic; see [1, 2, 3, 5, 6, 7, 8, 10, 13, 15, 16, 17, 20, 21, 22]. In order to state the matrix extension problem and our main results on this topic, let us introduce some notation and definitions first.

Let $\mathsf{p}(z) = \sum_{k\in\mathbb{Z}} p_k z^k$, $z \in \mathbb{C}\backslash\{0\}$ be a Laurent polynomial with complex coefficients $p_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$. We say that p has *symmetry* if its coefficient sequence $\{p_k\}_{k\in\mathbb{Z}}$ has symmetry; more precisely, there exist $\varepsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ such that

$$p_{c-k} = \varepsilon p_k \qquad \forall\, k \in \mathbb{Z}. \tag{1.1}$$

If $\varepsilon = 1$, then p is symmetric about the point $c/2$; if $\varepsilon = -1$, then p is antisymmetric about the point $c/2$. Symmetry of a Laurent polynomial can be conveniently expressed using a symmetry operator $\mathcal{S}$ defined by

$$\mathcal{S}\mathsf{p}(z) := \frac{\mathsf{p}(z)}{\mathsf{p}(1/z)}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{1.2}$$

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When p is not identically zero, it is evident that (1.1) holds if and only if $\mathcal{S}\mathsf{p}(z) = \varepsilon z^c$. For the zero polynomial, it is very natural that $\mathcal{S}0$ can be assigned any symmetry pattern; that is, for every occurrence of $\mathcal{S}0$ appearing in an identity in this paper, $\mathcal{S}0$ is understood to take an appropriate choice of εz^c for some $\varepsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ so that the identity holds. If P is an $r \times s$ matrix of Laurent polynomials with symmetry, then we can apply the operator $\mathcal{S}$ to each entry of P; that is, $\mathcal{S}\mathsf{P}$ is an $r \times s$ matrix such that $[\mathcal{S}\mathsf{P}]_{j,k} := \mathcal{S}([\mathsf{P}]_{j,k})$, where $[\mathsf{P}]_{j,k}$ denotes the (j,k)-entry of the matrix P throughout the paper.

For two matrices P and Q of Laurent polynomials with symmetry, even though all the entries in P and Q have symmetry, their sum $\mathsf{P} + \mathsf{Q}$, difference $\mathsf{P} - \mathsf{Q}$, or product PQ, if well defined, generally may not have symmetry anymore. This is one of the difficulties for matrix extension with symmetry. In order for $\mathsf{P} \pm \mathsf{Q}$ or PQ to possess some symmetry, the symmetry patterns of P and Q should be compatible. For example, if $\mathcal{S}\mathsf{P} = \mathcal{S}\mathsf{Q}$ (that is, both P and Q have the same symmetry pattern), then indeed $\mathsf{P} \pm \mathsf{Q}$ has symmetry and $\mathcal{S}(\mathsf{P} \pm \mathsf{Q}) = \mathcal{S}\mathsf{P} = \mathcal{S}\mathsf{Q}$. In the following, we discuss the compatibility of symmetry patterns of matrices of Laurent polynomials. For an $r \times s$ matrix $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}} P_k z^k$, throughout the paper we denote

$$\mathsf{P}^*(z) := \sum_{k\in\mathbb{Z}} P_k^* z^{-k} \quad \text{with} \quad P_k^* := \overline{P_k}^T, \qquad k \in \mathbb{Z}, \tag{1.3}$$

where $\overline{P_k}^T$ denotes the transpose of the complex conjugate of the constant matrix P_k in $\mathbb{C}$. We say that *the symmetry of* P *is compatible* or P *has compatible symmetry* if

$$\mathcal{S}\mathsf{P}(z) = (\mathcal{S}\theta_1)^*(z)\mathcal{S}\theta_2(z) \tag{1.4}$$

for some $1 \times r$ and $1 \times s$ row vectors θ_1 and θ_2 of Laurent polynomials with symmetry. For an $r \times s$ matrix P and an $s \times t$ matrix Q of Laurent polynomials, we say that (P, Q) *has mutually compatible symmetry* if

$$\mathcal{S}\mathsf{P}(z) = (\mathcal{S}\theta_1)^*(z)\mathcal{S}\theta(z) \quad \text{and} \quad \mathcal{S}\mathsf{Q}(z) = (\mathcal{S}\theta)^*(z)\mathcal{S}\theta_2(z) \tag{1.5}$$

for some $1 \times r$, $1 \times s$, $1 \times t$ row vectors $\theta_1, \theta, \theta_2$ of Laurent polynomials with symmetry. If (P, Q) has mutually compatible symmetry as in (1.5), then it is easy to verify that their product PQ has compatible symmetry and in fact $\mathcal{S}(\mathsf{PQ}) = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$.

For a matrix of Laurent polynomials, another important property is the support of its coefficient sequence. For $\mathsf{P} = \sum_{k\in\mathbb{Z}} P_k z^k$ such that $P_k = \mathbf{0}$ for all $k \in \mathbb{Z}\backslash[m, n]$ with $P_m \neq \mathbf{0}$ and $P_n \neq \mathbf{0}$, we define its coefficient support to be $\operatorname{coeffsupp}(\mathsf{P}) := [m, n]$ and the length of its coefficient support to be $|\operatorname{coeffsupp}(\mathsf{P})| := n - m$. In particular, we define $\operatorname{coeffsupp}(\mathbf{0}) := \emptyset$, the empty set, and $|\operatorname{coeffsupp}(\mathbf{0})| := -\infty$. Also, we use $\operatorname{coeff}(\mathsf{P}, k) := P_k$ to denote the coefficient matrix (vector) P_k of z^k in P. In this paper, $\mathbf{0}$ always denotes a general zero matrix whose size can be determined in the context.

The Laurent polynomials that we shall consider in this paper have their coefficients in a subfield $\mathbb{F}$ of the complex field $\mathbb{C}$. Let $\mathbb{F}$ denote a subfield of $\mathbb{C}$ such that $\mathbb{F}$ is closed under the operations of complex conjugate of $\mathbb{F}$ and square roots of positive numbers in $\mathbb{F}$. In other words, the subfield $\mathbb{F}$ of $\mathbb{C}$ satisfies the following properties:

$$\bar{x} \in \mathbb{F} \quad \text{and} \quad \sqrt{y} \in \mathbb{F} \qquad \forall\, x, y \in \mathbb{F} \quad \text{with} \quad y > 0. \tag{1.6}$$

Two particular examples of such subfields $\mathbb{F}$ are $\mathbb{F} = \mathbb{R}$ (the field of real numbers) and $\mathbb{F} = \mathbb{C}$ (the field of complex numbers). A nontrivial example is the field of all algebraic numbers, i.e., the algebraic closure $\overline{\mathbb{Q}}$ of the rational numbers $\mathbb{Q}$.

Now, we introduce the general matrix extension problem with symmetry. Throughout the paper, r and s denote two positive integers such that $1 \leqslant r \leqslant s$. Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in $\mathbb{F}$ such that $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible, where I_r denotes the $r \times r$ identity matrix. The matrix extension problem with symmetry is to find an $s \times s$ square matrix P_e of Laurent polynomials with coefficients in $\mathbb{F}$ and with symmetry such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$ (that is, the submatrix of the first r rows of P_e is the given matrix P), the symmetry of P_e is compatible, and $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$ (that is, P_e is paraunitary). Moreover, in many applications, it is often highly desirable that the coefficient support of P_e can be controlled by that of P in some way.

In this paper, we study this general matrix extension problem with symmetry and we completely solve this problem as follows.

THEOREM 1. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ such that* (1.6) *holds. Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in $\mathbb{F}$ such that the symmetry of P is compatible and $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$. Then there exists an $s \times s$ square matrix P_e, which can be constructed by Algorithm* 2 *in section* 3 *from the given matrix P, of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(i) $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$*; that is, the submatrix of the first r rows of P_e is P;*

(ii) P_e *is paraunitary:* $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ *for all* $z \in \mathbb{C}\backslash\{0\}$*;*

(iii) *the symmetry of P_e is compatible;*

(iv) *the coefficient support of P_e is controlled by that of P in the following sense:*

$$|\mathrm{coeffsupp}([\mathsf{P}_e]_{j,k})| \leqslant \max_{1 \leqslant n \leqslant r} |\mathrm{coeffsupp}([\mathsf{P}]_{n,k})|, \qquad 1 \leqslant j,k \leqslant s. \tag{1.7}$$

Theorem 1 on matrix extension with symmetry is built on a stronger result which represents any given paraunitary matrix having compatible symmetry by a simple cascade structure. The following result leads to a proof of Theorem 1 and completely characterizes any paraunitary matrix P in Theorem 1.

THEOREM 2. *Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in a subfield $\mathbb{F}$ of $\mathbb{C}$ such that* (1.6) *holds. Then $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible as in* (1.4) *if and only if there exist $s \times s$ matrices $\mathsf{P}_0, \dots, \mathsf{P}_{J+1}$ of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(i) P *can be represented as a product of* $\mathsf{P}_0, \dots, \mathsf{P}_{J+1}$*:*

$$\mathsf{P}(z) = [I_r, \mathbf{0}]\mathsf{P}_{J+1}(z)\mathsf{P}_J(z)\cdots\mathsf{P}_1(z)\mathsf{P}_0(z); \tag{1.8}$$

(ii) $\mathsf{P}_j, 1 \leqslant j \leqslant J$, *are elementary:* $\mathsf{P}_j(z)\mathsf{P}_j^*(z) = I_s$ *and* $\mathrm{coeffsupp}(\mathsf{P}_j) \subseteq [-1,1]$*;*

(iii) $(\mathsf{P}_{j+1}, \mathsf{P}_j)$ *has mutually compatible symmetry for all* $0 \leqslant j \leqslant J$*;*

(iv) $\mathsf{P}_0 = \mathsf{U}_{\mathcal{S}\theta_2}^*$ *and* $\mathsf{P}_{J+1} = \mathrm{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$*, where* $\mathsf{U}_{\mathcal{S}\theta_1}$*,* $\mathsf{U}_{\mathcal{S}\theta_2}$ *are products of a permutation matrix with a diagonal matrix of monomials, as defined in* (3.2)*;*

(v) $J \leqslant \max_{1 \leqslant m \leqslant r, 1 \leqslant n \leqslant s} \lceil |\mathrm{coeffsupp}([\mathsf{P}]_{m,n})|/2 \rceil$*, where* $\lceil \cdot \rceil$ *is the ceiling function.*

The representation in (1.8) (without symmetry) is often called the cascade structure in the engineering literature; see [14, 15, 21]. In the context of wavelet analysis, matrix extension without symmetry was discussed by Lawton, Lee, and Shen in their interesting paper [16] and a simple algorithm was proposed there to derive a desired matrix P_e from a given row vector P of Laurent polynomials without symmetry. In [21], Vaidyanathan studied the matrix extension without symmetry for filter banks with the perfect reconstruction property. References [16, 21] mainly deal with the special case that P is a row vector (that is, $r = 1$ in our case) without symmetry, and the coefficient support of the derived matrix P_e indeed can be controlled by that of P. The algorithms in [16, 21] for the special case $r = 1$ can be employed to handle a general $r \times s$ matrix P without symmetry; see [16, 19, 21] for detail. However, for the general case $r > 1$, it is no longer clear whether the coefficient support of the derived matrix P_e obtained by the algorithms in [16, 21] can still be controlled by that of P.

Several special cases of matrix extension with symmetry have been considered in the literature. For $\mathbb{F} = \mathbb{R}$ and $r = 1$, matrix extension with symmetry was considered in [17]. For $r = 1$, matrix extension with symmetry was studied in [8] and a simple algorithm is given there. In the context of wavelet analysis, several particular cases of matrix extension with symmetry related to the construction of wavelets and multiwavelets have been investigated in [2, 7, 8, 10, 14, 15, 17]. However, for the general case of an $r \times s$ matrix, the approaches on matrix extension with symmetry in [8, 17] for the particular case $r = 1$ cannot be employed to handle the general case. The algorithms in [8, 17] are very difficult to generalize to the general case $r > 1$, partially due to the complicated relations of the symmetry patterns between different rows of P. For the general case of matrix extension with symmetry, it becomes much harder to control the coefficient support of the derived matrix P_e, comparing with the special case $r = 1$. Extra effort is needed in this case for deriving P_e so that its coefficient support can be controlled by that of P.

The contributions of this paper lie in the following aspects. First, we satisfactorily solve the general matrix extension problem with symmetry for any r, s such that $1 \leqslant r \leqslant s$. More importantly, we obtain a complete representation of any $r \times s$ paraunitary matrix P having compatible symmetry with $1 \leqslant r \leqslant s$. This representation leads to a step-by-step algorithm for deriving a desired matrix P_e from a given matrix P. Second, we obtain an optimal result in the sense of (1.7) on controlling the coefficient support of the desired matrix P_e derived from a given matrix P by our algorithm. This is of importance in both theory and application, since short support of a filter or a multiwavelet is a highly desirable property and short support usually means a fast algorithm and simple implementation in practice. Third, we introduce the notion of compatibility of symmetry, which plays a critical role in the study of the general matrix extension problem with symmetry for the multirow case ($r \geqslant 1$). Fourth, we provide a complete analysis and a systematic construction algorithm for symmetric orthonormal multiwavelets. Finally, most of the literature on the matrix extension problem considers only Laurent polynomials with coefficients in the special field $\mathbb{C}$ (see [8, 16]) or $\mathbb{R}$ (see [1, 17]). In this paper, our setting is under a general field $\mathbb{F}$, which can be any subfield of $\mathbb{C}$ satisfying (1.6).

The structure of this paper is as follows. In section 2, we shall discuss an application of our main results on matrix extension with symmetry to the construction of symmetric orthonormal multiwavelets in wavelet analysis (or the design of symmetric filter banks in electronic engineering). Examples will be provided to illustrate our algorithms. In section 3, we shall present a step-by-step algorithm which leads to constructive proofs of Theorems 1 and 2. Finally, we shall prove Theorems 1 and 2 in section 4.

2. Application to symmetric orthonormal multiwavelets. In this section, we shall discuss the application of our results on matrix extension with symmetry to orthonormal multiwavelets with symmetry in wavelet analysis (or d-band symmetric paraunitary filter banks in electronic engineering). In order to do so, let us introduce some definitions first.

We say that d is *a dilation factor* if d is an integer with $|\mathsf{d}| > 1$. Throughout this section, d denotes a dilation factor. For simplicity of presentation, we further assume that d is positive, while multiwavelets and filter banks with a negative dilation factor can be handled similarly by a slight modification of the statements in this paper.

Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ such that (1.6) holds. A low-pass filter $a_0 : \mathbb{Z} \to \mathbb{F}^{r\times r}$ with multiplicity r is a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$. The *symbol* of the filter a_0 is defined to be $\mathsf{a}_0(z) := \sum_{k\in\mathbb{Z}} a_0(k)z^k$, which is a matrix of Laurent polynomials with coefficients in $\mathbb{F}$. Moreover, the d-*band subsymbols* of a_0 are defined by $\mathsf{a}_{0;\gamma}(z) := \sqrt{\mathsf{d}} \sum_{k\in\mathbb{Z}} a_0(\gamma + \mathsf{d}k)z^k$, $\gamma \in \mathbb{Z}$. We say that a_0 (or a_0) is *a* d-*band orthogonal filter* if

$$\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\mathsf{a}^*_{0;\gamma}(z) = I_r, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{2.1}$$

For $f \in L_1(\mathbb{R})$, the Fourier transform used in this paper is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$ and can be naturally extended to $L_2(\mathbb{R})$ functions. For a d-band orthogonal low-pass filter a_0, we assume that there exists *an orthogonal* d-*refinable function vector* $\phi = [\phi_1, \dots, \phi_r]^T$ associated with the low-pass filter a_0, with compactly supported functions $\phi_1, \dots, \phi_r$ in $L_2(\mathbb{R})$ such that

$$\hat{\phi}(\mathsf{d}\xi) = \mathsf{a}_0(e^{-i\xi})\hat{\phi}(\xi), \qquad \xi \in \mathbb{R} \qquad \text{with} \quad \|\hat{\phi}(0)\| = 1, \tag{2.2}$$

and

$$\langle \phi(\cdot - k), \phi\rangle := \int_{\mathbb{R}} \phi(x-k)\overline{\phi(x)}^T dx = \delta(k)I_r, \qquad k \in \mathbb{Z}, \tag{2.3}$$

where δ denotes the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$.

To construct an orthonormal multiwavelet basis (or an orthogonal filter bank with the perfect reconstruction property), one has to design high-pass filters $a_1, \dots, a_{\mathsf{d}-1} : \mathbb{Z} \to \mathbb{F}^{r\times r}$ such that the polyphase matrix

$$\mathcal{P}(z) = \begin{bmatrix} \mathsf{a}_{0;0}(z) & \cdots & \mathsf{a}_{0;\mathsf{d}-1}(z) \\ \mathsf{a}_{1;0}(z) & \cdots & \mathsf{a}_{1;\mathsf{d}-1}(z) \\ \vdots & \ddots & \vdots \\ \mathsf{a}_{\mathsf{d}-1;0}(z) & \cdots & \mathsf{a}_{\mathsf{d}-1;\mathsf{d}-1}(z) \end{bmatrix} \tag{2.4}$$

is paraunitary, that is, $\mathcal{P}(z)\mathcal{P}^*(z) = I_{\mathsf{d}r}$, where each $\mathsf{a}_{m;\gamma}$ is a subsymbol of a_m for $m, \gamma = 0, \dots, \mathsf{d}-1$, respectively. Symmetry of the filters in a filter bank is a very desirable property in many applications. We say that the low-pass filter a_0 (or a_0) has symmetry if

$$\mathsf{a}_0(z) = \operatorname{diag}(\varepsilon_1 z^{\mathsf{d}c_1}, \dots, \varepsilon_r z^{\mathsf{d}c_r})\mathsf{a}_0(1/z)\operatorname{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}) \tag{2.5}$$

for some $\varepsilon_1, \dots, \varepsilon_r \in \{-1, 1\}$ and $c_1, \dots, c_r \in \mathbb{R}$ such that $\mathsf{d}c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$. To design a symmetric filter bank with the perfect reconstruction property, from a given d-band orthogonal low-pass filter a_0, one has to construct high-pass filters $a_1, \dots, a_{\mathsf{d}-1} : \mathbb{Z} \to \mathbb{F}^{r\times r}$ such that all of them have symmetry that is compatible with the symmetry of a_0 in (2.5) and the polyphase matrix $\mathcal{P}$ in (2.4) is paraunitary. Define multiwavelet function vectors $\psi^m = [\psi^m_1, \dots, \psi^m_r]^T$ associated with the high-pass filters a_m, $m = 1, \dots, \mathsf{d}-1$, by

$$\widehat{\psi^m}(\mathsf{d}\xi) := \mathsf{a}_m(e^{-i\xi})\hat{\phi}(\xi), \qquad \xi \in \mathbb{R},\ m = 1, \dots, \mathsf{d}-1. \tag{2.6}$$

It is well known that $\{\psi^1,\dots,\psi^{\mathsf{d}-1}\}$ generates an orthonormal multiwavelet basis in $L_2(\mathbb{R})$; that is, $\{\mathsf{d}^{j/2}\psi^m_\ell(\mathsf{d}^j\cdot -k) \ : \ j,k\in\mathbb{Z};\ m=1,\dots,\mathsf{d}-1;\ \ell=1,\dots,r\}$ is an orthonormal basis of $L_2(\mathbb{R})$. For example, see [3, 7, 9, 12, 18, 20] and the references therein.

If a_0 has symmetry as in (2.5) and if 1 is a simple eigenvalue of $\mathsf{a}_0(1)$, then it is well known that the d-refinable function vector ϕ in (2.2) associated with the low-pass filter a_0 has the following symmetry:

$$\phi_1(c_1-\cdot)=\varepsilon_1\phi_1,\quad \phi_2(c_2-\cdot)=\varepsilon_2\phi_2,\quad \dots,\quad \phi_r(c_r-\cdot)=\varepsilon_r\phi_r. \tag{2.7}$$

Under the symmetry condition in (2.5), to apply Theorem 1, we first show that there exists a suitable paraunitary matrix U acting on $\mathsf{P}_{\mathsf{a}_0}:=[\mathsf{a}_{0;0},\dots,\mathsf{a}_{0;\mathsf{d}-1}]$ so that $\mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry. Note that $\mathsf{P}_{\mathsf{a}_0}$ itself may not have any symmetry.

LEMMA 1. *Let* $\mathsf{P}_{\mathsf{a}_0}:=[\mathsf{a}_{0;0},\dots,\mathsf{a}_{0;\mathsf{d}-1}]$, *where* $\mathsf{a}_{0;0},\dots,\mathsf{a}_{0;\mathsf{d}-1}$ *are* d*-band subsymbols of a* d*-band orthogonal filter* a_0 *satisfying* (2.5). *Then there exists a* $\mathsf{d}r\times\mathsf{d}r$ *paraunitary matrix* U *such that* $\mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ *has compatible symmetry.*

Proof. From (2.5), we deduce that

$$[\mathsf{a}_{0;\gamma}(z)]_{\ell,j}=\varepsilon_\ell\varepsilon_j z^{R^\gamma_{\ell,j}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}(z^{-1})]_{\ell,j},\quad \gamma=0,\dots,\mathsf{d}-1,\quad \ell,j=1,\dots,r, \tag{2.8}$$

where $\gamma,Q^\gamma_{\ell,j}\in\Gamma:=\{0,\dots,\mathsf{d}-1\}$ and $R^\gamma_{\ell,j}$, $Q^\gamma_{\ell,j}$ are uniquely determined by

$$\mathsf{d}c_\ell-c_j-\gamma=\mathsf{d}R^\gamma_{\ell,j}+Q^\gamma_{\ell,j}\quad\text{with}\quad R^\gamma_{\ell,j}\in\mathbb{Z},\ Q^\gamma_{\ell,j}\in\Gamma. \tag{2.9}$$

Since $\mathsf{d}c_\ell-c_j\in\mathbb{Z}$ for all $\ell,j=1,\dots,r$, we have $c_\ell-c_j\in\mathbb{Z}$ for all $\ell,j=1,\dots,r$ and therefore, $Q^\gamma_{\ell,j}$ is independent of ℓ. Consequently, by (2.8), for every $1\leqslant j\leqslant r$, the jth column of the matrix $\mathsf{a}_{0;\gamma}$ is a flipped version of the jth column of the matrix $\mathsf{a}_{0;Q^\gamma_{\ell,j}}$. Let $\kappa_{j,\gamma}\in\mathbb{Z}$ be an integer such that $|\mathrm{coeffsupp}([\mathsf{a}_{0;\gamma}]_{:,j}+z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j})|$ is the smallest possible number. Define $\mathsf{P}:=[\mathsf{b}_{0;0},\dots,\mathsf{b}_{0;\mathsf{d}-1}]$ as follows:

$$[\mathsf{b}_{0;\gamma}]_{:,j}:=\begin{cases}[\mathsf{a}_{0;\gamma}]_{:,j}, & \gamma=Q^\gamma_{\ell,j},\\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j}+z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma<Q^\gamma_{\ell,j},\\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j}-z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma>Q^\gamma_{\ell,j},\end{cases} \tag{2.10}$$

where $[\mathsf{a}_{0;\gamma}]_{:,j}$ denotes the jth column of $\mathsf{a}_{0;\gamma}$. Let U denote the unique transform matrix corresponding to (2.10) such that $\mathsf{P}:=[\mathsf{b}_{0;0},\dots,\mathsf{b}_{0;\mathsf{d}-1}]=[\mathsf{a}_{0;0},\dots,\mathsf{a}_{0;\mathsf{d}-1}]\mathsf{U}$. It is evident that U is paraunitary and $\mathsf{P}=\mathsf{P}_{\mathsf{a}_0}\mathsf{U}$. We now show that P has compatible symmetry. Indeed, by (2.8) and (2.10),

$$[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,j}=\mathrm{sgn}(Q^\gamma_{\ell,j}-\gamma)\varepsilon_\ell\varepsilon_j z^{R^\gamma_{\ell,j}+\kappa_{j,\gamma}}, \tag{2.11}$$

where $\mathrm{sgn}(x)=1$ for $x\geqslant 0$ and $\mathrm{sgn}(x)=-1$ for $x<0$. By (2.9) and noting that $Q^\gamma_{\ell,j}$ is independent of ℓ, we have

$$\frac{[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,j}}{[\mathcal{S}\mathsf{b}_{0;\gamma}]_{n,j}}=\varepsilon_\ell\varepsilon_n z^{R^\gamma_{\ell,j}-R^\gamma_{n,j}}=\varepsilon_\ell\varepsilon_n z^{c_\ell-c_n}$$

for all $1\leqslant \ell,n\leqslant r$, which is equivalent to saying that P has compatible symmetry. ∎

Now, for a d-band orthogonal low-pass filter a_0 satisfying (2.5), we have the following algorithm to construct high-pass filters $\mathsf{a}_1,\dots,\mathsf{a}_{\mathsf{d}-1}$ such that they form a symmetric paraunitary filter bank with the perfect reconstruction property.

ALGORITHM 1. Input an orthogonal d-band filter a_0 with symmetry as in (2.5).

(1) Construct U as in (2.10) such that $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry: $\mathcal{S}\mathsf{P} = [\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}]^T \mathcal{S}\theta$ for some $k_1, \dots, k_r \in \mathbb{Z}$ and some $1 \times \mathsf{d}r$ row vector θ of Laurent polynomials with symmetry.
(2) Derive P_e as in Theorem 1 from P by Algorithm 2 (see section 3).
(3) Let $\mathcal{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant \mathsf{d}-1}$ as in (2.4). Define high-pass filters

$$\mathsf{a}_m(z) := \frac{1}{\sqrt{\mathsf{d}}}\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{m;\gamma}(z^{\mathsf{d}})z^{\gamma}, \qquad m = 1, \dots, \mathsf{d}-1. \tag{2.12}$$

Output a symmetric filter bank $\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}\}$ with the perfect reconstruction property, i.e., $\mathcal{P}$ in (2.4) is paraunitary and all filters a_m, $m = 1, \dots, \mathsf{d}-1$, have symmetry:

$$\mathsf{a}_m(z) = \operatorname{diag}(\varepsilon_1^m z^{\mathsf{d}c_1^m}, \dots, \varepsilon_r^m z^{\mathsf{d}c_r^m})\mathsf{a}_m(1/z)\operatorname{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}), \tag{2.13}$$

where $c_\ell^m := (k_\ell^m - k_\ell) + c_\ell \in \mathbb{R}$ and all $\varepsilon_\ell^m \in \{-1, 1\}$, $k_\ell^m \in \mathbb{Z}$, for $\ell = 1, \dots, r$ and $m = 1, \dots, \mathsf{d}-1$, are determined by the symmetry pattern of P_e as follows:

$$[\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}, \varepsilon_1^1 z^{k_1^1}, \dots, \varepsilon_r^1 z^{k_r^1}, \dots, \varepsilon_1^{\mathsf{d}-1} z^{k_1^{\mathsf{d}-1}}, \dots, \varepsilon_r^{\mathsf{d}-1} z^{k_r^{\mathsf{d}-1}}]^T \mathcal{S}\theta := \mathcal{S}\mathsf{P}_e. \tag{2.14}$$

Proof. Rewrite $\mathsf{P}_e = (\mathsf{b}_{m;\gamma})_{0\leqslant m,\gamma\leqslant \mathsf{d}-1}$ as a $\mathsf{d} \times \mathsf{d}$ block matrix with $r \times r$ blocks $\mathsf{b}_{m;\gamma}$. Since P_e has compatible symmetry as in (2.14), we have $[\mathcal{S}\mathsf{b}_{m;\gamma}]_{\ell,:} = \varepsilon_\ell^m \varepsilon_\ell z^{k_\ell^m - k_\ell}[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,:}$ for $\ell = 1, \dots, r$ and $m = 1, \dots, \mathsf{d}-1$. By (2.11), we have

$$[\mathcal{S}\mathsf{b}_{m;\gamma}]_{\ell,j} = \operatorname{sgn}(Q_{\ell,j}^\gamma - \gamma)\varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^\gamma + \kappa_{j,\gamma} + k_\ell^m - k_\ell}, \qquad \ell, j = 1, \dots, r. \tag{2.15}$$

By (2.15) and the definition of U^* in (2.10), we deduce that

$$[\mathsf{a}_{m;\gamma}(z)]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^\gamma + k_\ell^m - k_\ell}[\mathsf{a}_{m;Q_{\ell,j}^\gamma}(z^{-1})]_{\ell,j}. \tag{2.16}$$

This implies that $[\mathcal{S}\mathsf{a}_m]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{\mathsf{d}(k_\ell^m - k_\ell + c_\ell) - c_j}$, which is equivalent to (2.13) with $c_\ell^m := k_\ell^m - k_\ell + c_\ell$ for $m = 1, \dots, \mathsf{d}-1$ and $\ell = 1, \dots, r$. ∎

Since the high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}$ satisfy (2.13), it is easy to verify that each $\psi^m = [\psi_1^m, \dots, \psi_r^m]^T$ defined in (2.6) also has the following symmetry:

$$\psi_1^m(c_1^m - \cdot) = \varepsilon_1^m \psi_1^m, \quad \psi_2^m(c_2^m - \cdot) = \varepsilon_2^m \psi_2^m, \quad \dots, \quad \psi_r^m(c_r^m - \cdot) = \varepsilon_r^m \psi_r^m. \tag{2.17}$$

In the following, let us present several examples to demonstrate our results and illustrate our algorithms.

Example 1. Let $\mathsf{d} = 2$ and $r = 2$. A 2-band orthogonal low-pass filter a_0 with multiplicity 2 in [5] is given by

$$\mathsf{a}_0(z) = \frac{1}{40}\begin{bmatrix} 12(1+z^{-1}) & 16\sqrt{2}z^{-1} \\ -\sqrt{2}(z^2 - 9z - 9 + z^{-1}) & -2(3z - 10 + 3z^{-1}) \end{bmatrix}.$$

The low-pass filter a_0 satisfies (2.5) with $c_1 = -1, c_2 = 0$, and $\varepsilon_1 = \varepsilon_2 = 1$. Using Lemma 1, we obtain $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}]$ and U as follows:

$$\mathsf{P}_{\mathsf{a}_0} = \frac{1}{20}\left[\begin{array}{cc|cc} 6\sqrt{2} & 0 & \frac{6\sqrt{2}}{z} & \frac{16}{z} \\ 9 - z & 10\sqrt{2} & 9 - \frac{1}{z} & -3\sqrt{2}(1 + \frac{1}{z}) \end{array}\right], \quad \mathsf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ z & 0 & -z & 0 \\ 0 & 0 & 0 & \sqrt{2}z \end{bmatrix}.$$

Then $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [1,z]^T[1,z^{-1},-1,1]$ and is given by

$$\mathsf{P} = \frac{\sqrt{2}}{20}\begin{bmatrix} 6\sqrt{2} & 0 & 0 & 8\sqrt{2} \\ 4(1+z) & 10 & 5(1-z) & -3(1+z) \end{bmatrix}.$$

Applying Algorithm 2, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = \frac{\sqrt{2}}{20}\begin{bmatrix} 6\sqrt{2} & 0 & 0 & 8\sqrt{2} \\ 4(1+z) & 10 & 5(1-z) & -3(1+z) \\ 4(1+z) & -10 & 5(1-z) & -3(1+z) \\ 4\sqrt{2}(1-z) & 0 & 5\sqrt{2}(z+1) & 3\sqrt{2}(z-1) \end{bmatrix}.$$

We have $\mathcal{S}\mathsf{P}_e = [1,z,z,-z]^T[1,z^{-1},-1,1]$ and $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 4$. Now, from the polyphase matrix $\mathcal{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant 1}$, we derive a high-pass filter a_1 as follows:

$$\mathsf{a}_1(z) = \frac{1}{40}\begin{bmatrix} -\sqrt{2}(z^2-9z-9+z^{-1}) & -2(3z+10+3z^{-1}) \\ 2(z^2-9z+9-z^{-1}) & 6\sqrt{2}(z-z^{-1}) \end{bmatrix}.$$

Then the high-pass filter a_1 satisfies (2.13) with $c_1^1 = c_2^1 = 0$ and $\varepsilon_1^1 = 1,\ \varepsilon_2^1 = -1$.

Example 2. Let $\mathsf{d} = 3$ and $r = 2$. A 3-band orthogonal low-pass filter a_0 with multiplicity 2 in [12] is given by

$$\mathsf{a}_0(z) = \frac{1}{540}\begin{bmatrix} a_{11}(z) + a_{11}(z^{-1}) & a_{12}(z) + z^{-1}a_{12}(z^{-1}) \\ a_{21}(z) + z^3 a_{21}(z^{-1}) & a_{22}(z) + z^2 a_{22}(z^{-1}) \end{bmatrix},$$

where

$$\begin{aligned} a_{11}(z) &= 90 + (55-5\sqrt{41})z - (8+2\sqrt{41})z^2 + (7\sqrt{41}-47)z^4, \\ a_{12}(z) &= 145 + 5\sqrt{41} + (1-\sqrt{41})z^2 + (34-4\sqrt{41})z^3, \\ a_{21}(z) &= (111+9\sqrt{41})z^2 + (69-9\sqrt{41})z^4, \\ a_{22}(z) &= 90z + (63-3\sqrt{41})z^2 + (3\sqrt{41}-63)z^3. \end{aligned}$$

The low-pass filter a_0 satisfies (2.5) with $c_1 = 0,\ c_2 = 1$, and $\varepsilon_1 = \varepsilon_2 = 1$. From $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}, \mathsf{a}_{0;2}]$, the matrix U constructed by Lemma 1 is given by

$$\mathsf{U} := \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & z & 0 & -z & 0 \\ 0 & z & 0 & 0 & 0 & -z \end{bmatrix}.$$

Let

$$\begin{aligned} &c_0 = 11-\sqrt{41}, && t_{12} = 5(7-\sqrt{41}), && c_{12} = 10(29+\sqrt{41}), && t_{13} = -5c_0, \\ &t_{16} = 3c_0, && t_{15} = 3(3\sqrt{41}-13), && t_{25} = 6(7+3\sqrt{41}), && t_{26} = 6(21-\sqrt{41}), \\ &t_{53} = 400\sqrt{6}/c_0, && t_{55} = 12\sqrt{6}(\sqrt{41}-1), && t_{56} = 6\sqrt{6}(4+\sqrt{41}), && c_{66} = 3\sqrt{6}(3+7\sqrt{41}). \end{aligned}$$

Then $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [1,z]^T[1,1,1,z^{-1},-1,-1]$ and is given by

$$\mathsf{P} = \frac{\sqrt{6}}{1080}\begin{bmatrix} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z-z^{-1}) & t_{16}(z-z^{-1}) \\ 0 & 0 & 180(1+z) & 180\sqrt{2} & t_{25}(1-z) & t_{26}(1-z) \end{bmatrix},$$

where $b_{12}(z) = t_{12}(z+z^{-1})+c_{12}$ and $b_{13}(z) = t_{13}(z-2+z^{-1})$. Applying Algorithm 2, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = \frac{\sqrt{6}}{1080}\left[\begin{array}{cccccc} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z-\frac{1}{z}) & t_{16}(z-\frac{1}{z}) \\ 0 & 0 & 180(1+z) & 180\sqrt{2} & t_{25}(1-z) & t_{26}(1-z) \\ \hline 360 & -\frac{b_{12}(z)}{\sqrt{2}} & -\frac{b_{13}(z)}{\sqrt{2}} & 0 & \frac{t_{15}}{\sqrt{2}}(\frac{1}{z}-z) & \frac{t_{16}}{\sqrt{2}}(\frac{1}{z}-z) \\ 0 & 0 & 90\sqrt{2}(1+z) & -360 & \frac{t_{25}}{\sqrt{2}}(1-z) & \frac{t_{26}}{\sqrt{2}}(1-z) \\ \hline 0 & \sqrt{6}t_{13}(1-z) & t_{53}(1-z) & 0 & t_{55}(1+z) & t_{56}(1+z) \\ 0 & \frac{\sqrt{6}t_{12}}{2}(\frac{1}{z}-z) & \frac{\sqrt{6}t_{13}}{2}(\frac{1}{z}-z) & 0 & b_{65}(z) & b_{66}(z) \end{array}\right],$$

where $b_{65}(z) = -\sqrt{6}(5t_{15}(z+z^{-1})+3c_{12})/10$ and $b_{66}(z) = -\sqrt{6}t_{16}(z+z^{-1})/2+c_{66}$. Note that $\mathcal{S}\mathsf{P}_e = [1, z, 1, z, -z, -1]^T[1, 1, 1, z^{-1}, -1, -1]$ and the coefficient support of P_e satisfies $\text{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \text{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 6$. From the polyphase matrix $\mathcal{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant 2}$, we derive two high-pass filters $\mathsf{a}_1, \mathsf{a}_2$ as follows:

$$\mathsf{a}_1(z) = \frac{\sqrt{2}}{1080}\begin{bmatrix} a^1_{11}(z)+a^1_{11}(z^{-1}) & a^1_{12}(z)+z^{-1}a^1_{12}(z^{-1}) \\ a^1_{21}(z)+z^3a^1_{21}(z^{-1}) & a^1_{22}(z)+z^2a^1_{22}(z^{-1}) \end{bmatrix},$$
$$\mathsf{a}_2(z) = \frac{\sqrt{6}}{1080}\begin{bmatrix} a^2_{11}(z)-z^3a^2_{11}(z^{-1}) & a^2_{12}(z)-z^2a^2_{12}(z^{-1}) \\ a^2_{21}(z)-a^2_{21}(z^{-1}) & a^2_{22}(z)-z^{-1}a^2_{22}(z^{-1}) \end{bmatrix},$$

where

$$\begin{aligned}
a^1_{11}(z) &= (47-7\sqrt{41})z^4+2(4+\sqrt{41})z^2+5(\sqrt{41}-11)z+180,\\
a^1_{12}(z) &= 2(2\sqrt{41}-17)z^3+(\sqrt{41}-1)z^2-5(29+\sqrt{41}),\\
a^1_{21}(z) &= 3(37+3\sqrt{41})z+3(23-3\sqrt{41})z^{-1},\\
a^1_{22}(z) &= -180z+3(21-\sqrt{41})-3(21-\sqrt{41})z^{-1},\\
a^2_{11}(z) &= (43+17\sqrt{41})z+(67-7\sqrt{41})z^{-1},\\
a^2_{12}(z) &= 11\sqrt{41}-31-(79+\sqrt{41})z^{-1},\\
a^2_{21}(z) &= (47-7\sqrt{41})z^4+2(4+\sqrt{41})z^2-3(29+\sqrt{41})z,\\
a^2_{22}(z) &= 2(2\sqrt{41}-17)z^3+(\sqrt{41}-1)z^2+3(3+7\sqrt{41}).
\end{aligned}$$

Then the high-pass filters a_1, a_2 satisfy (2.13) with $c^1_1 = 0,\ c^1_2 = 1,\ \varepsilon^1_1 = \varepsilon^1_2 = 1$ and $c^2_1 = 1,\ c^2_2 = 0,\ \varepsilon^2_1 = \varepsilon^2_2 = -1$.

As demonstrated by the following example, our Algorithm 1 also applies to low-pass filters with symmetry patterns other than those in (2.5).

Example 3. Let $\mathsf{d} = 3$ and $r = 2$. A 3-band orthogonal low-pass filter a_0 with multiplicity 2 in [9] is given by

$$\mathsf{a}_0(z) = \frac{1}{702}\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix},$$

where

$$\begin{aligned}
a_{11}(z) &= (11-14\sqrt{17})z^2+(29+8\sqrt{17})z+234+(85-16\sqrt{17})z^{-1}-(17+2\sqrt{17})z^{-2},\\
a_{12}(z) &= (5\sqrt{17}-16)z^3+(2+\sqrt{17})z^2+238-11\sqrt{17}+(136+29\sqrt{17})z^{-1},\\
a_{21}(z) &= (136+29\sqrt{17})z^2+(238-11\sqrt{17})z+(2+\sqrt{17})z^{-1}+(5\sqrt{17}-16)z^{-2},\\
a_{22}(z) &= (-17-2\sqrt{17})z^3+(85-16\sqrt{17})z^2+234z+29+8\sqrt{17}+(11-14\sqrt{17})z^{-1}.
\end{aligned}$$

This low-pass filter a_0 does not satisfy (2.5). However, we can employ a very simple orthogonal transform $E := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ to a_0 so that the symmetry in (2.5) holds. That is, for $\widetilde{\mathsf{a}}_0(z) := E\mathsf{a}_0(z)E$, it is easy to verify that $\widetilde{\mathsf{a}}_0$ satisfies (2.5) with $c_1 = c_2 = 1/2$ and $\varepsilon_1 = 1, \varepsilon_2 = -1$. Construct $\mathsf{P}_{\widetilde{\mathsf{a}}_0} := [\widetilde{\mathsf{a}}_{0;0}, \widetilde{\mathsf{a}}_{0;1}, \widetilde{\mathsf{a}}_{0;2}]$ from $\widetilde{\mathsf{a}}_0$. The matrix U constructed by Lemma 1 from $\mathsf{P}_{\widetilde{\mathsf{a}}_0}$ is given by

$$\mathsf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} z^{-1} & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & z^{-1} & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Then $\mathsf{P} := \mathsf{P}_{\widetilde{\mathsf{a}}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [z^{-1}, -z^{-1}]^T[1,-1,-1,1,1,-1]$ and is given by

$$\mathsf{P} = c\begin{bmatrix} 234(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & 0 & 117\sqrt{2}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \end{bmatrix},$$

where $c = \frac{\sqrt{6}}{1404}$ and t_{jk}'s are constants defined as follows:

$$\begin{aligned} &t_{12} = 3(11-\sqrt{17}), && t_{13} = 3(\sqrt{17}-89), && t_{16} = 15\sqrt{2}(2+\sqrt{17}), \\ &t_{21} = 13(\sqrt{17}-17), && t_{22} = 6(2+\sqrt{17}), && t_{23} = 6(37-\sqrt{17}), \\ &t_{24} = -13(1+\sqrt{17}), && t_{25} = -13\sqrt{2}(8+\sqrt{17}), && t_{26} = -3\sqrt{2}(7+10\sqrt{17}). \end{aligned}$$

Applying Algorithm 2 to P, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = c\left[\begin{array}{cccccc} 234(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & 0 & 117\sqrt{2}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \\ \hline t_{31}(1-\frac{1}{z}) & t_{32}(1+\frac{1}{z}) & t_{33}(1+\frac{1}{z}) & t_{34}(1-\frac{1}{z}) & t_{35}(1-\frac{1}{z}) & t_{36}(1+\frac{1}{z}) \\ t_{41}(1+\frac{1}{z}) & t_{42}(1-\frac{1}{z}) & t_{43}(1-\frac{1}{z}) & t_{44}(1+\frac{1}{z}) & -\sqrt{2}t_{41}(1+\frac{1}{z}) & t_{46}(1-\frac{1}{z}) \\ \hline \frac{2}{\sqrt{3}}t_{44} & 0 & 0 & -2\sqrt{3}t_{41} & -\frac{4}{\sqrt{6}}t_{44} & 0 \\ 0 & t_{62} & t_{63} & 0 & 0 & t_{66} \end{array}\right],$$

where all t_{jk}'s are constants given by

$$\begin{aligned} &t_{31} = -\sqrt{26}(61+25\sqrt{17})/4, && t_{32} = -3\sqrt{26}(397+23\sqrt{17})/52, \\ &t_{33} = 3\sqrt{26}(553+23\sqrt{17})/52, && t_{34} = 25\sqrt{26}(1+\sqrt{17})/4, \\ &t_{35} = \sqrt{13}(25\sqrt{17}-43)/2, && t_{36} = 15\sqrt{13}(23\sqrt{17}-19)/26, \\ &t_{41} = 9\sqrt{26}(1-3\sqrt{17})/4, && t_{42} = -3\sqrt{26}(383+29\sqrt{17})/52, \\ &t_{43} = 3\sqrt{26}(29\sqrt{17}+227)/52, && t_{44} = 27\sqrt{26}(1+\sqrt{17})/4, \\ &t_{46} = 3\sqrt{13}(145\sqrt{17}-61)/26, && t_{62} = 9\sqrt{78}(41\sqrt{17}-9)/26, \\ &t_{63} = 9\sqrt{78}(11\sqrt{17}+9)/26, && t_{66} = 27\sqrt{3}(\sqrt{17}+15)/\sqrt{13}. \end{aligned}$$

Note that P_e satisfies $\mathcal{S}\mathsf{P}_e = [z^{-1}, -z^{-1}, -z^{-1}, z^{-1}, 1, -1]^T[1,-1,-1,1,1,-1]$ and we have $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 6$. From the polyphase matrix $\mathcal{P} := \mathsf{P}_e\mathsf{U}^*$, we derive two high-pass filters $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ as follows:

$$\widetilde{\mathsf{a}}_1(z) = \frac{\sqrt{26}}{36504}\begin{bmatrix} a_{11}^1(z) - za_{11}^1(z^{-1}) & a_{12}^1(z) + za_{12}^1(z^{-1}) \\ a_{21}^1(z) + za_{21}^1(z^{-1}) & a_{22}^1(z) - za_{22}^1(z^{-1}) \end{bmatrix},$$
$$\widetilde{\mathsf{a}}_2(z) = \frac{\sqrt{78}}{4056}\begin{bmatrix} a_{11}^2(z) & a_{12}^2(z) \\ a_{21}^2(z) & a_{22}^2(z) \end{bmatrix},$$

where

$$\begin{aligned}
a_{11}^1(z) &= (433 - 128\sqrt{17})z^3 + 13(25\sqrt{17} - 43)z^2 - (1226 + 197\sqrt{17})z,\\
a_{12}^1(z) &= (128\sqrt{17} - 433)z^3 + 15(23\sqrt{17} - 19)z^2 - (758 + 197\sqrt{17})z,\\
a_{21}^1(z) &= 3(133 - 44\sqrt{17})z^3 + 117(3\sqrt{17} - 1)z^2 - 3(73\sqrt{17} + 94)z,\\
a_{22}^1(z) &= 3(44\sqrt{17} - 133)z^3 + 3(145\sqrt{17} - 61)z^2 - 3(250 + 73\sqrt{17})z,\\
a_{11}^2(z) &= 13(1 + \sqrt{17})(z^3 - 2z^2 + z); a_{12}^2(z) = 13(3\sqrt{17} - 1)(z^3 - z),\\
a_{21}^2(z) &= (9 + 11\sqrt{17})(z^3 - z),\\
a_{22}^2(z) &= (41\sqrt{17} - 9)(z^3 + 24z^2/137 + 18\sqrt{17}z^2/137 + z).
\end{aligned}$$

Then the high-pass filters $\widetilde{\mathsf{a}}_1$ and $\widetilde{\mathsf{a}}_2$ satisfy (2.13) with $c_1^1 = c_2^1 = 1/2$, $\varepsilon_1^1 = -1$, $\varepsilon_2^1 = 1$ and $c_1^2 = c_2^2 = 3/2$, $\varepsilon_1^1 = 1$, $\varepsilon_2^1 = -1$, respectively.

Let $\mathsf{a}_1, \mathsf{a}_2$ be two high-pass filters constructed from $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ by $\mathsf{a}_1(z) := E\widetilde{\mathsf{a}}_1(z)E$ and $\mathsf{a}_2(z) := E\widetilde{\mathsf{a}}_2(z)E$. Then due to the orthogonality of E, $\{\mathsf{a}_0, \mathsf{a}_1, \mathsf{a}_2\}$ still forms a d-band filter bank with the perfect reconstruction property, but their symmetry patterns are different from those of $\widetilde{\mathsf{a}}_0, \widetilde{\mathsf{a}}_1$, and $\widetilde{\mathsf{a}}_2$.

3. An algorithm for matrix extension with symmetry. In this section, we present a step-by-step algorithm on matrix extension with symmetry to derive a desired matrix P_e in Theorem 2 from a given matrix P. Our algorithm has three steps: initialization, support reduction, and finalization. The step of initialization reduces the symmetry pattern of P to a standard form. The step of support reduction is the main body of the algorithm, producing a sequence of elementary matrices $\mathsf{A}_1, \dots, \mathsf{A}_J$ that reduce the length of the coefficient support of P to 0. The step of finalization generates the desired matrix P_e as in Theorem 2. More precisely, our algorithm written in the form of pseudocode for Theorem 2 is as follows.

ALGORITHM 2. Input P as in Theorem 2 with $\mathcal{S}\mathsf{P} = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$ for some $1 \times r$ and $1 \times s$ row vectors θ_1 and θ_2 of Laurent polynomials with symmetry.

1. *Initialization.* Let $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^*\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$. Then the symmetry pattern of Q is

$$\mathcal{S}\mathsf{Q} = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}], \tag{3.1}$$

where all nonnegative integers $r_1, \dots, r_4, s_1, \dots, s_4$ are uniquely determined by $\mathcal{S}\mathsf{P}$.

2. *Support reduction.* Let $\mathsf{P}_0 := \mathsf{U}_{\mathcal{S}\theta_2}^*$ and $J := 1$.

while (|coeffsupp(Q)| > 0) do %% outer while loop
  Let $\mathsf{Q}_0 := \mathsf{Q}$, $[k_1, k_2] :=$ coeffsupp(Q), and $\mathsf{A}_J := I_s$.
  if $k_2 = -k_1$, then
    for j from 1 to r do
      Let $\mathsf{q} := [\mathsf{Q}_0]_{j,:}$ and $\mathsf{p} := [\mathsf{Q}]_{j,:}$ be the jth rows of Q_0 and Q, respectively.
      Let $[\ell_1, \ell_2] :=$ coeffsupp(q), $\ell := \ell_2 - \ell_1$, and $\mathsf{B}_j := I_s$.
      if coeffsupp(q) = coeffsupp(p) and $\ell \geqslant 2$ and ($\ell_1 = k_1$ or $\ell_2 = k_2$), then

$\mathsf{B}_j := \mathsf{B}_{\mathsf{q}}$. $\mathsf{A}_J := \mathsf{A}_J\mathsf{B}_j$. $\mathsf{Q}_0 := \mathsf{Q}_0\mathsf{B}_j$.
`end if`
`end for`
Q_0 takes the form in (3.7).
Let $\mathsf{B}_{(-k_2,k_2)} := I_s$, $\mathsf{Q}_1 := \mathsf{Q}_0$, $j_1 := 1$, and $j_2 := r_3 + r_4 + 1$.
`while` $j_1 \leqslant r_1 + r_2$ `and` $j_2 \leqslant r$ `do` %% inner while loop
Let $\mathsf{q}_1 := [\mathsf{Q}_1]_{j_1,:}$ and $\mathsf{q}_2 := [\mathsf{Q}_1]_{j_2,:}$.
`if` $\operatorname{coeff}(\mathsf{q}_1, k_1) = \mathbf{0}$, `then` $j_1 := j_1 + 1$. `end if`
`if` $\operatorname{coeff}(\mathsf{q}_2, k_2) = \mathbf{0}$, `then` $j_2 := j_2 + 1$. `end if`
`if` $\operatorname{coeff}(\mathsf{q}_1, k_1) \neq \mathbf{0}$ `and` $\operatorname{coeff}(\mathsf{q}_2, k_2) \neq \mathbf{0}$, `then`
$\mathsf{B}_{(-k_2,k_2)} := \mathsf{B}_{(-k_2,k_2)}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$. $\mathsf{Q}_1 := \mathsf{Q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$. $\mathsf{A}_J := \mathsf{A}_J\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$.
$j_1 := j_1 + 1$. $j_2 := j_2 + 1$.
`end if`
`end while` %% end inner while loop
`end if`
Q_1 takes the form in (3.7) with either $\operatorname{coeff}(\mathsf{Q}_1, -k) = \mathbf{0}$ or $\operatorname{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$.
Let $\mathsf{A}_J := \mathsf{A}_J\mathsf{B}_{\mathsf{Q}_1}$ and $\mathsf{Q} := \mathsf{Q}\mathsf{A}_J$.
Then $\mathcal{S}\mathsf{Q} = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$.
Replace $s_1, \dots, s_4$ by $s_1', \dots, s_4'$, respectively. Let $\mathsf{P}_J := \mathsf{A}_J^*$ and $J := J + 1$.
`end while` %% end outer while loop

3. *Finalization.* $\mathsf{Q} = \operatorname{diag}(F_1, F_2, F_3, F_4)$ for some $r_j \times s_j$ constant matrices F_j in $\mathbb{F}$, $j = 1, \dots, 4$. Let $U := \operatorname{diag}(U_{F_1}, U_{F_2}, U_{F_3}, U_{F_4})$ so that $\mathsf{Q}U = [I_r, \mathbf{0}]$. Define $\mathsf{P}_J := U^*$ and $\mathsf{P}_{J+1} := \operatorname{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$.
Output a desired matrix P_e satisfying all the properties in Theorem 2.

In the following subsections, we present detailed constructions of the matrices $\mathsf{U}_{\mathcal{S}\theta}$, B_{q}, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_{\mathsf{Q}_1}$, and U_F appearing in Algorithm 2.

3.1. Initialization. Let θ be a $1\times n$ row vector of Laurent polynomials with symmetry such that $\mathcal{S}\theta = [\varepsilon_1 z^{c_1}, \dots, \varepsilon_n z^{c_n}]$ for some $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and $c_1, \dots, c_n \in \mathbb{Z}$. Then, the symmetry of any entry in the vector $\theta\operatorname{diag}(z^{-\lceil c_1/2\rceil}, \dots, z^{-\lceil c_n/2\rceil})$ belongs to $\{\pm 1, \pm z^{-1}\}$. Thus, there is a permutation matrix E_θ to regroup these four types of symmetries together so that

$$\mathcal{S}(\theta\mathsf{U}_{\mathcal{S}\theta}) = [\mathbf{1}_{n_1}, -\mathbf{1}_{n_2}, z^{-1}\mathbf{1}_{n_3}, -z^{-1}\mathbf{1}_{n_4}], \tag{3.2}$$

where $\mathsf{U}_{\mathcal{S}\theta} := \operatorname{diag}(z^{-\lceil c_1/2\rceil}, \dots, z^{-\lceil c_n/2\rceil})E_\theta$, $\mathbf{1}_m$ denotes the $1\times m$ row vector $[1, \dots, 1]$, and $n_1, \dots, n_4$ are nonnegative integers uniquely determined by $\mathcal{S}\theta$. Since P satisfies (1.4), it is easy to see that $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^*\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ has the symmetry pattern as in (3.1). Note that $\mathsf{U}_{\mathcal{S}\theta_1}$ and $\mathsf{U}_{\mathcal{S}\theta_2}$ do not increase the length of the coefficient support of P.

3.2. Support reduction. Denote $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^*\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ as in Algorithm 2. The outer `while` loop in the step of support reduction produces a sequence of elementary paraunitary matrices $\mathsf{A}_1, \dots, \mathsf{A}_J$ that reduce the length of the coefficient support of Q gradually to 0. The construction of each A_j has three parts: $\{\mathsf{B}_1, \dots, \mathsf{B}_r\}$, $\mathsf{B}_{(-k,k)}$, and $\mathsf{B}_{\mathsf{Q}_1}$. The first part $\{\mathsf{B}_1, \dots, \mathsf{B}_r\}$ (see the `for` loop) is constructed recursively for each of the r rows of Q so that $\mathsf{Q}_0 := \mathsf{Q}\mathsf{B}_1 \cdots \mathsf{B}_r$ has a special form as in (3.7). If both $\operatorname{coeff}(\mathsf{Q}_0, -k) \neq \mathbf{0}$ and $\operatorname{coeff}(\mathsf{Q}_0, k) \neq \mathbf{0}$, then the second part $\mathsf{B}_{(-k,k)}$ (see the inner `while` loop) is further constructed so that $\mathsf{Q}_1 := \mathsf{Q}_0\mathsf{B}_{(-k,k)}$ takes the form in (3.7) with at least one of $\operatorname{coeff}(\mathsf{Q}_1, -k)$ and $\operatorname{coeff}(\mathsf{Q}_1, k)$ being $\mathbf{0}$. $\mathsf{B}_{\mathsf{Q}_1}$ is constructed to handle the case that $\operatorname{coeffsupp}(\mathsf{Q}_1) = [-k, k-1]$ or $\operatorname{coeffsupp}(\mathsf{Q}_1) = [-k+1, k]$ so that $\operatorname{coeffsupp}(\mathsf{Q}_1\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-k+1, k-1]$.

Let $\mathtt{q}$ denote an arbitrary row of Q with $|\text{coeffsupp}(\mathtt{q})| \geqslant 2$. We first explain how to construct $\mathsf{B}_{\mathtt{q}}$ for a given row $\mathtt{q}$ such that $\mathsf{B}_{\mathtt{q}}$ reduces the length of the coefficient support of $\mathtt{q}$ by 2 and keeps its symmetry pattern. Note that in the `for` loop, B_j is simply $\mathsf{B}_{\mathtt{q}}$ with $\mathtt{q}$ being the current jth row of $\mathsf{Q}\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$, where $\mathsf{B}_0 := I_s$.

By (3.1), we have $\mathcal{S}\mathtt{q} = \varepsilon z^c[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some $\varepsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$. For $\varepsilon = -1$, there is a permutation matrix E_ε such that $\mathcal{S}(\mathtt{q}E_\varepsilon) = z^c[\mathbf{1}_{s_2}, -\mathbf{1}_{s_1}, z^{-1}\mathbf{1}_{s_4}, -z^{-1}\mathbf{1}_{s_3}]$. For $\varepsilon = 1$, we let $E_\varepsilon := I_s$. Then, $\mathtt{q}E_\varepsilon$ must take the form in either (3.3) or (3.4) with $\mathtt{f}_1 \neq \mathbf{0}$ as follows:

$$\begin{aligned}\mathtt{q}E_\varepsilon = {} & [\mathtt{f}_1, -\mathtt{f}_2, \mathtt{g}_1, -\mathtt{g}_2]z^{\ell_1} + [\mathtt{f}_3, -\mathtt{f}_4, \mathtt{g}_3, -\mathtt{g}_4]z^{\ell_1+1} + \sum_{\ell=\ell_1+2}^{\ell_2-2} \text{coeff}(\mathtt{q}E_\varepsilon, \ell) z^\ell \\ & + [\mathtt{f}_3, \mathtt{f}_4, \mathtt{g}_1, \mathtt{g}_2]z^{\ell_2-1} + [\mathtt{f}_1, \mathtt{f}_2, \mathbf{0}, \mathbf{0}]z^{\ell_2},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathtt{q}E_\varepsilon = {} & [\mathbf{0}, \mathbf{0}, \mathtt{f}_1, -\mathtt{f}_2]z^{\ell_1} + [\mathtt{g}_1, -\mathtt{g}_2, \mathtt{f}_3, -\mathtt{f}_4]z^{\ell_1+1} + \sum_{\ell=\ell_1+2}^{\ell_2-2} \text{coeff}(\mathtt{q}E_\varepsilon, \ell) z^\ell \\ & + [\mathtt{g}_3, \mathtt{g}_4, \mathtt{f}_3, \mathtt{f}_4]z^{\ell_2-1} + [\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]z^{\ell_2}.\end{aligned} \tag{3.4}$$

If $\mathtt{q}E_\varepsilon$ takes the form in (3.4), we further construct a permutation matrix $E_{\mathtt{q}}$ such that $[\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]E_{\mathtt{q}} = [\mathtt{f}_1, \mathtt{f}_2, \mathtt{g}_1, \mathtt{g}_2]$ and we define $\mathsf{U}_{\mathtt{q},\varepsilon} := E_\varepsilon E_{\mathtt{q}} \,\text{diag}(I_{s-s_{\mathtt{g}}}, z^{-1}I_{s_{\mathtt{g}}})$, where $s_{\mathtt{g}}$ is the size of the row vector $[\mathtt{g}_1, \mathtt{g}_2]$. Then $\mathtt{q}\mathsf{U}_{\mathtt{q},\varepsilon}$ takes the form in (3.3). For $\mathtt{q}E_\varepsilon$ of form (3.3), we simply let $\mathsf{U}_{\mathtt{q},\varepsilon} := E_\varepsilon$. In this way, $\mathtt{q}_0 := \mathtt{q}\mathsf{U}_{\mathtt{q},\varepsilon}$ always takes the form in (3.3) with $\mathtt{f}_1 \neq \mathbf{0}$.

Note that $\mathsf{U}_{\mathtt{q},\varepsilon}\mathsf{U}_{\mathtt{q},\varepsilon}^* = I_s$ and $\|\mathtt{f}_1\| = \|\mathtt{f}_2\|$ if $\mathtt{q}_0\mathtt{q}_0^* = 1$, where $\|\mathtt{f}\| := \sqrt{\mathtt{f}\mathtt{f}^*}$. Now we construct an $s \times s$ paraunitary matrix $\mathsf{B}_{\mathtt{q}_0}$ to reduce the coefficient support of $\mathtt{q}_0$ as in (3.3) from $[\ell_1, \ell_2]$ to $[\ell_1 + 1, \ell_2 - 1]$ as follows:

$$\mathsf{B}_{\mathtt{q}_0}^* := \frac{1}{c}\left[\begin{array}{c|c|c|c} \mathtt{f}_1(z + \frac{c_0}{c_{\mathtt{f}_1}} + \frac{1}{z}) & \mathtt{f}_2(z - \frac{1}{z}) & \mathtt{g}_1(1 + \frac{1}{z}) & \mathtt{g}_2(1 - \frac{1}{z}) \\ cF_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline -\mathtt{f}_1(z - \frac{1}{z}) & -\mathtt{f}_2(z - \frac{c_0}{c_{\mathtt{f}_1}} + \frac{1}{z}) & -\mathtt{g}_1(1 - \frac{1}{z}) & -\mathtt{g}_2(1 + \frac{1}{z}) \\ \mathbf{0} & cF_2 & \mathbf{0} & \mathbf{0} \\ \hline \frac{c_{\mathtt{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1 + z) & -\frac{c_{\mathtt{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1 - z) & c_{\mathtt{g}_1'}\mathtt{g}_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & cG_1 & \mathbf{0} \\ \hline \frac{c_{\mathtt{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1 - z) & -\frac{c_{\mathtt{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1 + z) & \mathbf{0} & c_{\mathtt{g}_2'}\mathtt{g}_2' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & cG_2 \end{array}\right], \tag{3.5}$$

where $c_{\mathtt{f}_1} := \|\mathtt{f}_1\|$, $c_{\mathtt{g}_1} := \|\mathtt{g}_1\|$, $c_{\mathtt{g}_2} := \|\mathtt{g}_2\|$, $c_0 := \text{coeff}(\mathtt{q}_0, \ell_1 + 1)\text{coeff}(\mathtt{q}_0^*, -\ell_2)/c_{\mathtt{f}_1}$,

$$c_{\mathtt{g}_1'} := \begin{cases} \frac{-2c_{\mathtt{f}_1} - \overline{c_0}}{c_{\mathtt{g}_1}} & \text{if } \mathtt{g}_1 \neq \mathbf{0}, \\ c & \text{otherwise}, \end{cases} \qquad c_{\mathtt{g}_2'} := \begin{cases} \frac{2c_{\mathtt{f}_1} - \overline{c_0}}{c_{\mathtt{g}_2}} & \text{if } \mathtt{g}_2 \neq \mathbf{0}, \\ c & \text{otherwise}, \end{cases}$$

$c := (4c_{\mathtt{f}_1}^2 + 2c_{\mathtt{g}_1}^2 + 2c_{\mathtt{g}_2}^2 + |c_0|^2)^{1/2}$, and $[\frac{\mathtt{f}_j^*}{\|\mathtt{f}_j\|}, F_j^*] = U_{\mathtt{f}_j}$, $[\mathtt{g}_j'^*, G_j^*] = U_{\mathtt{g}_j}$ for $j = 1, 2$ are unitary constant extension matrices in $\mathbb{F}$ for vectors $\mathtt{f}_j, \mathtt{g}_j$ in $\mathbb{F}$, respectively (see section 4 for a concrete construction of such unitary matrices $U_{\mathtt{f}_j}$ and $U_{\mathtt{g}_j}$). Here, the role of a unitary constant matrix $U_{\mathtt{f}}$ in $\mathbb{F}$ is to reduce the number of nonzero entries in $\mathtt{f}$ such that $\mathtt{f}U_{\mathtt{f}} = [\|\mathtt{f}\|, 0, \ldots, 0]$. The operations for the emptyset $\emptyset$ are defined by $\|\emptyset\| = \emptyset$, $\emptyset + A = A$, and $\emptyset \cdot A = \emptyset$ for any object A.

Define $\mathsf{B}_{\mathtt{q}} := \mathsf{U}_{\mathtt{q},\varepsilon}\mathsf{B}_{\mathtt{q}_0}\mathsf{U}_{\mathtt{q},\varepsilon}^*$. Then $\mathsf{B}_{\mathtt{q}}$ is paraunitary. Due to the particular form of $\mathsf{B}_{\mathtt{q}_0}$ as in (3.5), direct computations yield the following very important properties of the paraunitary matrix $\mathsf{B}_{\mathtt{q}}$:

(P1) $\mathcal{S}\mathsf{B}_{\mathsf{q}} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$, coeffsupp$(\mathsf{B}_{\mathsf{q}}) = [-1,1]$, and coeffsupp$(\mathsf{q}\mathsf{B}_{\mathsf{q}}) = [\ell_1+1, \ell_2-1]$. That is, B_{q} has compatible symmetry with coefficient support on $[-1,1]$ and B_{q} reduces the length of the coefficient support of q exactly by 2. Moreover, $\mathcal{S}(\mathsf{q}\mathsf{B}_{\mathsf{q}}) = \mathcal{S}\mathsf{q}$.

(P2) If $(\mathsf{p}, \mathsf{q}^*)$ has mutually compatible symmetry and $\mathsf{p}\mathsf{q}^* = 0$, then $\mathcal{S}(\mathsf{p}\mathsf{B}_{\mathsf{q}}) = \mathcal{S}(\mathsf{p})$ and coeffsupp$(\mathsf{p}\mathsf{B}_{\mathsf{q}}) \subseteq$ coeffsupp(p). That is, B_{q} keeps the symmetry pattern of p and does not increase the length of the coefficient support of p.

Next, let us explain the construction of $\mathsf{B}_{(-k,k)}$. For coeffsupp$(\mathsf{Q}) = [-k,k]$ with $k \geqslant 1$, Q is of the following form:

$$\tag{3.6}
\begin{aligned}
\mathsf{Q} = &\left[\begin{array}{cc|cc} F_{11} & -F_{21} & G_{31} & -G_{41} \\ -F_{12} & F_{22} & -G_{32} & G_{42} \\ \hline \mathbf{0} & \mathbf{0} & F_{31} & -F_{41} \\ \mathbf{0} & \mathbf{0} & -F_{32} & F_{42} \end{array}\right] z^{-k} + \left[\begin{array}{cc|cc} F_{51} & -F_{61} & G_{71} & -G_{81} \\ -F_{52} & F_{61} & -G_{72} & G_{82} \\ \hline G_{11} & -G_{21} & F_{71} & -F_{81} \\ -G_{12} & G_{22} & -F_{72} & F_{82} \end{array}\right] z^{-k+1} \\
&+ \sum_{n=2-k}^{k-2} \operatorname{coeff}(\mathsf{Q}, n) + \left[\begin{array}{cc|cc} F_{51} & F_{61} & G_{31} & G_{41} \\ F_{52} & F_{61} & G_{32} & G_{42} \\ \hline G_{51} & G_{61} & F_{71} & F_{81} \\ G_{52} & G_{62} & F_{72} & F_{82} \end{array}\right] z^{k-1} + \left[\begin{array}{cc|cc} F_{11} & F_{21} & \mathbf{0} & \mathbf{0} \\ F_{12} & F_{22} & \mathbf{0} & \mathbf{0} \\ \hline G_{11} & G_{21} & F_{31} & F_{41} \\ G_{12} & G_{22} & F_{32} & F_{42} \end{array}\right] z^{k},
\end{aligned}$$

with all F_{jk}'s and G_{jk}'s being constant matrices in $\mathbb{F}$ and $F_{11}, F_{22}, F_{31}, F_{42}$ being of size $r_1 \times s_1$, $r_2 \times s_2$, $r_3 \times s_3$, $r_4 \times s_4$, respectively. Due to properties (P1) and (P2) of B_{q}, the `for` loop in Algorithm 2 reduces Q in (3.6) to $\mathsf{Q}_0 := \mathsf{Q}\mathsf{B}_1 \cdots \mathsf{B}_r$ as follows:

$$\tag{3.7}
\left[\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \widetilde{G}_{31} & -\widetilde{G}_{41} \\ \mathbf{0} & \mathbf{0} & -\widetilde{G}_{32} & \widetilde{G}_{42} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{-k} + \cdots + \left[\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \widetilde{G}_{11} & \widetilde{G}_{21} & \mathbf{0} & \mathbf{0} \\ \widetilde{G}_{12} & \widetilde{G}_{22} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{k}.$$

If either coeff$(\mathsf{Q}_0, -k) = \mathbf{0}$ or coeff$(\mathsf{Q}_0, k) = \mathbf{0}$, then the inner `while` loop does nothing and $\mathsf{B}_{(-k,k)} = I_s$. If both coeff$(\mathsf{Q}_0, -k) \neq \mathbf{0}$ and coeff$(\mathsf{Q}_0, k) \neq \mathbf{0}$, then $\mathsf{B}_{(-k,k)}$ is constructed recursively from pairs $(\mathsf{q}_1, \mathsf{q}_2)$ with $\mathsf{q}_1, \mathsf{q}_2$ being two rows of Q_0 satisfying coeff$(\mathsf{q}_1, -k) \neq \mathbf{0}$ and coeff$(\mathsf{q}_2, k) \neq \mathbf{0}$. The construction of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ with respect to such a pair $(\mathsf{q}_1, \mathsf{q}_2)$ in the inner `while` loop is as follows.

Similar to the discussion before (3.3), there is a permutation matrix $E_{(\mathsf{q}_1,\mathsf{q}_2)}$ such that $\widetilde{\mathsf{q}}_1 := \mathsf{q}_1 E_{(\mathsf{q}_1,\mathsf{q}_2)}$ and $\widetilde{\mathsf{q}}_2 := \mathsf{q}_2 E_{(\mathsf{q}_1,\mathsf{q}_2)}$ take the following form:

$$\tag{3.8}
\begin{aligned}
\begin{bmatrix} \widetilde{\mathsf{q}}_1 \\ \widetilde{\mathsf{q}}_2 \end{bmatrix} = &\left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \widetilde{g}_3 & -\widetilde{g}_4 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{-k} + \left[\begin{array}{cccc} \widetilde{f}_5 & -\widetilde{f}_6 & \widetilde{g}_7 & -\widetilde{g}_8 \\ \hline \varepsilon\widetilde{g}_1 & -\varepsilon\widetilde{g}_2 & \varepsilon\widetilde{f}_7 & -\varepsilon\widetilde{f}_8 \end{array}\right] z^{-k+1} \\
&+ \sum_{n=2-k}^{k-2} \operatorname{coeff}\left(\begin{bmatrix} \widetilde{\mathsf{q}}_1 \\ \widetilde{\mathsf{q}}_2 \end{bmatrix}, n\right) + \left[\begin{array}{cccc} \widetilde{f}_5 & \widetilde{f}_6 & \widetilde{g}_3 & \widetilde{g}_4 \\ \hline \widetilde{g}_5 & \widetilde{g}_6 & \widetilde{f}_7 & \widetilde{f}_8 \end{array}\right] z^{k-1} + \left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \widetilde{g}_1 & \widetilde{g}_2 & \mathbf{0} & \mathbf{0} \end{array}\right] z^{k},
\end{aligned}$$

where $\varepsilon \in \{-1,1\}$ and all $\widetilde{\mathsf{g}}_j$'s are nonzero row vectors. Note that $\|\widetilde{\mathsf{g}}_1\| = \|\widetilde{\mathsf{g}}_2\| =: c_{\widetilde{\mathsf{g}}_1}$ and $\|\widetilde{\mathsf{g}}_3\| = \|\widetilde{\mathsf{g}}_4\| =: c_{\widetilde{\mathsf{g}}_3}$. Construct an $s \times s$ paraunitary matrix $\mathsf{B}_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)}$ as in (3.9) with $c_0 :=$ coeff$(\widetilde{\mathsf{q}}_1, -k+1)$coeff$(\widetilde{\mathsf{q}}_2^*, -k)/c_{\widetilde{\mathsf{g}}_1}$, $c := (|c_0|^2 + 4c_{\widetilde{\mathsf{g}}_3}^2)^{1/2}$, and $[\frac{\widetilde{\mathsf{g}}_j^*}{\|\widetilde{\mathsf{g}}_j\|}, \widetilde{G}_j^*] = U_{\widetilde{\mathsf{g}}_j}$ being unitary constant extension matrices in $\mathbb{F}$ for vectors $\widetilde{\mathsf{g}}_j$ in $\mathbb{F}$, $j = 1, \ldots, 4$, respectively. Let $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)} := E_{(\mathsf{q}_1,\mathsf{q}_2)} \mathsf{B}_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)} E_{(\mathsf{q}_1,\mathsf{q}_2)}^T$. Similar to properties (P1) and (P2) of B_{q}, we have the following very important properties of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$:

(P3) $\mathcal{S}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$, the coefficient support of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ is on $[-1,1]$, coeffsupp$(\mathsf{q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-k+1, k-1]$, and coeffsupp$(\mathsf{q}_2\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-k+1, k-1]$. That is, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ has compatible symmetry with coefficient support on $[-1,1]$ and $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ reduces the length of both the coefficient supports of q_1 and q_2 by 2. Moreover, $\mathcal{S}(\mathsf{q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{q}_1$ and $\mathcal{S}(\mathsf{q}_2\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{q}_2$.

(P4) if both $(\mathsf{p},\mathsf{q}_1^*)$ and $(\mathsf{p},\mathsf{q}_2^*)$ have mutually compatible symmetry and $\mathsf{p}\mathsf{q}_1^* = \mathsf{p}\mathsf{q}_2^* = 0$, then $\mathcal{S}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{p}$ and $\operatorname{coeffsupp}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq \operatorname{coeffsupp}(\mathsf{p})$. That is, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ keeps the symmetry pattern of p and does not increase the length of the coefficient support of p.

$$
(3.9)\qquad \mathsf{B}^*_{(\tilde{\mathsf{q}}_1,\tilde{\mathsf{q}}_2)} := \frac{1}{c}\left[\begin{array}{c|c|c|c}
\frac{c_0}{c_{\tilde{\mathsf{g}}_1}}\tilde{\mathsf{g}}_1 & \mathbf{0} & \tilde{\mathsf{g}}_3(1+\frac{1}{z}) & \tilde{\mathsf{g}}_4(1-\frac{1}{z}) \\
c\widetilde{G}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline
\mathbf{0} & \frac{c_0}{c_{\tilde{\mathsf{g}}_1}}\tilde{\mathsf{g}}_2 & -\tilde{\mathsf{g}}_3(1-\frac{1}{z}) & -\tilde{\mathsf{g}}_4(1+\frac{1}{z}) \\
\mathbf{0} & c\widetilde{G}_2 & \mathbf{0} & \mathbf{0} \\ \hline
\frac{c_{\tilde{\mathsf{g}}_3}}{c_{\tilde{\mathsf{g}}_1}}\tilde{\mathsf{g}}_1(1+z) & -\frac{c_{\tilde{\mathsf{g}}_3}}{c_{\tilde{\mathsf{g}}_1}}\tilde{\mathsf{g}}_2(1-z) & -\frac{\overline{c_0}}{c_{\tilde{\mathsf{g}}_3}}\tilde{\mathsf{g}}_3 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & c\widetilde{G}_3 & \mathbf{0} \\ \hline
\frac{c_{\tilde{\mathsf{g}}_3}}{c_{\tilde{\mathsf{g}}_1}}\tilde{\mathsf{g}}_1(1-z) & -\frac{c_{\tilde{\mathsf{g}}_3}}{c_{\tilde{\mathsf{g}}_1}}\tilde{\mathsf{g}}_2(1+z) & \mathbf{0} & -\frac{\overline{c_0}}{c_{\tilde{\mathsf{g}}_3}}\tilde{\mathsf{g}}_4 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & c\widetilde{G}_4
\end{array}\right].
$$

Now, due to the properties (P3) and (P4) of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_{(-k,k)}$ constructed in the inner `while` loop reduces Q_0 of the form in (3.7), with both $\operatorname{coeff}(\mathsf{Q}_0,-k) \neq \mathbf{0}$ and $\operatorname{coeff}(\mathsf{Q}_0,k) \neq \mathbf{0}$, to $\mathsf{Q}_1 := \mathsf{Q}_0\mathsf{B}_{(-k,k)}$ of the form in (3.7) with either $\operatorname{coeff}(\mathsf{Q}_1,-k) = \operatorname{coeff}(\mathsf{Q}_1,k) = \mathbf{0}$ (for this case, simply let $\mathsf{B}_{\mathsf{Q}_1} := I_s$) or one of $\operatorname{coeff}(\mathsf{Q}_1,-k)$ and $\operatorname{coeff}(\mathsf{Q}_1,k)$ is nonzero. For the latter case, $\mathsf{B}_{\mathsf{Q}_1} := \operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ with matrices U_1,W_1 constructed with respect to $\operatorname{coeff}(\mathsf{Q}_1,k) \neq \mathbf{0}$ or $\mathsf{B}_{\mathsf{Q}_1} := \operatorname{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ with U_3,W_3 constructed with respect to $\operatorname{coeff}(\mathsf{Q}_1,-k) \neq \mathbf{0}$, where E is a permutation matrix. $\mathsf{B}_{\mathsf{Q}_1}$ is constructed so that $\operatorname{coeffsupp}(\mathsf{Q}_1\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-k+1,k-1]$. Let Q_1 take form in (3.7). The matrices U_1,W_1 or U_3,W_3, and E are constructed as follows.

Let $U_1 := \operatorname{diag}(U_{\widetilde{G}_1}, U_{\widetilde{G}_2})$ and $U_3 := \operatorname{diag}(U_{\widetilde{G}_3}, U_{\widetilde{G}_4})$ with

$$
(3.10)\qquad \widetilde{G}_1 := \begin{bmatrix} \widetilde{G}_{11} \\ \widetilde{G}_{12} \end{bmatrix},\quad \widetilde{G}_2 := \begin{bmatrix} \widetilde{G}_{21} \\ \widetilde{G}_{22} \end{bmatrix},\quad \widetilde{G}_3 := \begin{bmatrix} \widetilde{G}_{31} \\ \widetilde{G}_{32} \end{bmatrix},\quad \widetilde{G}_4 := \begin{bmatrix} \widetilde{G}_{41} \\ \widetilde{G}_{42} \end{bmatrix}.
$$

Here, for a nonzero matrix G with rank m, U_G is a unitary matrix such that $GU_G = [R,\mathbf{0}]$ for some matrix R of rank m. For $G = \mathbf{0}$, $U_G := I$ and for $G = \emptyset$, $U_G := \emptyset$. When $G_1G_1^* = G_2G_2^*$, U_{G_1} and U_{G_2} can be constructed such that $G_1U_{G_1} = [R,\mathbf{0}]$ and $G_2U_{G_2} = [R,\mathbf{0}]$ (see section 4 for more detail).

Let m_1, m_3 be the ranks of $\widetilde{G}_1$, $\widetilde{G}_3$, respectively ($m_1 = 0$ when $\operatorname{coeff}(\mathsf{Q}_1,k) = \mathbf{0}$ and $m_3 = 0$ when $\operatorname{coeff}(\mathsf{Q}_1,-k) = \mathbf{0}$). Note that $\widetilde{G}_1\widetilde{G}_1^* = \widetilde{G}_2\widetilde{G}_2^*$ or $\widetilde{G}_3\widetilde{G}_3^* = \widetilde{G}_4\widetilde{G}_4^*$ due to $\mathsf{Q}_1\mathsf{Q}_1^* = I_r$. The matrices $\mathsf{W}_1,\mathsf{W}_3$ are then constructed by

$$
(3.11)\qquad \mathsf{W}_1 := \left[\begin{array}{c|c|c|c}
\mathsf{U}_1 & & \mathsf{U}_2 & \\ \hline
 & I_{s_1-m_1} & & \\ \hline
\mathsf{U}_2 & & \mathsf{U}_1 & \\ \hline
 & & & I_{s_2-m_1}
\end{array}\right],\quad \mathsf{W}_3 := \left[\begin{array}{c|c|c|c}
\mathsf{U}_3 & & \mathsf{U}_4 & \\ \hline
 & I_{s_3-m_3} & & \\ \hline
\mathsf{U}_4 & & \mathsf{U}_3 & \\ \hline
 & & & I_{s_4-m_3}
\end{array}\right],
$$

where $\mathsf{U}_1(z) = -\mathsf{U}_2(-z) := \frac{1+z^{-1}}{2}I_{m_1}$ and $\mathsf{U}_3(z) = \mathsf{U}_4(-z) := \frac{1+z}{2}I_{m_3}$.

Let $\mathsf{W}_{\mathsf{Q}_1} := \operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})$ for the case that $\operatorname{coeff}(\mathsf{Q}_1,k) \neq \mathbf{0}$ or $\mathsf{W}_{\mathsf{Q}_1} := \operatorname{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)$ for the case that $\operatorname{coeff}(\mathsf{Q}_1,-k) \neq \mathbf{0}$. Then $\mathsf{W}_{\mathsf{Q}_1}$ is paraunitary. By the symmetry pattern and orthogonality of Q_1, $\mathsf{W}_{\mathsf{Q}_1}$ reduces the coefficient support of Q_1 to $[-k+1,k-1]$, i.e., $\operatorname{coeffsupp}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [-k+1,k-1]$. Moreover, $\mathsf{W}_{\mathsf{Q}_1}$ changes the symmetry pattern of Q_1 such that $\mathcal{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T\mathcal{S}\theta_1$ with

$$\mathcal{S}\theta_1 = [z^{-1}\mathbf{1}_{m_1}, \mathbf{1}_{s_1-m_1}, -z^{-1}\mathbf{1}_{m_1}, -\mathbf{1}_{s_2-m_1}, \mathbf{1}_{m_3}, z^{-1}\mathbf{1}_{s_3-m_3}, -\mathbf{1}_{m_3}, -z^{-1}\mathbf{1}_{s_4-m_3}].$$

E is then the permutation matrix such that

$$\mathcal{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1})E = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4},]^T\mathcal{S}\theta,$$

with $\mathcal{S}\theta = [\mathbf{1}_{s_1-m_1+m_3}, -\mathbf{1}_{s_2-m_1+m_3}, z^{-1}\mathbf{1}_{s_3-m_3+m_1}, -z^{-1}\mathbf{1}_{s_4-m_3+m_1}] = (\mathcal{S}\theta_1)E$.

4. Proofs of Theorems 1 and 2. In this section, we shall prove Theorems 1 and 2. The key ingredient is to prove that the coefficient supports of $\mathsf{A}_1, \dots, \mathsf{A}_J$ constructed in Algorithm 2 are all contained inside $[-1, 1]$.

Let us first present a detailed construction for the unitary matrices $U_{\mathtt{f}}$ and U_G that are used in Algorithm 2. For a $1 \times n$ row vector $\mathtt{f}$ in $\mathbb{F}$ such that $\|\mathtt{f}\| \neq 0$, we define $n_{\mathtt{f}}$ to be the number of nonzero entries in $\mathtt{f}$ and $\mathbf{e}_j := [0, \dots, 0, 1, 0, \dots, 0]$ to be the jth unit coordinate row vector in $\mathbb{R}^n$. Let $E_{\mathtt{f}}$ be a permutation matrix such that $\mathtt{f}E_{\mathtt{f}} = [f_1, \dots, f_{n_{\mathtt{f}}}, 0, \dots, 0]$ with $f_j \neq 0$ for $j = 1, \dots, n_{\mathtt{f}}$. We define

$$V_{\mathtt{f}} := \begin{cases} \frac{\bar{f}_1}{|f_1|} & \text{if } n_{\mathtt{f}} = 1, \\ \frac{\bar{f}_1}{|f_1|}\left(I_n - \frac{2}{\|v_{\mathtt{f}}\|^2} v_{\mathtt{f}}^* v_{\mathtt{f}}\right) & \text{if } n_{\mathtt{f}} > 1, \end{cases} \tag{4.1}$$

where $v_{\mathtt{f}} := \mathtt{f} - \frac{f_1}{|f_1|}\|\mathtt{f}\|\mathbf{e}_1$. Observing that $\|v_{\mathtt{f}}\|^2 = 2\|\mathtt{f}\|(\|\mathtt{f}\| - |f_1|)$, we can verify that $V_{\mathtt{f}}V_{\mathtt{f}}^* = I_n$ and $\mathtt{f}E_{\mathtt{f}}V_{\mathtt{f}} = \|\mathtt{f}\|\mathbf{e}_1$. Let $U_{\mathtt{f}} := E_{\mathtt{f}}V_{\mathtt{f}}$. Then $U_{\mathtt{f}}$ is unitary and satisfies $U_{\mathtt{f}} = [\frac{\mathtt{f}^*}{\|\mathtt{f}\|}, F^*]$ for some $(n-1) \times n$ matrix F in $\mathbb{F}$ such that $\mathtt{f}U_{\mathtt{f}} = [\|\mathtt{f}\|, 0, \dots, 0]$. We also define $U_{\mathtt{f}} := I_n$ if $\mathtt{f} = \mathbf{0}$ and $U_{\mathtt{f}} := \emptyset$ if $\mathtt{f} = \emptyset$. Here, $U_{\mathtt{f}}$ plays the role of reducing the number of nonzero entries in $\mathtt{f}$. More generally, for an $r \times n$ nonzero matrix G of rank m in $\mathbb{F}$, employing the above procedure to each row of G, we can obtain an $n \times n$ unitary matrix U_G such that $GU_G = [R, \mathbf{0}]$ for some $r \times m$ lower triangular matrix R of rank m. If $G_1G_1^* = G_2G_2^*$, then the above procedure produces two matrices U_{G_1}, U_{G_2} such that $G_1U_{G_1} = [R, \mathbf{0}]$ and $G_2U_{G_2} = [R, \mathbf{0}]$ for some lower triangular matrix R of full rank. It is important to notice that the constructions of $U_{\mathtt{f}}$ and U_G involve only the nonzero entries of $\mathtt{f}$ and nonzero columns of G, respectively. In other words, up to rearrangements, we have

$$\begin{aligned} &[U_{\mathtt{f}}]_{j,:} = ([U_{\mathtt{f}}]_{:,j})^T = \mathbf{e}_j \quad \text{if } [\mathtt{f}]_j = 0, \\ &[U_G]_{j,:} = ([U_G]_{:,j})^T = \mathbf{e}_j \quad \text{if } [G]_{:,j} = \mathbf{0}. \end{aligned} \tag{4.2}$$

Next, we establish the following lemma, which is needed later to show that the coefficient support of $(\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1, 1]$.

Lemma 2. *Let B be an $s \times s$ paraunitary matrix such that* coeffsupp$(\mathsf{B}) \subseteq [-1, 1]$ *and $\mathcal{S}\mathsf{B} = (\mathcal{S}\theta)^*\mathcal{S}\theta$ with $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some nonnegative integers $s_1, \dots, s_4$ such that $s_1 + \cdots + s_4 = s$. Then the following statements hold.*

(i) *Let p be a $1 \times s$ row vector of Laurent polynomials with symmetry such that $\mathsf{p}\mathsf{p}^* = 1$,* coeffsupp$(\mathsf{p}) = [k_1, k_2]$ *with $k_2 - k_1 \geqslant 2$, and $\mathcal{S}\mathsf{p} = \varepsilon z^c \mathcal{S}\theta$ for some $\varepsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$. Let $\mathsf{q} := \mathsf{p}\mathsf{B}$. If* coeffsupp$(\mathsf{q})$ = coeffsupp(p), *then* coeffsupp$(\mathsf{B}\mathsf{B}_{\mathsf{q}}) \subseteq [-1, 1]$, *where B_{q} is constructed with respect to q as in section* 3.

(ii) *Let $\mathsf{p}_1, \mathsf{p}_2$ be two $1 \times s$ row vectors of Laurent polynomials with symmetry such that $\mathsf{p}_{j_1}\mathsf{p}_{j_2}^* = \delta(j_1 - j_2)$ for $j_1, j_2 = 1, 2$, $\mathcal{S}\mathsf{p}_1 = \varepsilon_1\mathcal{S}\theta$ and $\mathcal{S}\mathsf{p}_2 = \varepsilon_2 z\mathcal{S}\theta$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, and* coeffsupp$(\mathsf{p}_1)$ = coeffsupp$(\mathsf{p}_2) \subseteq [-k, k]$ *with*

$k \geqslant 1$. *Let* $\mathsf{q}_1 := \mathsf{p}_1\mathsf{B}$ *and* $\mathsf{q}_2 := \mathsf{p}_2\mathsf{B}$. *If* $\text{coeffsupp}(\mathsf{q}_1) = [-k, k-1]$ *and* $\text{coeffsupp}(\mathsf{q}_2) = [-k+1, k]$, *then* $\text{coeffsupp}(\mathsf{B}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-1, 1]$, *where* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ *is constructed with respect to the pair* $(\mathsf{q}_1, \mathsf{q}_2)$ *as in section* 3.

Proof. Due to $\mathcal{S}\mathsf{p} = \varepsilon z^c \mathcal{S}\theta$, as we discussed in section 3, there is a $\mathsf{U}_{\mathsf{p},\varepsilon}$ such that $\mathsf{p}\mathsf{U}_{\mathsf{p},\varepsilon}$ takes the form in (3.3). Since $\mathsf{U}_{\mathsf{p},\varepsilon}$ is a product of a permutation matrix and a diagonal matrix of monomials, we shall consider the case that $\mathsf{U}_{\mathsf{p},\varepsilon} = I_s$, while the proofs for other cases of $\mathsf{U}_{\mathsf{p},\varepsilon}$ can be obtained accordingly. Then p takes the standard form in (3.3) with $\mathtt{f}_1 \neq \mathbf{0}$. In this case, $s_1 > 0$ and $s_2 > 0$ due to $\|\mathtt{f}_1\| = \|\mathtt{f}_2\| \neq 0$. By our assumptions, $\mathsf{q} := \mathsf{p}\mathsf{B}$ must take the following form:

$$\mathsf{q} := \mathsf{p}\mathsf{B} = [\widetilde{\mathtt{f}}_1, -\widetilde{\mathtt{f}}_2, \widetilde{\mathtt{g}}_1, -\widetilde{\mathtt{g}}_2]z^{k_1} + [\widetilde{\mathtt{f}}_3, -\widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_3, -\widetilde{\mathtt{g}}_4]z^{k_1+1} + \sum_{n=k_1+2}^{k_2-2} \text{coeff}(\mathsf{p}\mathsf{B}, n) z^n$$
$$+ [\widetilde{\mathtt{f}}_3, \widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2]z^{k_2-1} + [\widetilde{\mathtt{f}}_1, \widetilde{\mathtt{f}}_2, \mathbf{0}, \mathbf{0}]z^{k_2},$$

with $\widetilde{\mathtt{f}}_1 \neq \mathbf{0}$. Then B_q is given by (3.5) with $\mathtt{f}_1$, $\mathtt{f}_2$, $\mathtt{g}_1$, $\mathtt{g}_2$, F_1, F_2, G_1, G_2 being replaced by $\widetilde{\mathtt{f}}_1$, $\widetilde{\mathtt{f}}_2$, $\widetilde{\mathtt{g}}_1$, $\widetilde{\mathtt{g}}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{G}_1$, $\widetilde{G}_2$, respectively, and all constants $c_{\widetilde{\mathtt{f}}_1}, c_{\widetilde{\mathtt{g}}_1}, c_{\widetilde{\mathtt{g}}_2}, c_0, c, c_{\widetilde{\mathtt{g}}'_1}, c_{\widetilde{\mathtt{g}}'_2}$ being defined accordingly.

Also, due to the symmetry pattern and $\text{coeffsupp}(\mathsf{B}) \subseteq [-1, 1]$, B is of the form

$$\mathsf{B} = \left[\begin{array}{c|c|c|c} A_1(z+\frac{1}{z})+D_1 & A_3(z-\frac{1}{z}) & B_3(1+\frac{1}{z}) & B_4(1-\frac{1}{z}) \\ \hline A_2(z-\frac{1}{z}) & A_4(z+\frac{1}{z})+D_2 & C_3(1-\frac{1}{z}) & C_4(1+\frac{1}{z}) \\ \hline B_1(1+z) & C_1(1-z) & A_5(z+\frac{1}{z})+D_3 & A_7(z-\frac{1}{z}) \\ \hline B_2(1-z) & C_2(1+z) & A_6(z-\frac{1}{z}) & A_8(z+\frac{1}{z})+D_4 \end{array}\right], \tag{4.3}$$

where A_j's, B_j's, C_j's, and D_j's are all constant matrices in $\mathbb{F}$ and D_j is of size $s_j \times s_j$ for $j = 1, \ldots, 4$. Let $\mathcal{I} := \{1, s_1+1, (1-\delta(s_3))(s_1+s_2+1), (1-\delta(s_4))(s_1+s_2+s_3+1)\}$ be an index set. It is easy to verify that $\text{coeffsupp}([\mathsf{B}\mathsf{B}_\mathsf{q}]_{:,j}) \subseteq [-1, 1]$ for all $j \notin \mathcal{I}$. Hence, by $\text{coeffsupp}(\mathsf{B}\mathsf{B}_\mathsf{q}) \subseteq [-2, 2]$, we need only compute $\text{coeff}([\mathsf{B}\mathsf{B}_\mathsf{q}]_{:,j}, 2)$ and $\text{coeff}([\mathsf{B}\mathsf{B}_\mathsf{q}]_{:,j}, -2)$ for those $j \in \mathcal{I}$. Let us show that $\text{coeff}([\mathsf{B}\mathsf{B}_\mathsf{q}]_{:,j}, 2) = \mathbf{0}$ for $j = 1$, i.e., the coefficient vector of z^2 for the first column of $\mathsf{B}\mathsf{B}_\mathsf{q}$ is $\mathbf{0}$. By $\text{coeff}(\mathsf{p}\mathsf{B}, k_1) = \text{coeff}(\mathsf{p}, k_1+1)\text{coeff}(\mathsf{B}, -1) + \text{coeff}(\mathsf{p}, k_1)\text{coeff}(\mathsf{B}, 0)$, we have

$$\begin{aligned} \widetilde{\mathtt{f}}_1 &= \mathtt{f}_3 A_1 + \mathtt{f}_4 A_2 + \mathtt{f}_1 D_1 + \mathtt{g}_1 B_1 - \mathtt{g}_2 B_2, \\ \widetilde{\mathtt{f}}_2 &= \mathtt{f}_3 A_3 + \mathtt{f}_4 A_4 + \mathtt{f}_2 D_2 - \mathtt{g}_1 C_1 + \mathtt{g}_2 C_2, \\ \widetilde{\mathtt{g}}_1 &= \mathtt{f}_3 B_3 + \mathtt{f}_4 C_3 + \mathtt{g}_3 A_5 + \mathtt{g}_4 A_6 + \mathtt{f}_1 B_3 - \mathtt{f}_2 C_3 + \mathtt{g}_1 D_3, \\ \widetilde{\mathtt{g}}_2 &= \mathtt{f}_3 B_4 + \mathtt{f}_4 C_4 + \mathtt{g}_3 A_7 + \mathtt{g}_4 A_8 - \mathtt{f}_1 B_4 + \mathtt{f}_2 C_4 + \mathtt{g}_2 D_4. \end{aligned} \tag{4.4}$$

Similarly, by $\text{coeff}(\mathsf{B}\mathsf{B}_\mathsf{q}, 2) = \text{coeff}(\mathsf{B}, 1)\text{coeff}(\mathsf{B}_\mathsf{q}, 1)$, we have

$$\text{coeff}([\mathsf{B}\mathsf{B}_\mathsf{q}]_{:,1}, 2) = \frac{1}{c}\begin{bmatrix} A_1 & A_3 & \mathbf{0} & \mathbf{0} \\ A_2 & A_4 & \mathbf{0} & \mathbf{0} \\ B_1 & -C_1 & A_5 & A_7 \\ -B_2 & C_2 & A_6 & A_8 \end{bmatrix}\begin{bmatrix} \widetilde{\mathtt{f}}_1^* \\ -\widetilde{\mathtt{f}}_2^* \\ \widetilde{\mathtt{g}}_1^* \\ -\widetilde{\mathtt{g}}_2^* \end{bmatrix} = \frac{1}{c}\begin{bmatrix} A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^* \\ A_2\widetilde{\mathtt{f}}_1^* - A_4\widetilde{\mathtt{f}}_2^* \\ B_1\widetilde{\mathtt{f}}_1^* + C_1\widetilde{\mathtt{f}}_2^* + A_5\widetilde{\mathtt{g}}_1^* - A_7\widetilde{\mathtt{g}}_2^* \\ -B_2\widetilde{\mathtt{f}}_1^* - C_1\widetilde{\mathtt{f}}_2^* + A_6\widetilde{\mathtt{g}}_1^* - A_8\widetilde{\mathtt{g}}_2^* \end{bmatrix}.$$

Due to $\mathsf{B}\mathsf{B}^* = I_s$, we obtain

$$\begin{cases} A_1A_1^* - A_3A_3^* = \mathbf{0},\ A_1A_2^* - A_3A_4^* = \mathbf{0}, \\ A_1D_1^* + D_1A_1^* + B_3B_3^* - B_4B_4^* = \mathbf{0}, \\ D_1A_2^* - A_3D_2^* + B_3C_3^* - B_4C_4^* = \mathbf{0}, \\ A_1B_1^* + A_3C_1^* + B_3A_5^* - B_4A_7^* = \mathbf{0}, \\ -A_1B_2^* - A_3C_2^* + B_3A_6^* - B_4A_8^* = \mathbf{0}. \end{cases}$$

Applying the identities above to $A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^*$ and using (4.4), we get

$$\begin{aligned}A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^* &= A_1(\mathtt{f}_3A_1 + \mathtt{f}_4A_2 + \mathtt{f}_1D_1 + \mathtt{g}_1B_1 - \mathtt{g}_2B_2)^* \\ &\quad - A_3(\mathtt{f}_3A_3 + \mathtt{f}_4A_4 + \mathtt{f}_2D_2 - \mathtt{g}_1C_1 + \mathtt{g}_2C_2)^* \\ &= (A_1A_1^* - A_3A_3^*)\mathtt{f}_3^* + (A_1A_2^* - A_3A_4^*)\mathtt{f}_4^* + (A_1D_1^*)\mathtt{f}_1^* \\ &\quad + (-A_3D_2^*)\mathtt{f}_2^* + (A_1B_1^* + A_3C_1^*)\mathtt{g}_1^* - (A_1B_2^* + A_3C_2^*)\mathtt{g}_2^* \\ &= -(D_1A_1^* + B_3B_3^* - B_4B_4^*)\mathtt{f}_1^* - (D_1A_2^* + B_3C_3^* - B_4C_4^*)\mathtt{f}_2^* \\ &\quad - (B_3A_5^* - B_4A_7^*)\mathtt{g}_1^* - (B_3A_6^* - B_4A_8^*)\mathtt{g}_2^* \\ &= -D_1(\mathtt{f}_1A_1 + \mathtt{f}_2A_2)^* - B_3(\mathtt{f}_1B_3 + \mathtt{f}_2C_3 + \mathtt{g}_1A_5 + \mathtt{g}_2A_6)^* \\ &\quad + B_4(\mathtt{f}_1B_4 + \mathtt{f}_2C_4 + \mathtt{g}_1A_7 + \mathtt{g}_2A_8)^* = \mathbf{0},\end{aligned}$$

where the last above identity follows by $\operatorname{coeff}(\mathsf{pB}, k_2+1) = \operatorname{coeff}(\mathsf{pB}, k_1-1) = \mathbf{0}$. Similarly, we can show that $A_2\widetilde{\mathtt{f}}_1^* - A_4\widetilde{\mathtt{f}}_2^* = \mathbf{0}$, $B_1\widetilde{\mathtt{f}}_1^* + C_1\widetilde{\mathtt{f}}_2^* + A_5\widetilde{\mathtt{g}}_1^* - A_7\widetilde{\mathtt{g}}_2^* = \mathbf{0}$, and $-B_2\widetilde{\mathtt{f}}_1^* - C_1\widetilde{\mathtt{f}}_2^* + A_6\widetilde{\mathtt{g}}_1^* - A_8\widetilde{\mathtt{g}}_2^* = \mathbf{0}$. Hence, $\operatorname{coeff}([\mathsf{BB_q}]_{:,1}, 2) = \mathbf{0}$. By similar computations as above and using the paraunitary property of B, we have $\operatorname{coeff}([\mathsf{BB_q}]_{:,j}, \pm 2) = \mathbf{0}$ for all $j \in \mathcal{I}$. Therefore, we conclude that $\operatorname{coeffsupp}(\mathsf{BB_q}) \subseteq [-1,1]$. Item (i) holds.

For item (ii), up to a permutation matrix $E_{(\mathsf{q}_1,\mathsf{q}_2)}$ as in section 3, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ takes the form in (3.9). Since B takes the form in (4.3), to show that the coefficient support of $\mathsf{BB}_{(-k,k)}$ is contained inside $[-1,1]$, we need to show that all the coefficient vectors $A_1\widetilde{\mathtt{g}}_1^* - A_3\widetilde{\mathtt{g}}_2^*$, $A_2\widetilde{\mathtt{g}}_1^* - A_4\widetilde{\mathtt{g}}_2^*$, $A_5\widetilde{\mathtt{g}}_3^* - A_7\widetilde{\mathtt{g}}_4^*$, and $A_6\widetilde{\mathtt{g}}_3^* - A_8\widetilde{\mathtt{g}}_4^*$ are zero vectors. Again, using the paraunitary property of B and expressing $\widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2, \widetilde{\mathtt{g}}_3, \widetilde{\mathtt{g}}_4$ in terms of the original vectors from $\mathsf{p}_1, \mathsf{p}_2$ similar to (4.4), we conclude that $\operatorname{coeffsupp}(\mathsf{BB}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-1,1]$. ∎

With the result of Lemma 2, the next lemma shows that the coefficient support of $\mathsf{B} := (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1,1]$. Moreover, it shows that the coefficient support of $\mathsf{A} := \mathsf{BB}_{\mathsf{Q}_1}$ is also contained inside $[-1,1]$.

LEMMA 3. *Suppose* Q *is an* $r \times s$ *matrix of Laurent polynomials such that* $\mathsf{QQ}^* = I_r$, $\mathcal{S}\mathsf{Q}$ *satisfies* (3.1), *and* $\operatorname{coeffsupp}(\mathsf{Q}) = [k_1, k_2]$ *with* $k_2 - k_1 \geqslant 1$. *Then there exists an* $s \times s$ *paraunitary matrix* A *of Laurent polynomials with symmetry such that*

- (i) $\operatorname{coeffsupp}(\mathsf{A}) \subseteq [-1,1]$ *and* $|\operatorname{coeffsupp}(\mathsf{QA})| \leqslant |\operatorname{coeffsupp}(\mathsf{Q})| - |\operatorname{coeffsupp}(\mathsf{A})|$;
- (ii) *if the* j*th column* $\mathsf{p} := [\mathsf{Q}]_{:,j}$ *of* Q *satisfies* $\operatorname{coeff}(\mathsf{p}, k_1) = \operatorname{coeff}(\mathsf{p}, k_2) = \mathbf{0}$, *then, up to a permutation matrix,* $[\mathsf{A}]_{j,:} = ([\mathsf{A}]_{:,j})^T = \mathbf{e}_j$. *That is, any entry in the* j*th row or* j*th column of* A *is zero except that the* (j,j)*-entry* $[\mathsf{A}]_{j,j} = 1$;
- (iii) $\mathcal{S}\mathsf{A} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$ *for some nonnegative integers* $s_1', \ldots, s_4'$ *such that* $s_1' + s_2' + s_3' + s_4' = s$.

Proof. Let $\mathsf{A} = (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}\mathsf{B}_{\mathsf{Q}_1}$ be constructed as in Algorithm 2, where $\mathsf{Q}_1 := \mathsf{Q}(\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$, $\mathsf{B}_{(-k,k)}$ is constructed in the inner `while` loop of Algorithm 2, and $\mathsf{B}_1, \ldots, \mathsf{B}_r$ is constructed in the `for` loop of Algorithm 2. If $k_2 \neq -k_1$, then $\mathsf{B}_1 = \cdots = \mathsf{B}_r = \mathsf{B}_{(-k,k)} = I_s$ and A is simply $\mathsf{B}_{\mathsf{Q}_1}$, where $\mathsf{Q}_1 = \mathsf{Q}$ is of the form in (3.7) with either $\operatorname{coeff}(\mathsf{Q}_1, -k) = \mathbf{0}$ or $\operatorname{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$. In this case, by the construction of $\mathsf{B}_{\mathsf{Q}_1}$ as in section 3, all items in Lemma 3 hold. We are already done. So, without loss of generality, we assume that $k_2 = -k_1 = k$.

We first show that the coefficient support of $\mathsf{B}_1 \cdots \mathsf{B}_r$ is contained inside $[-1,1]$. Let $\mathsf{p}_j := [\mathsf{Q}]_{j,:}$, $\mathsf{B}_0 := I_s$, and $\mathsf{q}_j := \mathsf{p}_j\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$ for $j = 1, \ldots, r$. Suppose we already show that $\operatorname{coeffsupp}(\mathsf{B}_0 \cdots \mathsf{B}_{j-1}) \subseteq [-1,1]$ for $j \geqslant 1$. Then, according to Algorithm 2, $\mathsf{B}_j = \mathsf{B}_{\mathsf{q}_j}$ if $\operatorname{coeffsupp}(\mathsf{p}_j) = \operatorname{coeffsupp}(\mathsf{q}_j)$, $|\operatorname{coeffsupp}(\mathsf{q}_j)| \geqslant 2$, and one of $\operatorname{coeff}(\mathsf{q}_j, k)$ and $\operatorname{coeff}(\mathsf{q}_j, -k)$ is nonzero; otherwise $\mathsf{B}_j = I_s$. Note that $\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$ is paraunitary and satisfies $\mathcal{S}(\mathsf{B}_0 \cdots \mathsf{B}_{j-1}) = (\mathcal{S}\theta)^*\mathcal{S}\theta$ with $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$. By item (i) of Lemma 2, the coefficient support of $\mathsf{B}_0 \cdots \mathsf{B}_{j-1}\mathsf{B}_j$ is also contained inside $[-1,1]$. By induction, the coefficient support of $\mathsf{B}_1 \cdots \mathsf{B}_r$ is contained inside

$[-1,1]$. Moreover, $\mathsf{B}_1\cdots\mathsf{B}_r$ takes the form in (4.3). Next, since $\mathsf{B}_{(-k,k)}$ is constructed recursively from pairs $(\mathsf{q}_1,\mathsf{q}_2)$ of $\mathsf{Q}_0 := \mathsf{Q}(\mathsf{B}_1\cdots\mathsf{B}_r)$, by applying induction again and using item (ii) of Lemma 2, we conclude that the coefficient support of $\mathsf{B} := (\mathsf{B}_1\cdots\mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1,1]$.

Due to the properties (P1), (P2) of B_q and (P3), (P4) of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_1,\dots,\mathsf{B}_r$ and $\mathsf{B}_{(-k,k)}$ reduce Q of the form in (3.6) to $\mathsf{Q}_1 = \mathsf{Q}(\mathsf{B}_1\cdots\mathsf{B}_r)\mathsf{B}_{(-k,k)} = \mathsf{Q}\mathsf{B}$ of the form in (3.7) with at least one of $\operatorname{coeff}(\mathsf{Q}_1,-k)$ and $\operatorname{coeff}(\mathsf{Q}_1,k)$ being $\mathbf{0}$. As constructed in section 3, $\mathsf{B}_{\mathsf{Q}_1} = I_s$ for the case that $\operatorname{coeff}(\mathsf{Q}_1,-k) = \operatorname{coeff}(\mathsf{Q}_1,k) = \mathbf{0}$, or $\mathsf{B}_{\mathsf{Q}_1} = \operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ for the case $\operatorname{coeff}(\mathsf{Q}_1,k) \neq \mathbf{0}$, or $\mathsf{B}_{\mathsf{Q}_1} := \operatorname{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ for the case that $\operatorname{coeff}(\mathsf{Q}_1,-k) \neq \mathbf{0}$. We next show that $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1,1]$.

Let Q take the form in (3.6) and Q_1 take the form in (3.7) with $\operatorname{coeff}(\mathsf{Q}_1,k) \neq \mathbf{0}$. Then $\mathsf{B}_{\mathsf{Q}_1} := \operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ with U_1, W_1, and E being constructed as in section 3. Note that B takes the form in (4.3). Define

$$[G_1,G_2,F_3,F_4,G_5,G_6,F_7,F_8] := \begin{bmatrix} G_{11} & G_{21} & F_{31} & F_{41} & G_{51} & G_{61} & F_{71} & F_{81} \\ G_{12} & G_{22} & F_{32} & F_{42} & G_{52} & G_{62} & F_{72} & F_{82} \end{bmatrix}.$$

By $\operatorname{coeff}(\mathsf{Q}_1,k) = \operatorname{coeff}(\mathsf{Q},k-1)\operatorname{coeff}(\mathsf{B},1) + \operatorname{coeff}(\mathsf{Q},k)\operatorname{coeff}(\mathsf{B},0)$, we have

$$\begin{aligned} \widetilde{G}_1 &= G_5A_1 + G_6A_2 + F_7B_1 - F_8B_2 + G_1D_1 + F_3B_1 + F_4B_2, \\ \widetilde{G}_2 &= G_5A_3 + G_6A_4 - F_7C_1 + F_8C_2 + G_2D_2 + F_3C_1 + F_4C_2, \\ \mathbf{0} &= F_7A_5 + F_8A_6 + G_1B_3 + G_2C_3 + F_3D_3 =: \widetilde{F}_3, \\ \mathbf{0} &= F_7A_7 + F_8A_8 + G_1B_4 + G_2C_4 + F_4D_4 =: \widetilde{F}_4, \end{aligned} \tag{4.5}$$

where $\widetilde{G}_1, \widetilde{G}_2$ are matrices defined in (3.10). Then $U_1 = \operatorname{diag}(U_{\widetilde{G}_1}, U_{\widetilde{G}_2})$ and W_1 is defined as in (3.11). By the coefficient supports of B and $\mathsf{B}_{\mathsf{Q}_1}$, we need only check that $\operatorname{coeff}(\mathsf{B}\operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}),-2) = \mathbf{0}$. Let $V_{11}, V_{12}, V_{21}, V_{22}$ be diagonal matrices of size $s_1\times s_1$, $s_1\times s_2$, $s_2\times s_1$, $s_2\times s_2$, respectively, and satisfy $\operatorname{diag}(V_{j\ell}) = [\mathbf{1}_{m_1}, \mathbf{0}]$ for $j,\ell = 1,2$, where m_1 is the rank of $\widetilde{G}_1$. Then

$$\begin{aligned} &\operatorname{coeff}(\mathsf{B}\operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}),-2) = \operatorname{coeff}(\mathsf{B},-1)\cdot\operatorname{coeff}(\operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}),-1) \\ &= \begin{bmatrix} A_1 & -A_3 & B_3 & -B_4 \\ -A_2 & A_4 & -C_3 & C_4 \\ \mathbf{0} & \mathbf{0} & A_5 & -A_7 \\ \mathbf{0} & \mathbf{0} & -A_6 & A_8 \end{bmatrix} \begin{bmatrix} U_{\widetilde{G}_1}V_{11} & U_{\widetilde{G}_1}V_{12} & \mathbf{0} & \mathbf{0} \\ U_{\widetilde{G}_2}V_{21} & U_{\widetilde{G}_2}V_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

Thus, we need to show $A_1U_{\widetilde{G}_1}V_{1j} - A_3U_{\widetilde{G}_2}V_{2j} = \mathbf{0}$ and $A_2U_{\widetilde{G}_1}V_{1j} - A_4U_{\widetilde{G}_2}V_{2j} = \mathbf{0}$, for $j = 1,2$, which is equivalent to showing that $V_{j1}U^*_{\widetilde{G}_1}A^*_1 - V_{j2}U^*_{\widetilde{G}_2}A^*_3 = \mathbf{0}$ and $V_{j1}U^*_{\widetilde{G}_1}A^*_2 - V_{j2}U^*_{\widetilde{G}_2}A^*_4 = \mathbf{0}$ for $j = 1,2$. Since $\widetilde{G}_1U_{\widetilde{G}_1} = [R,\mathbf{0}]$ and $\widetilde{G}_2U_{\widetilde{G}_2} = [R,\mathbf{0}]$, for some lower triangular matrix R of full rank m_1, it is equivalent to proving that $\widetilde{G}_1A^*_1 - \widetilde{G}_2A^*_3 = \mathbf{0}$ and $\widetilde{G}_1A^*_2 - \widetilde{G}_2A^*_4 = \mathbf{0}$. By (4.5), we have

$$\begin{aligned} \widetilde{G}_1A^*_1 - \widetilde{G}_2A^*_3 &= \widetilde{G}_1A^*_1 - \widetilde{G}_2A^*_3 + \widetilde{F}_3B^*_3 - \widetilde{F}_4B^*_4 \\ &= (G_5A_1 + G_6A_2 + F_7B_1 - F_8B_2 + G_1D_1 + F_3B_1 + F_4B_2)A^*_1 \\ &\quad - (G_5A_3 + G_6A_4 - F_7C_1 + F_8C_2 + G_2D_2 + F_3C_1 + F_4C_2)A^*_3 \\ &\quad + (F_7A_5 + F_8A_6 + G_1B_3 + G_2C_3 + F_3D_3)B^*_3 \\ &\quad - (F_7A_7 + F_8A_8 + G_1B_4 + G_2C_4 + F_4D_4)B^*_4 \end{aligned}$$

$$
\begin{aligned}
&= G_5(A_1A_1^* - A_3A_3^*) + G_6(A_2A_1^* - A_4A_3^*) \\
&\quad + F_7(B_1A_1^* + C_1A_3^* + A_5B_3^* - A_7B_4^*) - F_8(B_2A_1^*C_2A_3^* - A_6B_3^* + A_8B_4^*) \\
&\quad + G_1(D_1A_1^* + B_3B_3^* - B_4B_4^*) + G_2(-D_2A_3^* + C_3B_3^* - C_4B_4^*) \\
&\quad + F_3(B_1A_1^* - C_1A_3^* + D_3B_3^*) + F_4(B_2A_1^* - C_2A_3^* - D_4B_4^*) = \mathbf{0},
\end{aligned}
$$

where the last identity follows from $\mathsf{B}\mathsf{B}^* = I_s$ and $\mathrm{coeff}(\mathsf{Q}\mathsf{B}, k+1) = \mathbf{0}$. Similarly, $\widetilde{G}_1A_2^* - \widetilde{G}_2A_4^* = \mathbf{0}$. The computation for showing $\mathrm{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1,1]$ with $\mathsf{B}_{\mathsf{Q}_1} = \mathrm{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ is similar. Consequently, $\mathrm{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1,1]$. Therefore, item (i) holds. Item (ii) is due to the property (4.2) of $U_{\mathtt{f}}$ and U_G.

Note that $\mathcal{S}\mathsf{B} = (\mathcal{S}\theta)^*\mathcal{S}\theta$ with $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$. By the construction of $\mathsf{B}_{\mathsf{Q}_1}$, $\mathcal{S}\mathsf{B}_{\mathsf{Q}_1} = (\mathcal{S}\theta)^*[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$ for some nonnegative integers $s_1', \ldots, s_4'$ depending on the rank of $\widetilde{G}_1$ or $\widetilde{G}_3$ (see section 3). Consequently, item (iii) holds. This also completes the proof of Algorithm 2. ∎

Proof of Theorems 1 and 2. The sufficiency part of Theorem 2 is obvious. We need only show the necessary part. Suppose $\mathcal{S}\mathsf{P} = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$. Let $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^*\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ and $\mathrm{coeffsupp}(\mathsf{Q}) := [k_1, k_2]$. Then $\mathcal{S}\mathsf{Q}$ satisfies (3.1). By Lemma 3, the step of support reduction in Algorithm 2 produces a sequence of paraunitary matrices $\mathsf{A}_1, \ldots, \mathsf{A}_J$ with coefficient support contained inside $[-1,1]$ such that $\mathsf{Q}\mathsf{A}_1 \cdots \mathsf{A}_J = [I_r, \mathbf{0}]$. Due to item (i) of Lemma 3, $J \leqslant \lceil \frac{k_2-k_1}{2} \rceil$. Let $\mathsf{P}_j := \mathsf{A}_j^*$, $\mathsf{P}_0 := \mathsf{U}_{\mathcal{S}\theta_2}^*$, and $\mathsf{P}_{J+1} := \mathrm{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$. Then $\mathsf{P}_e := \mathsf{P}_{J+1}\mathsf{P}_J \cdots \mathsf{P}_1\mathsf{P}_0$ satisfies $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$. By item (iii) of Lemma 3, $(\mathsf{P}_{j+1}, \mathsf{P}_j)$ has mutually compatible symmetry for all $0 \leqslant j \leqslant J$. The claim that $|\mathrm{coeffsupp}([\mathsf{P}_e]_{k,j})| \leqslant \max_{1 \leqslant n \leqslant r} |\mathrm{coeffsupp}([\mathsf{P}]_{n,j})|$ for $1 \leqslant j, k \leqslant s$ follows from item (ii) of Lemma 3. Hence, all claims in Theorems 1 and 2 have been verified. ∎

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