"A human being should be able to change a diaper, plan an invasion, butcher a hog, conn a ship, design a building, write a sonnet, balance accounts, build a wall, set a bone, comfort the dying, take orders, give orders, cooperate, act alone, solve equations, analyze a new problem, pitch manure, program a computer, cook a tasty meal, fight efficiently, die gallantly. Specialization is for insects."

Robert A. Heinlein

University of Alberta

Interpolating Refinable Function Vectors and Matrix Extension with Symmetry

by

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To my parents

Abstract

In this thesis, we are interested in the construction of interpolating refinable function vectors with certain desirable properties, matrix extension with symmetry, and construction of wavelet generators from such refinable function vectors via our matrix extension algorithms with symmetry.

In Chapters 1 and 2, we introduce the definition of interpolating refinable function vectors in dimension one and high dimensions, characterize such interpolating refinable function vectors in terms of their masks, and derive their sum rule structure explicitly. We study biorthogonal refinable function vectors from interpolating refinable function vectors. We also study the symmetry property of an interpolating refinable function vector and characterize a symmetric interpolating refinable function vector in any dimension with respect to certain symmetry group in terms of its mask. Examples of interpolating refinable function vectors with some desirable properties, such as orthogonality, symmetry, compact support, and so on, are constructed according to our characterization results.

In Chapters 3 and 4, we turn to the study of general matrix extension problems with symmetry for the construction of orthogonal and biorthogonal multiwavelets. We give characterization theorems and develop step-by-step

algorithms for matrix extension with symmetry. To illustrate our results, we apply our algorithms to several examples of interpolating refinable function vectors with orthogonality or biorthogonality obtained in Chapter 1.

In Chapter 5, we discuss some possible future research topics on the subjects of matrix extension with symmetry in high dimensions and frequency-based nonstationary tight wavelet frames with directionality. We demonstrate that one can construct a frequency-based tight wavelet frame with symmetry and show that directional analysis can be easily achieved under the framework of tight wavelet frames. Potential applications and research directions of such tight wavelet frames with directionality are discussed.

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Lastly and most importantly, to my parents, Rui-Zhen Zhuang and Yu-Rong Zhang, for their unflagging love and support throughout my life. To them I dedicate this thesis.

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Chapter 1

Univariate Interpolating Refinable Function Vectors of Type $(\mathsf{d}, \Gamma_r, 0)$

We first discuss some motivations of this thesis and present a short summary of our main results and contributions.

Refinable function vectors play a central role in both wavelet analysis and its applications. In sampling theory, shift-invariant spaces generated by compactly supported refinable function vectors are often used in sampling theorems in signal processing. The most desirable properties of a refinable function vector in sampling theorems are interpolation and orthogonality so that inner products can be realized by sampling and thus prefiltering is on longer needed in signal processing. In computer graphics, a refinable function vector is the limit function vector of a subdivision scheme, which is widely used in smooth curve and surface generation. For a subdivision scheme, smoothness and symmetry of the limit function vectors are of paramount importance for the reason that smooth curves and surfaces with good visual effect are preferred and symmetry of a refinable function vector is indispensable in the implementation of a subdivision scheme in the design of free-form curves and surfaces. In multiresolution analysis (MRA), multiwavelets can be derived from a refinable function vector associated with a low-pass filter (or mask). In electronic engineering, such a multiwavelet system is associated with a filter bank with the perfect reconstruction property. Symmetry and short support are two desirable properties for such a filter bank since short support usually means fast algorithms and symmetry generally provides better visual results in image compression and denoising, not to mention the reduction of the computational cost using a symmetric system. Though in different applications, different properties of a refinable function vectors are desired, those properties (orthogonality, biorthogonality, symmetry, interpolation, smoothness, and compact support) usually mutually conflict to each other. It is not easy (in some cases, it is impossible) to have all these nice properties together for a refinable function vector. Also, the construction of multiwavelets from refinable function vectors can be formulated as some matrix extension problems. When integrated with symmetry, the matrix extension problems with symmetry become far more complicated and extra effort is needed to guarantee symmetry of the extension matrices so that from which multiwavelets can also possess the property of symmetry.

Motivated by wavelet applications in sampling theory, subdivision schemes, and image/signal processing, Chapters 1 and 2 study the interpolating refinable function vectors and provide complete mathematical characterizations for such interpolating refinable function vectors in terms of their masks (see Theorems 1.1 and 2.1). Based on our characterization theorems, we show that we can construct families of univariate interpolating refinable function vectors that have properties of symmetry, interpolation, orthogonality, and compact support, simultaneously (see Examples 1.4 and 1.5). Such examples provide Shannon-like sampling theorems which make them very attractive in sampling theorems and signal processing. Moreover, in high dimensions, we study the symmetry property of an interpolating refinable function vector, which involves some symmetry groups that are highly nontrivial compared to symmetry groups in one dimension. We also provide a characterization theorem of an interpolating refinable function vector to be symmetry with respect to some symmetry group in terms of its mask (see Theorem 2.3). Based on this theorem, we present several examples in dimension two that are D_4-symmetric and C^1 smoothness (see Examples 2.1 and 2.2). Our general construction and examples with nice properties of symmetry, smoothness, and interpolation may find their application for designing suitable surfaces which are preferred in geometric modeling and for new sampling theorems in signal processing.

On the other hand, motivated by the applications of the general matrix extension problem in electronic engineering, in system sciences, and especially in the construction of multiwavelets from interpolating refinable function vectors with symmetry and orthogonality (biorthogonality), Chapters 3 and 4 study the general matrix extension problems with symmetry in dimension one. For matrix extension with symmetry, deriving the extension matrices with symmetry is not an easy task due to different symmetry patterns of different columns for a given matrix of Laurent polynomials. In Chapter 3, we introduce the notion of compatibility of symmetry and successfully solve the matrix extension problem with symmetry for paraunitary matrices. We provide a complete characterization theorem for any paraunitary matrix (see Theoerm 3.1). More importantly, we provide a step-by-step algorithm (see Algorithm 3.1) to derive the desired extension matrix such that it has compatible symmetry and its length of coefficient support is controlled in an optimal way in the sense of (3.1.7). In Chapter 4, we discuss the matrix extension problem with symmetry for biorthogonal matrices. We also provide a step-by-step algorithm (see Algorithm 4.1) to derive the desired pair of biorthogonal matrices. We apply our algorithms to the construction of orthogonal and biorthogonal multiwavelets (see Sections 3.4 and 4.3) and show that our algorithms indeed guarantee the symmetry of orthogonal multiwavelets or biorthogonal multiwavelets from some orthogonal refinable function vectors with symmetry or pairs of dual refinable function vectors with symmetry. For those examples in Chapters 3 and 4, their associated filter banks having the perfect reconstruction property and symmetry are highly desirable in image compression/denoising and may provide better performance in applications compared to those classical families of wavelets.

In summary, we provide complete analysis, characterizations, and construction of families of interpolating refinable function vectors in both dimension one and high dimensions. We completely solve the matrix extension problems with symmetry and present step-by-step algorithms for deriving the extension matrices. Our results and algorithms provide engineers and applied mathematicians a powerful tool for constructing new wavelets and filter banks with the perfect reconstruction property, symmetry, and other nice properties.

1.1 Introduction

We say that $\phi := [\phi_1, \ldots, \phi_r]^T : \mathbb{R} \to \mathbb{C}^{r\times 1}$ is a d-*refinable function vector* if

$$\phi(x) = |\mathsf{d}| \sum_{k\in\mathbb{Z}} a(k)\phi(\mathsf{d}x - k), \qquad a.e.\ x \in \mathbb{R}, \tag{1.1.1}$$

where $a : \mathbb{Z} \to \mathbb{C}^{r\times r}$ is a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$, called the *(matrix) mask* with *multiplicity* r for the refinable function vector ϕ, and d denotes a *dilation factor*, which is an integer with $|\mathsf{d}| > 1$; for simplicity of presentation, we further assume that $\mathsf{d} > 1$ while the results for a negative dilation factor can be obtained similarly. When the multiplicity $r = 1$, the function vector ϕ is simply a scalar function and it is called a *scalar* d-*refinable function*.

In the frequency domain, the matrix refinement equation in (1.1.1) can be rewritten as

$$\widehat{\phi}(\mathsf{d}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R}, \tag{1.1.2}$$

where $\widehat{a}$ is the *Fourier series* of the mask a given by

$$\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}, \qquad \xi \in \mathbb{R}. \tag{1.1.3}$$

Here, i denotes the imaginary unit such that $i^2 = -1$ and the Fourier transform $\widehat{f}$ of $f \in L_1(\mathbb{R})$ is defined to be $\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$, which can be extended to square integrable functions ($L_2(\mathbb{R})$ functions) and tempered distributions. For simplicity, we also call $\widehat{a}$ *the mask for* ϕ.

For the scalar case ($r = 1$), Cavaretta et al. in [2] proved that there exists a unique compactly supported distribution solution to (1.1.1) under a very mild condition: $\widehat{a}(0) = 1$ and $\widehat{\phi}(0) = 1$. This result was extended to the vector case ($r > 1$) by Heil et al. in [40] (c.f. [70]) under the condition that 1 is a simple eigenvalue of $\widehat{a}(0)$ while all the other eigenvalues of $\widehat{a}(0)$ are less than 1 in modulus and $\overline{\widehat{\phi}(0)}^T \widehat{\phi}(0) = 1$. Consequently, for a given mask a, the existence of ϕ with refinability as in (1.1.1) can be easily checked by the condition on $\widehat{a}(0)$. In many applications, for example, image/signal processing, sampling theory, numerical algorithm, and so on, besides the refinability of a function vector ϕ, it is often desirable that ϕ also has other properties such as regularity, interpolation, symmetry, orthogonality, etc. To obtain such a function vector ϕ, extra conditions must be imposed on the mask a. We shall see that the main focus of this chapter and next chapter is on characterizing a refinable function vector ϕ with certain desired properties in terms of its mask a.

Refinability is important in many aspects. Here are two of the most important reasons. On one hand, it allows for the definition of a nested sequence of shift-invariant spaces $\{\mathscr{V}_j : j \in \mathbb{Z}\}$, which is the so-called multiresolution analysis (MRA, see [55, 57, 58]). MRA is the key to wavelet constructions and their associated fast wavelet algorithms. Once a multiresolution analysis is obtained, a (multi)wavelet ψ is generally derived from the refinable function vector ϕ via

$$\widehat{\psi}(\mathsf{d}\xi) := \widehat{b}(\xi)\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R} \tag{1.1.4}$$

for some $r \times r$ matrix $\widehat{b}$ of 2π-periodic trigonometric polynomials. According to various requirements of problems in applications, different desired properties of the wavelet function vector ψ and the refinable function vector ϕ are needed, which can be characterized by conditions on the mask a. We shall see in Chapters 3 and 4 on the subjects of matrix extension with symmetry, which plays an important role in deriving a proper wavelet mask b from the mask a so that ψ has symmetry once the corresponding refinable function vector ϕ also has symmetry.

On the other hand, functions composed from linear combinations of shifts of a refinable function ϕ can be computed using a simple subdivision scheme (or cascade algorithm, see Section 2.2 for a detailed definition). Consequently, such functions can be computed at any desired resolution and any desired position. Such an adaptive "Zoom-in" property makes subdivision curves and surfaces very attractive for interactive geometric modeling applications (see [15]). Among many subdivision schemes, interpolatory subdivision schemes and Hermite interpolatory subdivision schemes are of great interest in sampling theory, CAGD (Computer Aided Geometric Design) and computer graphics. See Figure 1.1 for an example of interpolatory subdivision schemes. In this chapter, we are interested in interpolatory subdivision schemes, which corresponds to refinable function vector with the interpolation property (see (1.1.5)).

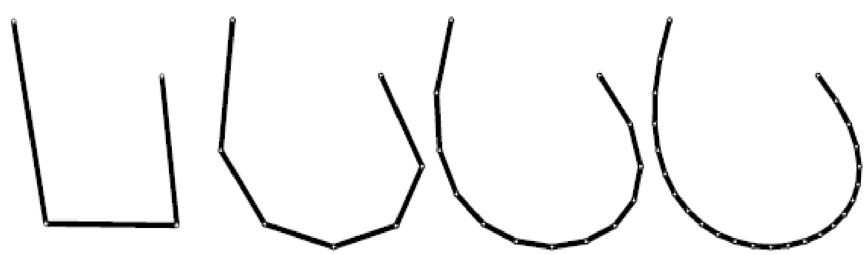

FIGURE 1.1: Example of an interpolatory subdivision scheme for curves in the plane. Initial points remain unchanged while additional points are added by rules related to an interpolatory mask.

Before giving the definition of *interpolating refinable function vectors*, let us discuss two particular important families of scalar refinable functions.

One is the family of Deslauriers-Dubuc interpolating refinable functions and another is the family of Daubechies orthogonal refinable functions.

We say that a compactly supported d-refinable function ϕ with mask a is *interpolating* if the function ϕ is continuous and $\phi(k) = \delta(k)$ for all $k \in \mathbb{Z}$, where δ denotes the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$. We say that a compactly supported d-refinable function ϕ with mask a is *orthogonal* if $\int_{\mathbb{R}} \phi(x-k)\overline{\phi(x)}\,dx = \delta(k)$ for all $k \in \mathbb{Z}$. By the refinement equation (1.1.1), one can easily see that the mask a of a scalar interpolating d-refinable function must be *an interpolatory mask* with the dilation factor d: $a(\mathsf{d}k) = \frac{1}{\mathsf{d}}\delta(k)$ for all $k \in \mathbb{Z}$, or equivalently, $\sum_{m=0}^{\mathsf{d}-1} \widehat{a}(\xi + 2\pi m/\mathsf{d}) = 1$. Similarly, the mask a for an orthogonal d-refinable function must be an *orthogonal mask* with dilation factor d: $\sum_{m=0}^{\mathsf{d}-1} |\widehat{a}(\xi + 2\pi m/\mathsf{d})|^2 = 1$.

The family of Deslauriers-Dubuc interpolatory masks [12] is a family of interpolatory masks $\{b_n : n \in \mathbb{N}\}$ with dilation factor $\mathsf{d} = 2$ such that b_n is supported on $[1-2n, 2n-1]$ and $\widehat{b_n}(\xi) = \cos^{2n}(\xi/2) \sum_{j=0}^{n-1} \binom{n-1+j}{j} \sin^{2j}(\xi/2)$. Also $\widehat{b_n}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$. It is well known [10, 11] that the family of Daubechies orthogonal masks $\{a_n : n \in \mathbb{N}\}$ is closely related to the mask $\{b_n : n \in \mathbb{N}\}$ via $|\widehat{a_n}(\xi)|^2 = \widehat{b_n}(\xi)$. That is, the Daubechies orthogonal mask $\widehat{a_n}$ of order n can be obtained from the interpolatory mask $\widehat{b_n}$ via the Riesz lemma. These two families of refinable function vectors have been extensively studied and applied in many application fields such as sampling theory, image/signal processing. Though these two families are connected via Riesz lemma, each of them cannot have the property of the other one. That is, the family of Deslauriers-Dubuc interpolating refinable functions is not orthogonal while the family of Daubechies orthogonal refinable functions is not interpolating.

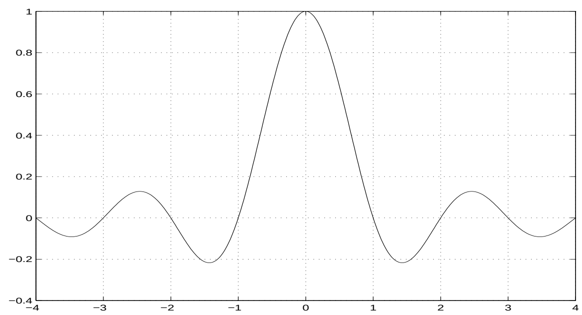

FIGURE 1.2: The sinc function.

An interesting example is the sinc function: $\mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$, see Figure 1.2. The sinc function is interpolating: $\mathrm{sinc}|_{\mathbb{Z}} = \delta$, which is used in the Shannon sampling theorem saying that for $f \in C(\mathbb{R}) \cap L_1(\mathbb{R})$ such that $\widehat{f}$ is supported inside $[-\pi, \pi]$ (that is, f is bandlimited with band π), $f(x) = \lim_{n\to\infty} \sum_{k=-n}^{n} f(k)\mathrm{sinc}(x-k)$, where the series converges uniformly for any $x \in \mathbb{R}$. The Shannon sampling theorem can be restated using shift-invariant spaces. For a function vector $f = [f_1, \ldots, f_r]^T$ in $L_2(\mathbb{R})$, we denote by $V(f)$ the smallest closed subspace in $L_2(\mathbb{R})$ containing $f_1(\cdot - k), \ldots, f_r(\cdot - k)$ for all $k \in \mathbb{Z}$. Due to the interpolation property of the sinc function, for any continuous function $f \in V(\mathrm{sinc}) \cap L_1(\mathbb{R})$, one always has $f = \sum_{k\in\mathbb{Z}} f(k)\mathrm{sinc}(\cdot - k)$. Note that $\widehat{\mathrm{sinc}} = \chi_{(-\pi,\pi]}$ is the characteristic function of the interval $(-\pi, \pi]$. It is well known in the literature ([66, 68] and references therein) that sinc is a 2-refinable function satisfying $\widehat{\mathrm{sinc}}(2\xi) = \widehat{a}(\xi)\widehat{\mathrm{sinc}}(\xi)$, where $\widehat{a}$ is a 2π-periodic function defined by $\widehat{a}(\xi) := \chi_{(-\pi/2,\pi/2]}(\xi)$ for all $\xi \in (-\pi, \pi]$. So, sinc is an interpolating 2-refinable function. Moreover, it is easy to verify that sinc is also orthogonal: $\langle \mathrm{sinc}, \mathrm{sinc}(\cdot - k)\rangle = \delta(k)$ for all $k \in \mathbb{Z}$. For a function $f \in V(\mathrm{sinc}) \cap L_2(\mathbb{R})$, one also has $f = \sum_{k\in\mathbb{Z}} \langle f, \mathrm{sinc}(\cdot - k)\rangle \mathrm{sinc}(\cdot - k)$. Consequently, for a function $f \in V(\mathrm{sinc}) \cap L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, the coefficients $\langle f, \mathrm{sinc}(\cdot - k)\rangle, k \in \mathbb{Z}$ can be realized by sampling f instead of inner products, which in signal processing means prefiltering is no longer needed.

Though sinc has the properties of both interpolation and orthogonality, it is not compactly supported and has a slow decay rate near ∞. Motivated by the wavelet applications in sampling theorems in signal processing, it is desirable to have compactly supported refinable functions that are both interpolating and orthogonal ([53, 62, 73]). However, it has been observed in [53, 62, 73] that for the dilation factor $\mathsf{d} = 2$, it is impossible to have a compactly supported scalar 2-refinable function such that it is both interpolating and orthogonal. In order to achieve both interpolation and orthogonality, it is natural to consider either the dilation factor $\mathsf{d} > 2$ or the multiplicity $r > 1$. For $\mathsf{d} = r = 2$, several interesting examples have been obtained in [53, 61, 62, 73] to show that one indeed can achieve both the interpolation and orthogonality properties of a refinable function vector simultaneously. In this chapter, we shall consider the general case of interpolating d-refinable function vectors and investigate their properties. More precisely, we are interested in a family of d-refinable function vectors with the interpolation property defined as follows.

We say that $\phi = [\phi_1, \ldots, \phi_r]^T : \mathbb{R} \to \mathbb{C}^{r\times 1}$ is an *interpolating refinable function vector of type* $(\mathsf{d}, \Gamma_r, 0)$ if ϕ is d-refinable, continuous, and interpolating, i.e., $\phi \in (C(\mathbb{R}))^{r\times 1}$ satisfies (1.1.1) and

$$\begin{aligned} \phi_\ell\left(\frac{m}{r} + j\right) =& \delta(j)\delta(\ell - 1 - m), \qquad \forall\, j \in \mathbb{Z}, \\ & m = 0, \ldots, r-1,\ \ell = 1, \ldots, r. \end{aligned} \tag{1.1.5}$$

Here, $\Gamma_r := \{0, \frac{1}{r}, \ldots, \frac{r-1}{r}\}$ and the meaning of 0 in $(\mathsf{d}, \Gamma_r, 0)$ will become obvious in Chapter 2. Comparing with the sinc function that interpolates on the integer lattice $\mathbb{Z}$, we use r functions $\phi_1, \ldots, \phi_r$ to interpolate the lattice $r^{-1}\mathbb{Z}$. For a function $f:\mathbb{R}\to\mathbb{C}$, it can be interpolated and approximated by

$$\begin{aligned}\widetilde{f} &= \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} f\Big(\frac{\ell-1}{r}+k\Big)\phi_\ell(\cdot-k)\\ &= \sum_{k\in\mathbb{Z}}\Big[f(k), f\Big(\frac{1}{r}+k\Big),\ldots,f\Big(\frac{r-1}{r}+k\Big)\Big]\phi(\cdot-k).\end{aligned}$$

Since ϕ is interpolating, $\widetilde{f}(k/r)=f(k/r)$ for all $k\in\mathbb{Z}$; that is, $\widetilde{f}$ agrees with f on the lattice $r^{-1}\mathbb{Z}$. Such an interpolation property is important in approximation and sampling theory. For more about the approximation property of such a function ϕ, one may refer to [41, 73].

The structure of this chapter is as follows. In Section 1.2, we shall characterize both compactly supported interpolating d-refinable function vectors and orthogonal interpolating d-refinable function vectors in terms of their masks. In Section 1.2, we also study the sum rule structure of the interpolatory masks of type $(\mathsf{d},\Gamma_r,0)$ for interpolating d-refinable function vectors with multiplicity r, which will play a central role in our construction of interpolatory masks of type $(\mathsf{d},\Gamma_r,0)$. In Section 1.3, based on the results of Section 1.2, several examples of interpolating refinable function vectors will be presented. Finally, in Section 1.4, we shall discuss biorthogonal multiwavelets derived from interpolating refinable function vectors and some examples will be presented. Conclusions and remarks shall be given in Section 1.5, in which we shall also discuss relations of results in this chapter to the next serval chapters. Most of the results in this chapter and next chapter have been published in [30, 37].

1.2 Analysis of Interpolating Refinable Function Vectors

In this section, we shall study the interpolating refinable function vectors of type $(\mathsf{d}, \Gamma_r, 0)$. Based on [24, Theorem 4.3], we shall provide a complete characterization for a compactly supported d-refinable function vector in terms of its mask. We also study the sum rule structure of such interpolating refinable function vectors. As a consequence, we obtain a criterion for a compactly supported interpolating refinable function vectors whose shifts are orthogonal.

Before we introduce our characterization theorem, we need a quantity $\nu_p(a, \mathsf{d})$, whose detailed definition is given in (2.2.8) (see Chapter 2). For a finitely supported matrix mask a, the quantity $\nu_2(a, \mathsf{d})$ can be numerically computed by finding the spectral radius of certain finite matrix using the algorithm in [44, 45, 51, 72]. The quantity $\nu_p(a, \mathsf{d})$ plays a very important role in characterizing the convergence of a vector cascade algorithm in a Sobolev space and in characterizing the L_p smoothness of a refinable function vector. In general, $\nu_p(a, \mathsf{d})$ provides a lower bound for the L_p smoothness exponent $\nu_p(\phi)$ (see (2.2.9)) of a refinable function vector ϕ with mask a and dilation factor d, that is, $\nu_p(a, \mathsf{d}) \leqslant \nu_p(\phi)$ always holds. Moreover, if the shifts of the refinable function vector ϕ associated with mask a and dilation factor d are stable in $L_p(\mathbb{R})$, then $\nu_p(\phi) = \nu_p(a, \mathsf{d})$. That is, in this case, $\nu_p(a, \mathsf{d})$ indeed characterizes the L_p smoothness exponent of a refinable function vector ϕ with mask a and dilation factor d. Although there is no general algorithm for the computation of $\nu_p(a, \mathsf{d})$, we can use $\nu_2(a, \mathsf{d})$ to estimate $\nu_p(a, \mathsf{d})$ by the inequalities: $\nu_2(a, \mathsf{d}) \geqslant \nu_p(a, \mathsf{d}) \geqslant \nu_2(a, \mathsf{d}) - (1/2 - 1/p)$ for all $p \geqslant 2$. One may refer to [22, 24, 31, 46, 49, 50, 51, 72] and many references therein for more details on the convergence of vector cascade algorithms and smoothness of refinable function vectors.

For $1 \leqslant \ell \leqslant r$, let $\mathbf{e}_\ell$ denote the ℓ-th unit coordinate column vector in $\mathbb{R}^r$; that is, $\mathbf{e}_\ell$ is the $r \times 1$ column vector whose only nonzero entry is located at the ℓth component with value 1[1]. For a function f, let $f^{(j)}$ denote its j-th derivative.

Now we have the following result characterizing a compactly supported interpolating d-refinable function vector in terms of its mask.

Theorem 1.1. *Let* d *and* r *be positive integers such that* $\mathsf{d} > 1$*. Let* $a : \mathbb{Z} \to \mathbb{C}^{r \times r}$ *be a finitely supported sequence of* $r \times r$ *matrices on* $\mathbb{Z}$*. Let* $\phi = [\phi_1, \dots, \phi_r]^T$ *be a compactly supported d-refinable function vector such that* $\widehat{\phi}(\mathsf{d}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$*. Then* ϕ *is interpolating, that is,* ϕ *is a continuous function vector and* (1.1.5) *holds if and only if the following statements hold:*

(i) $[1, \dots, 1]\widehat{\phi}(0) = 1$ (This is a normalization condition on the refinable function vector ϕ).

(ii) *a is an interpolatory mask of type* $(\mathsf{d}, \Gamma_r, 0)$*:* $[1, \dots, 1]\widehat{a}(0) = [1, \dots, 1]$ *and*

$$a(R_\ell + \mathsf{d}j)\mathbf{e}_{Q_\ell+1} = \mathsf{d}^{-1}\delta(j)\mathbf{e}_{\ell+1}, \ \forall\, j \in \mathbb{Z}; \ \ell = 0, 1, \dots, r-1, \quad (1.2.1)$$

where for each $\ell = 0, 1, \dots, r-1$*,* $R_\ell \in \mathbb{Z}$ *and* $Q_\ell \in \{0, 1, \dots, r-1\}$ *are defined to be*

$$R_\ell := \left\lfloor \frac{\mathsf{d}\ell}{r} \right\rfloor \quad \textit{and} \quad Q_\ell := r\Big(\frac{\mathsf{d}\ell}{r} - \left\lfloor \frac{\mathsf{d}\ell}{r} \right\rfloor\Big) = \mathsf{d}\ell \mod r. \qquad (1.2.2)$$

[1]For convention, $\mathbf{e}_\ell$ shall denote a $1 \times r$ row vector in Chapters 3 and 4.

Here $\lfloor x \rfloor$ denotes the largest integer that is not larger than x.

(iii) $\nu_\infty(a, \mathsf{d}) > 0$.

We have presented a complete proof in [30, Theorem 2.1] and since it is a special case of Theorem 2.1 of Chapter 2, we shall omit the proof here.

As a consequence of Theorem 1.1, we have the following result characterizing compactly supported *orthogonal* interpolating refinable function vectors.

Corollary 1.2. *Let d and r be positive integers such that $\mathsf{d} > 1$. Let $a : \mathbb{Z} \to \mathbb{C}^{r\times r}$ be a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$ and ϕ be a compactly supported d-refinable function vector such that $\widehat{\phi}(\mathsf{d}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. Then ϕ is an orthogonal interpolating function vector; that is, ϕ is continuous, (1.1.5) holds and*

$$\int_{\mathbb{R}} \phi(x-j)\overline{\phi(x)}^T\, dx = \frac{1}{r}\delta(j) I_r \qquad \forall\, j \in \mathbb{Z}, \tag{1.2.3}$$

if and only if, (i)–(iii) of Theorem 1.1 hold and a is an orthogonal mask:

$$\sum_{m=0}^{\mathsf{d}-1} \widehat{a}(\xi + 2\pi m/\mathsf{d})\overline{\widehat{a}(\xi + 2\pi m/\mathsf{d})}^T = I_r. \tag{1.2.4}$$

Proof. Necessity. Suppose that ϕ is an orthogonal interpolating d-refinable function vector. Then in particular, ϕ is an interpolating d-refinable function vector. Hence, by Theorem 1.1, (i)–(iii) hold. Now we show that (1.2.3) implies (1.2.4). Since ϕ is a compactly supported d-refinable function vector satisfying the refinement equation (1.1.1), noting that the mask

a is finitely supported, we deduce from (1.2.3) that

$$\frac{1}{r}\delta(j)I_r = \int_{\mathbb{R}} \phi(x-j)\overline{\phi(x)}^T\,dx = \frac{\mathsf{d}}{r}\sum_{k\in\mathbb{Z}} a(k)\overline{a(\mathsf{d}j+k)}^T,$$

which is equivalent to (1.2.4).

Sufficiency. Since (i)–(iii) of Theorem 1.1 hold, by Theorem 1.1, we see that ϕ is continuous and (1.1.5) holds. To complete the proof, we show that (1.2.3) holds. Since (iii) of Theorem 1.1 holds, we have $\nu_\infty(a,\mathsf{d}) > 0$. Since $\nu_2(a,\mathsf{d}) \geqslant \nu_\infty(a,\mathsf{d}) > 0$, by [24, Theorem 4.3], the vector cascade algorithm associated with mask a and dilation factor d converges in $L_2(\mathbb{R})$.

Let $f_0 := [g(r\cdot), g(r\cdot -1), \ldots, g(r\cdot -(r-1))]^T$ and $f_n := \mathsf{d}\sum_{k\in\mathbb{Z}} a(k) f_{n-1}(\mathsf{d}\cdot -k)$, where $g = \chi_{[0,1]}$, the characteristic function of the interval $[0,1]$. Denote $y := [1,\ldots,1] \in \mathbb{R}^{1\times r}$. By calculation, we have $\widehat{g}(\xi) = \frac{1-e^{-i\xi}}{i\xi}$. Therefore, $\widehat{g}(0) = 1$ and $\widehat{g}(2\pi k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$. By the same argument as in the proof of Theorem 2.1, we can check that $y\widehat{f_0}(0) = 1$ and $y\widehat{f_0}(2\pi k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$. Since $y\widehat{a}(0) = y$ by (ii) of Theorem 1.1, f_0 is a suitable initial function vector in $L_2(\mathbb{R})$. On the other hand, by (i), we have $y\widehat{\phi}(0) = 1$. Now by $\nu_2(a,\mathsf{d}) > 0$, we see that $\lim_{n\to\infty}\|f_n - \phi\|_{(L_2(\mathbb{R}))^{r\times 1}} = 0$. By induction on n, we can show that

$$\int_{\mathbb{R}} f_n(x-j)\overline{f_n(x)}^T\,dx = \frac{1}{r}\delta(j)I_r \qquad \forall\, j\in\mathbb{Z},\ n\in\mathbb{N}_0. \tag{1.2.5}$$

Now it is easy to conclude from $\lim_{n\to\infty}\|f_n - \phi\|_{(L_2(\mathbb{R}))^{r\times 1}} = 0$ and (1.2.5) that (1.2.3) is true. □

For a matrix mask a with multiplicity r, we say that a satisfies the *sum rules* of order κ with a dilation factor d ([22, 24, 43, 60]) if there exists a sequence $y \in (\ell_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{y}(0) \neq 0$ and

$$[\widehat{y}(\mathsf{d}\cdot)\widehat{a}(\cdot)]^{(j)}(2\pi m/\mathsf{d}) = \delta(m)\widehat{y}^{(j)}(0) \quad \forall\, j = 0,\dots,\kappa-1 \quad \text{and} \quad m = 0,\dots,\mathsf{d}-1. \tag{1.2.6}$$

Next, we shall study the structure of the vector $\widehat{y}$ in the definition of the sum rules in (1.2.6) for the particular family of interpolatory masks of type $(\mathsf{d}, \Gamma_r, 0)$ given in (1.2.1). Once the structure of $\widehat{y}$ is determined, the nonlinear equations in (1.2.6) become linear equations, which greatly facilities our construction of interpolatory masks of type $(\mathsf{d}, \Gamma_r, 0)$.

Theorem 1.3. *Let* d *and* r *be positive integers such that* $\mathsf{d} > 1$*. Let* $a : \mathbb{Z} \to \mathbb{C}^{r\times r}$ *be a finitely supported sequence of* $r \times r$ *matrices on* $\mathbb{Z}$*. Suppose that* a *is an interpolatory mask of type* $(\mathsf{d}, \Gamma_r, 0)$*; that is,* $[1,\dots,1]\widehat{a}(0) = [1,\dots,1]$ *and* (1.2.1) *holds. If* a *satisfies the sum rules of order* κ *in* (1.2.6) *with a sequence* $y \in (\ell_0(\mathbb{Z}))^{1\times r}$ *and* $\widehat{y}(0) = [1,\dots,1]$*, then*

$$\widehat{y}^{(j)}(0) = i^j r^{-j}[\delta(j), 1^j, 2^j, \dots, (r-1)^j], \qquad j = 0,\dots,\kappa-1. \tag{1.2.7}$$

In other words, $\widehat{y}(\xi) = Y(\xi) + O(|\xi|^\kappa)$ *with* $Y(\xi) := [1, e^{i\xi/r}, \dots, e^{i(r-1)\xi/r}]$.

Since Theorem 1.3 is a special case of Theorem 2.2, we shall also leave its proof to Chapter 2. The above characterizations greatly facilitate our design of interpolatory masks of type $(\mathsf{d}, \Gamma_r, 0)$ (see next section for such examples). In fact, under certain constrains, we can construct families of interpoatory masks of type $(\mathsf{d}, \Gamma_r, h)$ for any d, r and h. For details, see Section 2.4.

1.3 Examples

In this section, to illustrate our main results, we shall present several examples of interpolatory masks of type $(\mathsf{d}, \Gamma_r, 0)$, as well as several examples of masks for orthogonal interpolating refinable function vectors.

Example 1.1. Let $\mathsf{d} = r = 2$. Then we have an interpolatory mask a of type $(2, \Gamma_2, 0)$ satisfying the sum rules of order 3 and supported on $[-1, 2]$ as follows:

$$a(-1) = \frac{1}{16}\begin{bmatrix} 0 & 6 \\ 0 & -1 \end{bmatrix}, \quad a(0) = \frac{1}{16}\begin{bmatrix} 8 & 6 \\ 0 & 3 \end{bmatrix},$$
$$a(1) = \frac{1}{16}\begin{bmatrix} 0 & 0 \\ 8 & 3 \end{bmatrix}, \quad a(2) = \frac{1}{16}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have $\nu_2(a, 2) \approx 1.839036$. Therefore, $\nu_\infty(a, 2) \geqslant \nu_2(a, 2) - 1/2 \approx 1.339036 > 0$. By Theorem 1.1, its associated refinable function vector $\phi = [\phi_1, \phi_2]^T$ is interpolating. Moreover, $\phi_1(-x) = \phi_1(x)$ and $\phi_2(1 - x) = \phi_2(x)$ for all $x \in \mathbb{R}$. See Figure 1.3 for the graph of the interpolating 2-refinable function vector ϕ associated with the mask a.

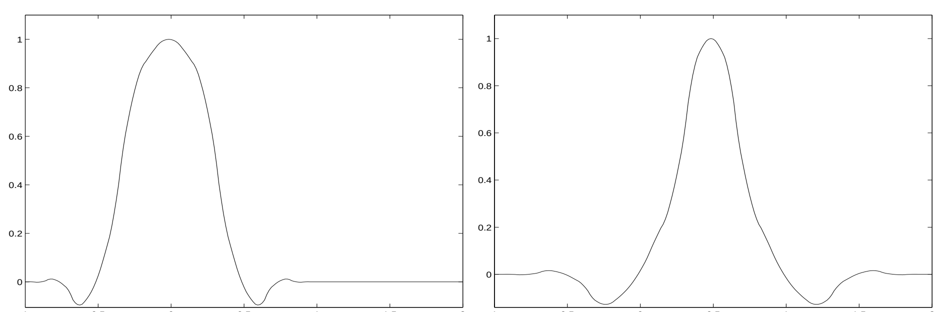

FIGURE 1.3: The graphs of ϕ_1 (left) and ϕ_2 (right) in Example 1.1. $\nu_2(\phi) \approx 1.839036$, $\phi_1(-\cdot) = \phi_1$, and $\phi_2(1 - \cdot) = \phi_2$.

Example 1.2. Let $\mathsf{d} = 3$ and $r = 2$. Then we have an interpolatory mask a of type $(3, \Gamma_2, 0)$ satisfying the sum rules of order 4. The mask a is supported on $[-2,3]$ and is given by

$$a(-2)=\frac{1}{243}\begin{bmatrix}-21 & 0\\ 4 & 0\end{bmatrix},\quad a(-1)=\frac{1}{243}\begin{bmatrix}30 & 60\\ -4 & -5\end{bmatrix},\quad a(0)=\frac{1}{243}\begin{bmatrix}81 & 84\\ 0 & 14\end{bmatrix},$$
$$a(1)=\frac{1}{243}\begin{bmatrix}14 & 0\\ 84 & 81\end{bmatrix},\quad a(2)=\frac{1}{243}\begin{bmatrix}-5 & -4\\ 60 & 30\end{bmatrix},\quad a(3)=\frac{1}{243}\begin{bmatrix}0 & 4\\ 0 & -21\end{bmatrix}. \tag{1.3.1}$$

We have $\nu_2(a,3)\approx 1.348473$. Therefore, $\nu_\infty(a,3)\geqslant \nu_2(a,3)-1/2\approx 0.848473>0$. By Theorem 1.1, its associated refinable function vector $\phi=[\phi_1,\phi_2]^T$ is interpolating. See Figure 1.4 for the graph of the interpolating 3-refinable function vector ϕ associated with the mask a.

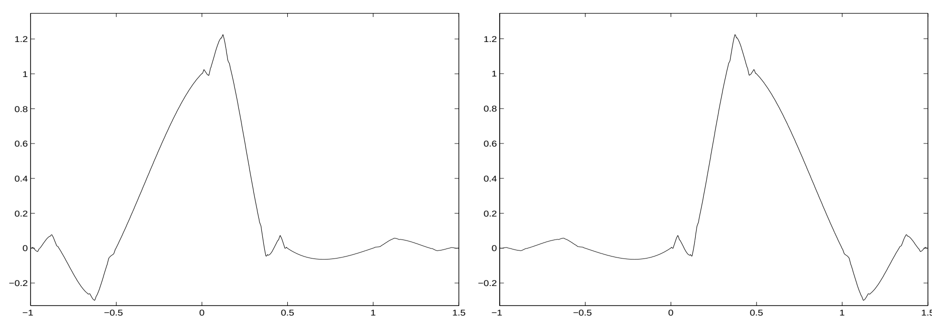

FIGURE 1.4: The graphs of ϕ_1 (left) and ϕ_2 (right) in Example 1.2. $\nu_2(\phi)\approx 1.348473$ and $\phi_1=\phi_2(1/2-\cdot)$.

Example 1.3. Let $\mathsf{d}=r=3$. Then we have an interpolatory mask a of type $(3,\Gamma_3,0)$ satisfying the sum rules of order 6. The mask a is supported on $[-2,3]$ and is given by

$$a(-2)=\frac{1}{2187}\begin{bmatrix}0 & -176 & -175\\ 0 & 55 & 50\\ 0 & -8 & -7\end{bmatrix},\quad a(-1)=\frac{1}{2187}\begin{bmatrix}0 & 280 & 560\\ 0 & -56 & -70\\ 0 & 7 & 8\end{bmatrix},$$

$$a(0) = \frac{1}{2187}\begin{bmatrix} 729 & 700 & 440 \\ 0 & 175 & 440 \\ 0 & -14 & -22 \end{bmatrix}, \quad a(1) = \frac{1}{2187}\begin{bmatrix} 0 & -22 & -14 \\ 729 & 440 & 175 \\ 0 & 440 & 700 \end{bmatrix},$$

$$a(2) = \frac{1}{2187}\begin{bmatrix} 0 & 8 & 7 \\ 0 & -70 & -56 \\ 729 & 560 & 280 \end{bmatrix}, \quad a(3) = \frac{1}{2187}\begin{bmatrix} 0 & -7 & -8 \\ 0 & 50 & 55 \\ 0 & -175 & -176 \end{bmatrix}.$$

We have $\nu_2(a,3) \approx 2.589443$. Therefore, $\nu_\infty(a,3) \geqslant \nu_2(a,3) - 1/2 \approx 2.089443 > 0$. By Theorem 1.1, its associated refinable function vector $\phi = [\phi_1, \phi_2, \phi_3]^T$ is interpolating and belongs to $(C^2(\mathbb{R}))^{3\times 1}$. See Figure 1.5 for the graph of the interpolating 3-refinable function vector ϕ.

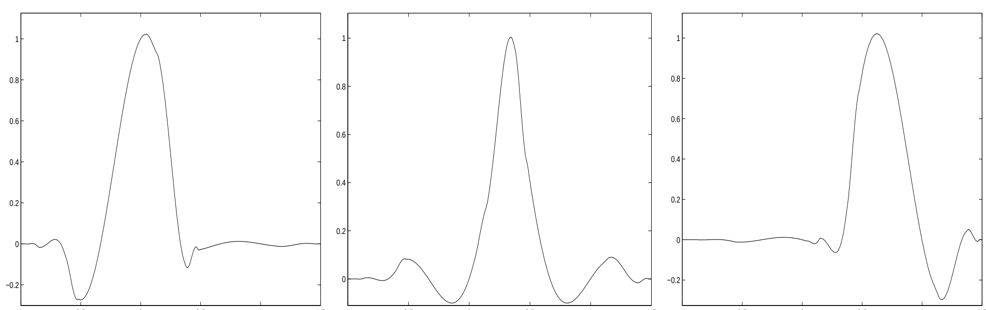

FIGURE 1.5: The graphs of ϕ_1 (left), ϕ_2 (middle), and ϕ_3 (right) in Example 1.3. $\nu_2(\phi) \approx 2.589443$.

Examples of orthogonal interpolating refinable function vectors of type $(2, \Gamma_2, 0)$ have been given in [53, 62, 73]. Next, let us present two examples of orthogonal interpolating refinable function vectors of type $(3, \Gamma_2, 0)$.

Example 1.4. Let $\mathsf{d} = 3$ and $r = 2$. The orthogonal and interpolatory mask a of type $(3, \Gamma_2, 0)$ is supported on $[-2, 3]$ and is given by

$$a(-2) = \begin{bmatrix} -\frac{17}{702} - \frac{\sqrt{17}}{351} & 0 \\ -\frac{8}{351} + \frac{5\sqrt{17}}{702} & 0 \end{bmatrix}, \qquad a(-1) = \begin{bmatrix} \frac{85}{702} - \frac{8\sqrt{17}}{351} & \frac{68}{351} + \frac{29\sqrt{17}}{702} \\ \frac{1}{351} + \frac{\sqrt{17}}{702} & \frac{11}{702} - \frac{7\sqrt{17}}{351} \end{bmatrix},$$
$$a(0) = \begin{bmatrix} \frac{1}{3} & \frac{119}{351} - \frac{11\sqrt{17}}{702} \\ 0 & \frac{29}{702} + \frac{4\sqrt{17}}{351} \end{bmatrix}, \qquad a(1) = \begin{bmatrix} \frac{29}{702} + \frac{4\sqrt{17}}{351} & 0 \\ \frac{119}{351} - \frac{11\sqrt{17}}{702} & \frac{1}{3} \end{bmatrix},$$
$$a(2) = \begin{bmatrix} \frac{11}{702} - \frac{7\sqrt{17}}{351} & \frac{1}{351} + \frac{\sqrt{17}}{702} \\ \frac{68}{351} + \frac{29\sqrt{17}}{702} & \frac{85}{702} - \frac{8\sqrt{17}}{351} \end{bmatrix}, \qquad a(3) = \begin{bmatrix} 0 & -\frac{8}{351} + \frac{5\sqrt{17}}{702} \\ 0 & -\frac{17}{702} - \frac{\sqrt{17}}{351} \end{bmatrix}.$$

The mask a satisfies the sum rules of order 2. We have $\nu_2(a, 3) \approx 1.046673$. Therefore, $\nu_\infty(a, 3) \geqslant \nu_2(a, 3) - 1/2 \approx 0.546673 > 0$. By Corollary 1.2, the refinable function vector $\phi = [\phi_1, \phi_2]^T$ associated with the mask a and dilation factor d is interpolating and orthogonal. See Figure 1.6 for the graph of the orthogonal interpolating 3-refinable function vector ϕ.

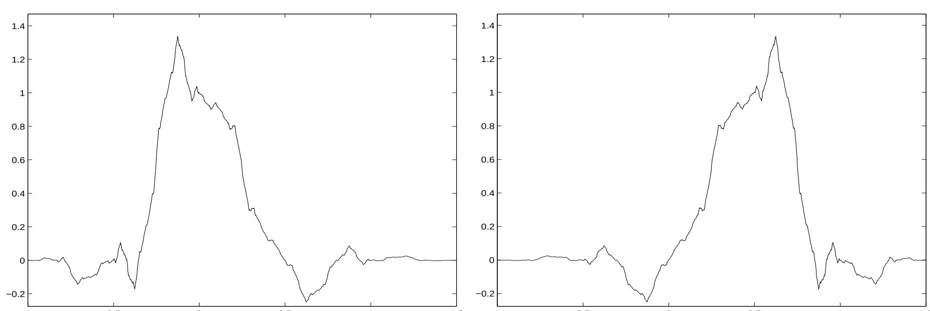

FIGURE 1.6: The graphs of ϕ_1 (left) and ϕ_2 (right) in Example 1.4. $\nu_2(\phi) \approx 1.046673$ and $\phi_1 = \phi_2(1/2 - \cdot)$.

Example 1.5. Let $\mathsf{d} = 3$ and $r = 2$. We have an orthogonal and interpolatory mask a of type $(3, \Gamma_2, 0)$ satisfying the sum rules of order 2. The mask a is supported on $[-4, 4]$ and

$$a(0) = \begin{bmatrix} \frac{1}{3} & \frac{29}{108} + \frac{\sqrt{41}}{108} \\ 0 & \frac{7}{60} - \frac{\sqrt{41}}{180} \end{bmatrix}, \qquad a(1) = \begin{bmatrix} \frac{11}{108} - \frac{\sqrt{41}}{108} & 0 \\ \frac{37}{180} + \frac{\sqrt{41}}{60} & \frac{1}{3} \end{bmatrix},$$

$$a(2) = \begin{bmatrix} -\frac{2}{135} - \frac{\sqrt{41}}{270} & \frac{1}{540} - \frac{\sqrt{41}}{540} \\ \frac{37}{180} + \frac{\sqrt{41}}{60} & \frac{7}{60} - \frac{\sqrt{41}}{180} \end{bmatrix}, \quad a(3) = \begin{bmatrix} 0 & \frac{17}{270} - \frac{\sqrt{41}}{135} \\ 0 & -\frac{7}{60} + \frac{\sqrt{41}}{180} \end{bmatrix},$$

$$a(4) = \begin{bmatrix} -\frac{47}{540} + \frac{7\sqrt{41}}{540} & 0 \\ \frac{23}{180} - \frac{\sqrt{41}}{60} & 0 \end{bmatrix},$$

while $a(-4), a(-3), a(-2), a(-1)$ can be obtained by the symmetry condition in (1.5.2). We have $\nu_2(a, 3) \approx 0.976503$. Therefore, $\nu_\infty(a, 3) \geqslant \nu_2(a, 3) - 1/2 \approx 0.476503 > 0$. By Corollary 1.2, its associated 3-refinable function vector $\phi = [\phi_1, \phi_2]^T$ is an orthogonal interpolating refinable function vector of type $(3, \Gamma_2, 0)$. See Figure 1.7 for the graphs of ϕ_1 and ϕ_2.

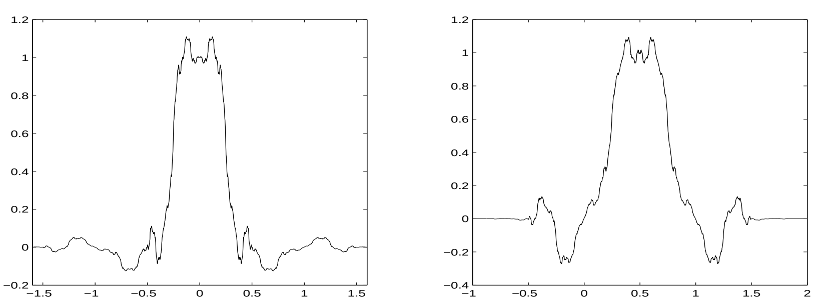

FIGURE 1.7: The graphs of ϕ_1 and ϕ_2 in Example 1.5. $\nu_2(\phi) \approx 0.976503$, $\phi_1(-\cdot) = \phi_1$, and $\phi_2(1 - \cdot) = \phi$.

1.4 Biorthogonal Refinable Function Vectors

It is of interest to construct biorthogonal refinable function vectors from interpolating refinable function vectors, due to their interesting interpolation property. In this section, let us discuss how to derive biorthogonal refinable function vectors from interpolating refinable function vectors that have been investigated and constructed in previous sections. To do so, let us introduce some necessary concepts.

For two $r \times 1$ vectors ϕ and $\widetilde{\phi}$ of compactly supported functions in $L_2(\mathbb{R})$, we say that $(\phi, \widetilde{\phi})$ is *a pair of dual function vectors* (or $\widetilde{\phi}$ is a dual function vector of ϕ) if

$$\int_{\mathbb{R}} \phi(x-j)\overline{\widetilde{\phi}(x)}^T dx = \frac{1}{r}\delta(j)I_r, \qquad j \in \mathbb{Z}. \tag{1.4.1}$$

For two $r \times r$ matrices $\widehat{a}$ and $\widehat{\widetilde{a}}$ of 2π-periodic trigonometric polynomials, we say that $(a, \widetilde{a})$ is *a pair of dual masks* (or $\widetilde{a}$ is a dual mask of a) with a dilation factor d if

$$\sum_{m=0}^{\mathsf{d}-1} \widehat{a}(\xi + 2\pi m/\mathsf{d})\overline{\widehat{\widetilde{a}}(\xi + 2\pi m/\mathsf{d})}^T = I_r. \tag{1.4.2}$$

If a is a dual mask of itself, then (1.4.2) becomes (1.2.4) and a is an orthogonal mask. Let ϕ and $\widetilde{\phi}$ be two compactly supported d-refinable function vectors with masks a and $\widetilde{a}$, respectively. Assume that $\widehat{\phi}(0)$ and $\widehat{\widetilde{\phi}}(0)$ are appropriately normalized so that $\overline{\widehat{\phi}(0)}^T \widehat{\widetilde{\phi}}(0) = 1$. Then it is known that $(\phi, \widetilde{\phi})$ is a pair of dual d-refinable function vectors in $L_2(\mathbb{R})$, if and only if, $(a, \widetilde{a})$ is a pair of dual masks and both $\nu_2(a, \mathsf{d}) > 0$ and $\nu_2(\widetilde{a}, \mathsf{d}) > 0$. In wavelet analysis, for a given mask a, it is of interest to construct a dual mask $\widetilde{a}$ of a such that $\widetilde{a}$ can attain the sum rules of any preassigned order $\widetilde{\kappa}$ with a sequence $\widetilde{y}$, that is, $\widehat{\widetilde{y}}(0) \neq 0$ and

$$\widehat{\widetilde{y}}(\mathsf{d}\xi)\widehat{\widetilde{a}}(\xi+2\pi m/\mathsf{d}) = \delta(m)\widehat{\widetilde{y}}(\xi)+O(|\xi|^{\widetilde{\kappa}}),\ \xi \to 0,\ m = 0,\ldots,\mathsf{d}-1. \quad (1.4.3)$$

A systematic way of constructing such desirable dual masks $\widetilde{a}$ has been developed in [4, 21, 22]. There are two key ingredients in the proposed CBC (coset by coset) algorithm in [21]. In the following, let us outline the main ideas of the CBC algorithm and use it to construct biorthogonal multiwavelets for the interpolating refinable function vectors obtained in this chapter.

The first key ingredient of the CBC algorithm in [4, 21, 22] is the following interesting fact, whose proof is given in [22], as well as [21] for the scalar case. For the purpose of completeness, we shall provide a self-contained proof here.

Theorem 1.4. *Let d be a dilation factor. Let $\widehat{a}$ be an $r \times r$ matrix of 2π-periodic trigonometric polynomials such that 1 is a simple eigenvalue of $\widehat{a}(0)$ and for every $j \in \mathbb{N}$, d^j is not an eigenvalue of $\widehat{a}(0)$. Suppose that $\widetilde{a}$ is a dual mask of a and $\widetilde{a}$ satisfies the sum rules of order $\widetilde{\kappa}$ in (1.4.3) with a sequence $\widetilde{y}$. Then up to a multiplicative constant, the values $\widehat{\widetilde{y}}^{(j)}(0), j = 0,\ldots,\widetilde{\kappa}-1$ are uniquely determined by the mask $\widehat{a}$ via the following recursive formula: $\widehat{\widetilde{y}}(0) = \widehat{\widetilde{y}}(0)\overline{\widehat{a}(0)}^T$ and for $j = 1,\ldots,\widetilde{\kappa}$,*

$$\widehat{\widetilde{y}}^{(j)}(0) = \left[\sum_{k=0}^{j-1} \frac{j!}{k!(j-k)!}\widehat{\widetilde{y}}^{(k)}(0)\overline{\widehat{a}^{(j-k)}(0)}^T\right][\mathsf{d}^j I_r - \overline{\widehat{a}(0)}^T]^{-1}. \quad (1.4.4)$$

In other words, if ϕ is a compactly supported d-refinable function vector satisfying $\widehat{\phi}(\mathsf{d}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\phi}(0) \neq 0$, then $\widehat{\widetilde{y}}(\xi) = c\overline{\widehat{\phi}(\xi)}^T + O(|\xi|^{\widetilde{\kappa}})$ as

$\xi \to 0$ *for some nonzero constant c.*

Proof. By (1.4.2), we deduce that

$$\begin{aligned}\overline{\widehat{\widetilde{y}}(\mathsf{d}\xi)}^T &= \sum_{m=0}^{\mathsf{d}-1} \widehat{a}(\xi + 2\pi m/\mathsf{d})\overline{\widehat{\widetilde{a}}(\xi + 2\pi m/\mathsf{d})}^T \overline{\widehat{\widetilde{y}}(\mathsf{d}\xi)}^T \\ &= \sum_{m=0}^{\mathsf{d}-1} \widehat{a}(\xi + 2\pi m/\mathsf{d})\overline{\widehat{\widetilde{y}}(\mathsf{d}\xi)\widehat{\widetilde{a}}(\xi + 2\pi m/\mathsf{d})}^T.\end{aligned}$$

Now by (1.4.3) we get

$$\overline{\widehat{\widetilde{y}}(\mathsf{d}\xi)}^T = \widehat{a}(\xi)\overline{\widehat{\widetilde{y}}(\xi)}^T + O(|\xi|^{\widetilde{\kappa}}), \qquad \xi \to 0.$$

That is, the vector $\widehat{\widetilde{y}}$ must satisfy

$$\widehat{\widetilde{y}}(\mathsf{d}\xi) = \widehat{\widetilde{y}}(\xi)\overline{\widehat{a}(\xi)}^T + O(|\xi|^{\widetilde{\kappa}}), \qquad \xi \to 0. \tag{1.4.5}$$

By Leibniz differentiation formula, it follows from (1.4.5) that

$$\mathsf{d}^j \widehat{\widetilde{y}}^{(j)}(0) = \widehat{\widetilde{y}}^{(j)}(0)\overline{\widehat{a}(0)}^T + \sum_{k=0}^{j-1} \frac{j!}{k!(j-k)!}\widehat{\widetilde{y}}^{(k)}(0)\overline{\widehat{a}^{(j-k)}(0)}^T, \qquad j = 0, \dots, \widetilde{\kappa}.$$

Since 1 is a simple eigenvalue of $\widehat{a}(0)$ and d^j is not an eigenvalue of $\widehat{a}(0)$ for all $j \in \mathbb{N}$, now the recursive formula in (1.4.4) can be easily deduced from the above relation. Moreover, the relation $\widehat{\widetilde{y}}(\xi) = c\overline{\widehat{\phi}(\xi)}^T + O(|\xi|^{\widetilde{\kappa}})$ follows directly from (1.4.5). ☐

By obtaining the values $\widehat{\widetilde{y}}^{(j)}(0)$, $j \in \mathbb{N}_0$ from a given mask a via the recursive formula in (1.4.4) of Theorem 1.4, the CBC algorithm reduces the system of nonlinear equations (in terms of both $\widetilde{a}(k)$, $k \in \mathbb{Z}$ and $\widehat{\widetilde{y}}^{(j)}(0)$, $j = 0, \dots, \widetilde{\kappa} - 1$) in (1.4.3) into a system of linear equations, since now $\widehat{\widetilde{y}}^{(j)}(0)$,

$j = 0, \dots, \widetilde{\kappa} - 1$ are known. On the other hand, both conditions in (1.4.2) and (1.4.3) can be equivalently rewritten in terms of the cosets of the masks a and $\widetilde{a}$. More precisely, it is easy to verify that (1.4.2) is equivalent to

$$\sum_{m=0}^{\mathsf{d}-1} \widehat{a}^m(\xi) \overline{\widehat{\widetilde{a}}^m(\xi)}^T = \mathsf{d}^{-1} I_r, \tag{1.4.6}$$

where $\widehat{\widetilde{a}}^m(\xi) := \sum_{k\in\mathbb{Z}} \widetilde{a}(m + \mathsf{d}k) e^{-i\xi(m+\mathsf{d}k)}$, and (1.4.3) is equivalent to

$$\widehat{\widetilde{y}}(\mathsf{d}\xi)\widehat{\widetilde{a}}^m(\xi) = \mathsf{d}^{-1}\widehat{\widetilde{y}}(\xi) + O(|\xi|^{\widetilde{\kappa}}), \qquad \xi \to 0, \ m = 0, \dots, d-1. \tag{1.4.7}$$

The second key ingredient lies in that using Therorem 1.4, the CBC algorithm reduces the big system of linear equations in both (1.4.3) and (1.4.2) into small systems of linear equations using the idea of coset by coset construction and the equations in (1.4.6) and (1.4.7). Moreover, the CBC algorithm in [22] guarantees that for any given positive integer $\widetilde{\kappa}$, there always exists a finitely supported dual mask $\widetilde{a}$ of a such that $\widetilde{a}$ satisfies the sum rules of order $\widetilde{\kappa}$, see [22, Theorem 3.4] and [4, 21] for more details on the CBC algorithm.

We also mention that due to Theorem 1.3, all biorthogonal multiwavelets derived from interpolating refinable function vectors in this section have the highest possible balancing order, that is, its balancing order matches the order of sum rules. See [26, 28] and references therein on balanced biorthogonal multiwavelets.

In the following, let us present several examples of dual masks for some given interpolatory masks constructed in this paper.

Example 1.6. Let $\mathsf{d} = r = 2$. Let a denote the mask given in Example 1.1. By (1.4.4) of Theorem 1.4 with $\widetilde{\kappa} = 3$, we have

$$\widehat{\widetilde{y}}(0) = \frac{1}{2}[3, 2], \quad \frac{-i}{1!}\widehat{\widetilde{y}}^{(1)}(0) = \frac{1}{2}[0, 2], \quad \frac{(-i)^2}{2!}\widehat{\widetilde{y}}^{(2)}(0) = \frac{1}{136}[3, 14].$$

By the CBC algorithm in [22], we have a dual mask $\widetilde{a}$ of a such that $\widetilde{a}$ satisfies the sum rules of order 3. The dual mask $\widetilde{a}$ is supported on $[-1, 3]$ and is given by

$$\widetilde{a}(-1) = \frac{1}{384}\begin{bmatrix} -28 & 112 \\ 21 & -36 \end{bmatrix}, \quad \widetilde{a}(0) = \frac{1}{384}\begin{bmatrix} 216 & 112 \\ -18 & 60 \end{bmatrix}, \quad \widetilde{a}(1) = \frac{1}{384}\begin{bmatrix} -28 & 0 \\ 330 & 60 \end{bmatrix},$$
$$\widetilde{a}(2) = \frac{1}{384}\begin{bmatrix} 0 & 0 \\ -18 & -36 \end{bmatrix}, \quad \widetilde{a}(3) = \frac{1}{384}\begin{bmatrix} 0 & 0 \\ 21 & 0 \end{bmatrix}.$$

By calculation, we have $\nu_2(\widetilde{a}, 2) \approx 1.117992$. So, the associated 2-refinable function vectors ϕ and $\widetilde{\phi}$ with masks a and $\widetilde{a}$ indeed satisfy the biorthogonal relation in (1.4.1). See Figure 1.8 for the graph of the dual 2-refinable function vector $\widetilde{\phi} = [\widetilde{\phi}_1, \widetilde{\phi}_2]^T$. Note that $\widetilde{\phi}_1(-x) = \widetilde{\phi}_1(x)$ and $\widetilde{\phi}_2(1 - x) = \widetilde{\phi}_2(x)$ for all $x \in \mathbb{R}$.

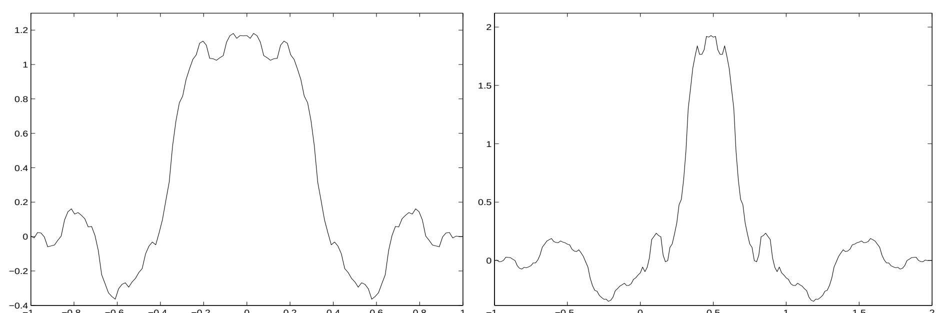

FIGURE 1.8: The graphs of $\widetilde{\phi}_1$ (left) and $\widetilde{\phi}_2$ (right) in Example 1.6 for the interpolating 2-refinable function vector in Example 1.1. $\nu_2(\widetilde{\phi}) \approx 1.117992$, $\widetilde{\phi}_1(-\cdot) = \widetilde{\phi}_1$, and $\widetilde{\phi}_2(1 - \cdot) = \widetilde{\phi}_2$.

Example 1.7. Let $\mathsf{d} = 3$ and $r = 2$. Let a denote the mask given in Example 1.2. By (1.4.4) of Theorem 1.4 with $\widetilde{\kappa} = 2$, we have

$$\widehat{\widetilde{y}}(0) = [1, 1], \quad \frac{-i}{1!}\widehat{\widetilde{y}}^{(1)}(0) = \frac{1}{774}[32, 355].$$

By the CBC algorithm in [22], we have a dual mask $\widetilde{a}$ of a such that $\widetilde{a}$ satisfies the sum rules of order 2. The dual mask $\widetilde{a}$ is supported on $[-2, 3]$ and is given by

$$\begin{aligned}
\widetilde{a}(-2) &= \frac{1}{34884}\begin{bmatrix} 1292 & -4773 \\ -969 & 1866 \end{bmatrix}, \quad \widetilde{a}(-1) = \frac{1}{34884}\begin{bmatrix} 2844 & 9682 \\ 386 & -1284 \end{bmatrix}, \\
\widetilde{a}(0) &= \frac{1}{34884}\begin{bmatrix} 17496 & 8715 \\ -2961 & 2590 \end{bmatrix}, \quad \widetilde{a}(1) = \frac{1}{34884}\begin{bmatrix} 2590 & -2961 \\ 8715 & 17496 \end{bmatrix}, \\
\widetilde{a}(2) &= \frac{1}{34884}\begin{bmatrix} -1284 & 386 \\ 9682 & 2844 \end{bmatrix}, \quad \widetilde{a}(3) = \frac{1}{34884}\begin{bmatrix} 1866 & -969 \\ -4773 & 1292 \end{bmatrix}.
\end{aligned} \tag{1.4.8}$$

By calculation, we have $\nu_2(\widetilde{a}, 3) \approx 0.736519$. Therefore, the associated 3-refinable function vectors ϕ and $\widetilde{\phi}$ with masks a and $\widetilde{a}$ indeed satisfy the biorthogonal relation in (1.4.1). See Figure 1.9 for the graph of the dual 3-refinable function vector $\widetilde{\phi} = [\widetilde{\phi}_1, \widetilde{\phi}_2]^T$. Note that $\widetilde{\phi}_1 = \widetilde{\phi}_2(1/2 - \cdot)$.

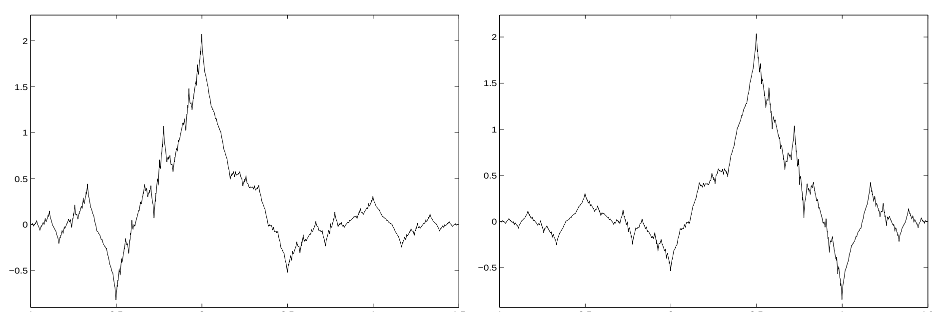

FIGURE 1.9: The graphs of $\widetilde{\phi}_1$ (left) and $\widetilde{\phi}_2$ (right) in Example 1.7 for the interpolating 3-refinable function vector in Example 1.2. $\nu_2(\widetilde{\phi}) \approx 0.736519$ and $\widetilde{\phi}_1 = \widetilde{\phi}_2(1/2 - \cdot)$.

1.5 Conclusions and Remarks

In this chapter, we present in Theorem 1.1 a complete characterization of an interpolating refinable function vector of type $(\mathsf{d}, \Gamma_r, 0)$ in terms of its mask. As a consequence, we have a criterion for orthogonal interpolating d-refinable function vectors in Corollary 1.2. We introduce the notion of an interpolatory mask of type $(\mathsf{d}, \Gamma_r, 0)$ and study its sum rule structure in Theorem 1.3. We address in Section 1.4 how to construct pairs of dual d-refinable function vectors using the CBC algorithm in [22] from the interpolatory masks of type $(\mathsf{d}, \Gamma_r, 0)$ obtained in this chapter. In next chapter, we shall introduce the notation of interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ in high dimensions and shall also characterize such refinable function vectors in terms of their masks.

Symmetry property is one of the most important and desirable properties in wavelet analysis (e.g. [11, 23, 34]). Though we provide several examples of interpolatory masks, we did not address the symmetry properties of a general interpolatory mask of type $(\mathsf{d}, \Gamma_r, 0)$ and its associated refinable function vector. For dilation factor $\mathsf{d} = 2$ and $r = 2$, we can show that an interpolating d-refinable function does possess a certain symmetry pattern once its associated interpolatory mask of type $(2, \Gamma_2, 0)$ has symmetry (see Corollary 2.6). However, when the dilation factor $\mathsf{d} > 2$ and the multiplicity $r > 1$, except for some special cases (e.g. [34]), it seems that little is known in the literature about the connections between the symmetry property of a matrix mask and that of its associated refinable function vector. For an interpolating refinable function vector $\phi = [\phi_1, \ldots, \phi_r]^T$ with an interpolatory mask a of type $(\mathsf{d}, \Gamma_r, 0)$, it is natural that each function ϕ_ℓ, $\ell = 1, \ldots, r$, is symmetric about the point $(\ell - 1)/r$, more precisely, $\phi_\ell(2(\ell-1)/r - \cdot) = \phi_\ell$.

For $\mathsf{d} > 2$ and $r > 1$, it is unclear to us so far under which kind of symmetry conditions on its interpolatory mask a, the interpolating refinable function vector ϕ is guaranteed to possess the desired symmetry property. That is, what is the right symmetry condition for an interpolatory mask of type $(\mathsf{d}, \Gamma_r, 0)$ so that its associated interpolating refinable function vector possesses certain desired symmetry. Nevertheless, when extra conditions imposed on d and r, we do have the following proposition that provides symmetry conditions for an interpolating refinable function vector of type $(\mathsf{d}, \Gamma_r, 0)$ in terms of its mask (see Appendix A for its proof).

Proposition 1.5. *Let $\phi = [\phi_1, \ldots, \phi_r]^T$ be an interpolating d-refinable function vector with mask a. Let $a_1, \ldots, a_r : \mathbb{Z} \to \mathbb{C}$ be defined by $a_\ell(j) := [a(Q_j)]_{\ell, R_j+1}$ for $\ell = 1, \ldots, r$, where $R_j, Q_j \in \mathbb{Z}$ are uniquely determined by $j = rQ_j + R_j, 0 \leqslant R_j \leqslant r-1$. Then*

(1) *if $\phi_\ell = \phi_{r-\ell+1}(\frac{r-1}{r} - \cdot)$ for $\ell = 1, \ldots, r$, then*

$$a_\ell(j) = a_{r-\ell+1}(-j + (r-1)\mathsf{d}), \qquad j \in \mathbb{Z}; \ell = 1, \ldots, r. \qquad (1.5.1)$$

(2) *if $\phi_\ell = \phi_\ell(\frac{2(\ell-1)}{r} - \cdot)$ for $\ell = 1, \ldots, r$, then*

$$a_\ell(j) = a_{r-\ell+1}(-j + 2(\ell-1)\mathsf{d}), \qquad j \in \mathbb{Z}; \ell = 1, \ldots, r. \qquad (1.5.2)$$

Conversely, if $\mathsf{d} - 1 = k_0 r$ for some integer $k_0 \geqslant 1$, (1.5.1) or (1.5.2) implies the symmetry of ϕ as in item (1) or (2), respectively.

Due to the above proposition, we can see that both the interpolating refinable function vectors in Examples 1.2, 1.4, and 1.5 of Section 1.3 have symmetry. One may consider the condition $\mathsf{d} - 1 = k_0 r$ for some integer

$k_0 \geqslant 1$ too restricted and unnatural, yet we point out this condition cannot be removed in the proposition. In fact, we can verify that (1.5.1) holds with $\mathsf{d} = r = 3$ in Example 1.3 in Section 1.3 and from the graph of ϕ, it seems ϕ has symmetry. However, a careful computation shows that ϕ is not symmetric. For example, $-\frac{4433}{177147} = \phi_1(14/27) \neq \phi_3(2/3 - 14/27) = \frac{920}{177147}$.

We mentioned that there is no compactly supported 2-refinable scalar function that is both interpolating and orthogonal, while orthogonality and interpolation can be easily achieved when considering compactly supported d-refinable function vectors. For $\mathsf{d} = r = 2$, Zhou in [73] pointed out there is also no compactly supported 2-refinable function vector that can have orthogonality, interpolation, and symmetry, simultaneously. Under our general setting, we show in Examples 1.4 and 1.5 that we can have these nice properties simultaneously when we consider dilation factors $\mathsf{d} > 2$. Once ϕ has symmetry and orthogonality, we can derive its corresponding orthogonal multiwavelets by employing techniques for matrix extension with symmetry, which we shall give a detailed study in Chapter 3.

In Section 1.4, we discuss how to derive biorthogonal refinable function vectors from an interpolating refinable function vector. Examples 1.6 and 1.7 are two such biorthogonal refinable function vectors derived from two interpolating refinable function vectors (Examples 1.1 and 1.2, respectively). We can see that the interpolating refinable function vectors and their dual biorthogonal refinable function vectors have the same symmetry patterns. Constructing their corresponding biorthogonal multiwavelets with symmetry are of interest and importance in applications. Such constructions can be also reduced to a matrix extension problem. We shall study the biorthogonal matrix extension problem in Chapter 4.

In one dimension, the symmetry patterns of a d-refinable function vectors are somewhat simple (symmetric, antisymmetric) compared to those in higher dimension. In higher dimensions, the symmetry of a refinable function vector becomes highly nontrivial which involves symmetry groups in high dimensions. We shall also study the symmetry of an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ in high dimensions in next chapter.

Chapter 2

Multivariate Interpolating Refinable Function Vectors of Type $(\mathsf{M}, \Gamma_N, h)$

2.1 Introduction

In this chapter, we shall study the interpolating refinable function vectors in a more general setting: interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ in any dimension.

We say that a $d \times d$ integer matrix M is *a dilation matrix* if $\lim_{n\to\infty} \mathsf{M}^{-n} = 0$, that is, all the eigenvalues of M are greater than 1 in modulus. A $d \times d$ dilation matrix is *isotropic* if it is similar to a diagonal matrix $\mathrm{diag}(\sigma_1, \ldots, \sigma_d)$

such that $|\sigma_1| = \cdots = |\sigma_d| = |\det \mathsf{M}|^{1/d}$. An M-*refinable function (or distribution) vector* $\phi = [\phi_1, \ldots, \phi_L]^T$ satisfies the vector refinement equation

$$\phi = |\det \mathsf{M}| \sum_{k\in\mathbb{Z}^d} a(k)\phi(\mathsf{M}\cdot -k), \tag{2.1.1}$$

where $a : \mathbb{Z}^d \to \mathbb{C}^{L\times L}$ is *a (matrix) mask* with multiplicity L for ϕ.

In the frequency domain, (2.1.1) can be rewritten as

$$\widehat{\phi}(\mathsf{M}^T\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad a.e.\ \xi \in \mathbb{R}^d, \tag{2.1.2}$$

where M^T denotes the transpose of the matrix M and for $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\widehat{f}$ is defined to be $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx$, $\xi \in \mathbb{R}^d$, which can be naturally extended to tempered distributions and $L_2(\mathbb{R}^d)$.

In this chapter, we shall generalize the results of univariate interpolating refinable function vectors of type $(\mathsf{d}, \Gamma_r, 0)$ in Chapter 1 to the multivariate refinable function vectors of type $(\mathsf{M}, \Gamma_N, h)$. Before giving the definition, let us introduce two examples. The first example is a univariate 2-refinable function vector $\phi = [\phi_0, \phi_1]^T \in (C^1(\mathbb{R}))^{2\times 1}$, which is a refinable Hermite interpolant from [63], whose mask is given by:

$$a(-1) = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}, a(0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, a(1) = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix}.$$

ϕ has the Hermite interpolation property: $\phi(k) = \delta(k)[1,0]^T, \phi'(k) = \delta(k)[0,1]^T, \forall k \in \mathbb{Z}$, where ϕ' is the first derivative of ϕ. ϕ consists of cubic splines. See Figure 2.1 for its graph.

Another example is a bivariate example of Goodman et al. [18], which

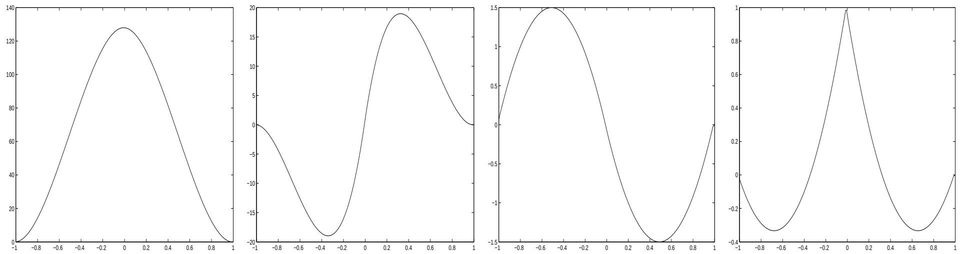

FIGURE 2.1: Refinable Hermite cubic splines. $\phi = [\phi_0, \phi_1]^T$ and $\phi' = [\phi_0', \phi_1']$ (left to right).

obtains an $M_{\sqrt{2}}$-refinable function vector $\phi = [\phi_0, \phi_1]^T$ (see Figure 2.2), where

$$M_{\sqrt{2}} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.1.3}$$

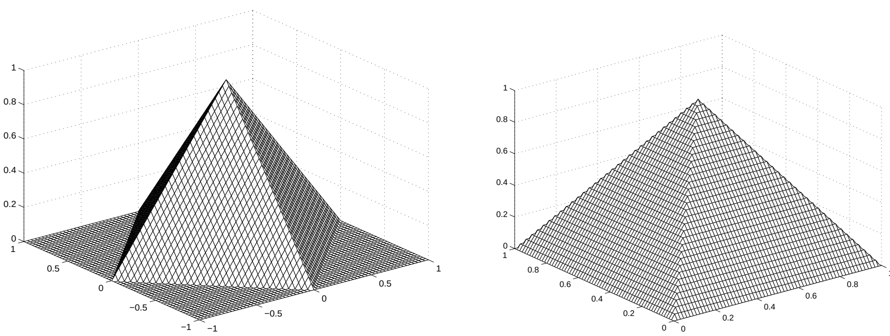

FIGURE 2.2: Two components ϕ_0 (left) and ϕ_1 (right) of the $M_{\sqrt{2}}$-refinable function vector in [18].

From Figure 2.2, it is evident that ϕ is a piecewise linear function vector satisfying the following interpolation property:

$$\phi_0(k) = \delta(k) \quad \text{and} \quad \phi_1((1/2, 1/2)^T + k) = \delta(k), \qquad k \in M_{\sqrt{2}}^{-1}\mathbb{Z}^2. \tag{2.1.4}$$

But ϕ in Figure 2.2 is not a function vector in $(C^1(\mathbb{R}^2))^{2\times 1}$. It is of interest to have compactly supported $M_{\sqrt{2}}$-refinable function vectors satisfying the interpolation property in (2.1.4) and having higher order of smoothness, such as C^1 smoothness.

In a moment, we shall see that the above two examples are just two special cases of our interpolating refinable function vectors of type $(\mathsf{M}, \Gamma_N, h)$.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}_0^d$, we denote $\mu! := \mu_1! \cdots \mu_d!$, $|x| := |x_1| + \cdots + |x_d|$ and $x^\mu := x_1^{\mu_1} \cdots x_d^{\mu_d}$. For a given Hermite order $h \in \mathbb{N}_0$, we denote $O_h := \{\mu : |\mu| = h, \mu \in \mathbb{N}_0^d\}$ and $\Lambda_h := \{\mu : |\mu| \leqslant h, \mu \in \mathbb{N}_0^d\}$. Also, $\#O_h$ and $\#\Lambda_h$ denote the cardinalities of the sets O_h and Λ_h, respectively. It is easy to see that $\#\Lambda_h = \binom{h+d}{d}$. Throughout this chapter, the elements in O_h and Λ_h will be always ordered in such a way that $\nu = (\nu_1, \ldots, \nu_d)$ is less than $\mu = (\mu_1, \ldots, \mu_d)$ if either $|\nu| < |\mu|$ or if $|\nu| = |\mu|, \nu_j = \mu_j$ for $j = 1, \ldots, \ell - 1$ and $\nu_\ell < \mu_\ell$ for some $1 \leqslant \ell \leqslant d$.

Let ∂_j denote the differentiation operator with respect to the j-th coordinate. For $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}_0^d$, ∂^μ is the differentiation operator $\partial_1^{\mu_1} \cdots \partial_d^{\mu_d}$ and $\mathscr{D}^{\Lambda_h} := (\partial^\mu)_{\mu \in \Lambda_h}$ is a $1 \times (\#\Lambda_h)$ row vector.

For a $d \times d$ matrix N, $S(N, O_h)$ is defined to be the following $(\#O_h) \times (\#O_h)$ matrix [25], uniquely determined by

$$\frac{(Nx)^\mu}{\mu!} = \sum_{\nu \in O_h} S(N, O_h)_{\mu,\nu} \frac{x^\nu}{\nu!}, \qquad \mu \in O_h. \tag{2.1.5}$$

Clearly, $S(N, \Lambda_h) := \operatorname{diag}(S(N, O_0), S(N, O_1), \ldots, S(N, O_h))$, which is a $(\#\Lambda_h) \times (\#\Lambda_h)$ matrix. It is obvious that $S(A, O_h)S(B, O_h) = S(AB, O_h)$.

For matrices $A = (a_{i,j})_{1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J}$ and $B = (b_{\ell,k})_{1 \leqslant \ell \leqslant L, 1 \leqslant k \leqslant K}$, the (right) *Kronecker product* $A \otimes B$ is defined to be $A \otimes B := (a_{i,j}B)_{1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J}$; its $((i-1)L + \ell, (j-1)K + k)$-entry is $a_{i,j}b_{\ell,k}$ and can be conveniently denoted by $[A \otimes B]_{i,j;\ell,k}$. Throughout this paper, for an $I \times J$ block matrix A with each block of size $L \times K$, we will use $[A]_{i,j}$ to denote the (i, j)-block of A

and $[A]_{i,j;\ell,k}$ to denote the (ℓ,k)-entry of the block $[A]_{i,j}$. It is easily shown that $(A+B)\otimes C=(A\otimes C)+(B\otimes C)$, $C\otimes(A+B)=(C\otimes A)+(C\otimes B)$, $(A\otimes B)(C\otimes E)=(AC)\otimes(BE)$ and $(A\otimes B)^T=A^T\otimes B^T$.

Let M be a $d\times d$ dilation matrix. Let N be a $d\times d$ invertible integer matrix. Then $\mathbb{Z}^d\subseteq N^{-1}\mathbb{Z}^d$. We denote Γ_N an ordered complete set of representatives of the cosets of $[N^{-1}\mathbb{Z}^d]/\mathbb{Z}^d$ with the first element of Γ_N being $\mathbf{0}$. Naturally, we require that M and N should be compatible by imposing the condition $\mathsf{M}N^{-1}\mathbb{Z}^d\subseteq N^{-1}\mathbb{Z}^d$. This is equivalent to saying that $N\mathsf{M}N^{-1}$ is an integer matrix. Let $h\in\mathbb{N}_0$ be a Hermite order. Let $\phi=(\phi_\gamma)_{\gamma\in\Gamma_N}$ be a $(\#\Gamma_N)(\#\Lambda_h)\times 1$ column vector of compactly supported distributions with each $\phi_\gamma=(\phi_{\gamma,\mu})_{\mu\in\Lambda_h}$ being a $(\#\Lambda_h)\times 1$ column vector. We say that ϕ is an *interpolating refinable function vector of type* (M,Γ_N,h) if

(i) ϕ is an M-refinable function vector associated with a mask a: $\widehat{\phi}(\mathsf{M}^T\xi)=\widehat{a}(\xi)\widehat{\phi}(\xi)$;

(ii) ϕ satisfies the Hermite interpolation property:

$$\begin{aligned}&\phi\in(C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}\ \text{ and for all } \beta,\gamma\in\Gamma_N, k\in\mathbb{Z}^d,\\&[\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\beta+k)=\delta(k)\delta(\beta-\gamma)I_{\#\Lambda_h}.\end{aligned}\tag{2.1.6}$$

The above notation seems a little bit complicated, but it essentially says that the components of the function vector ϕ interpolate all the derivatives up to order h on the lattice $N^{-1}\mathbb{Z}^d$. Let ϕ be an interpolating refinable function vector of type (M,Γ_N,h). For a function $f\in C^h(\mathbb{R}^d)$, defining

$$\widetilde{f}(x):=\sum_{\gamma\in\Gamma_N}\sum_{\mu\in\Lambda_h}\sum_{k\in\mathbb{Z}^d}[\partial^\mu f](k+\gamma)\phi_{\gamma,\mu}(x-k),\qquad x\in\mathbb{R}^d,\tag{2.1.7}$$

then $\partial^\mu \widetilde{f}(x) = \partial^\mu f(x)$ for all $\mu \in \Lambda_h$ and $x \in N^{-1}\mathbb{Z}^d$; that is, $\widetilde{f}$ agrees with f on the lattice $N^{-1}\mathbb{Z}^d$ with all derivatives up to order h.

Obviously, the sinc function is an interpolating refinable function of type $(2, \Gamma_1, 0)$. The refinable Hermite interpolant introduced above is an interpolating refinable function vector of type $(2, \Gamma_1, 1)$. The function vector from Goodman [18] is an interpolating refinable function vector of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 0)$. More generally, an M-refinable function ϕ satisfying $\phi|_{\mathbb{Z}^d} = \delta$ is simply an interpolating function of type $(\mathsf{M}, \Gamma_{I_d}, 0)$. All examples in [62, 73] are just interpolating refinable function vectors of type $(2, \Gamma_2, 0)$ and the one-dimensional examples in Chapter 1 correspond to interpolating refinable function vectors of type $(\mathsf{d}, \Gamma_r, 0)$ for any positive integers $\mathsf{d} > 1$ and $r \geqslant 1$. All examples of refinable Hermite interpolants from [36] are interpolating refinable function vectors of type $(\mathsf{M}, \Gamma_I, h)$ for some dilation matrix M and Hermite order h. We shall see more general examples of interpolating refinable function vectors of type $(\mathsf{M}, \Gamma_N, h)$ for some $N \neq I$ and $h \geqslant 0$.

The structure of this chapter is as follows. In Section 2.2, we shall introduce definitions for the sum rule of a mask, the quantity $\nu_p(a, \mathsf{M})$, the convergence of a cascade algorithm, etc. In Section 2.3, for an isotropic dilation matrix M, we shall characterize an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ in terms of its mask and study the underlying sum rule structure of its interpolatory mask of type $(\mathsf{M}, \Gamma_N, h)$. Due to the importance of the symmetry property in applications, we shall also discuss the symmetry property of an interpolating M-refinable function vector and its interpolatory mask. We shall postpone proofs of the results in this section until Section 2.5 for the sake of readability. In Section 2.4, using the results in Section 2.3, for any dilation factor d, integers $r \in \mathbb{N}$, and $h \in \mathbb{N}_0$, we construct a family of univariate interpolatory masks of type $(\mathsf{d}, \Gamma_r, h)$ with increasing orders of sum rules. Such a family includes the family of the famous Deslauriers-Dubuc interpolatory masks in [12] and the family of Hermite interpolatory masks in [22] as special cases. Conclusions and remarks shall be given in the last section.

2.2 Auxiliary Results

A *(vector) cascade algorithm* associated with a matrix mask a and a dilation matrix M is defined by $Q^n_{a,\mathsf{M}}\phi_0$, $n = 1, \ldots,$ where

$$Q_{a,\mathsf{M}}\phi_0 := |\det \mathsf{M}| \sum_{k \in \mathbb{Z}^d} a(k)\phi_0(\mathsf{M} \cdot -k)$$

and ϕ_0 is an appropriate initial function vector (see definition (2.2.4)).

For a matrix mask a with multiplicity L, we say that a satisfies the *sum rules* of order κ with a dilation matrix M if there exists a sequence $y \in (\ell_0(\mathbb{Z}^d))^{1\times L}$ such that $\widehat{y}(0) \neq 0$ and

$$\begin{aligned} &\partial^\mu[\widehat{y}(\mathsf{M}^T\cdot)\widehat{a}(\cdot)](0) = \partial^\mu \widehat{y}(0) \qquad \forall\, |\mu| < \kappa,\ \mu \in \mathbb{N}_0^d. \\ &\partial^\mu[\widehat{y}(\mathsf{M}^T\cdot)\widehat{a}(\cdot)](2\pi\gamma) = 0 \qquad \forall\, |\mu| < \kappa,\ \gamma \in [(\mathsf{M}^T)^{-1}\mathbb{Z}^d] \setminus \mathbb{Z}^d. \end{aligned} \tag{2.2.1}$$

Let Ω_M be a complete set of representatives of the cosets $\mathbb{Z}^d/[\mathsf{M}\mathbb{Z}^d]$ such that $\mathbf{0} \in \Omega_\mathsf{M}$. Then one can show that (2.2.1) is equivalent to

$$\widehat{y}(\mathsf{M}^T\xi)\widehat{a}^\omega(\xi) = |\det \mathsf{M}|^{-1}\widehat{y}(\xi) + O(\|\xi\|^\kappa), \qquad \xi \to 0,\ \omega \in \Omega_{\mathsf{M}^T}, \tag{2.2.2}$$

where $\widehat{a}^\omega(\xi) := \sum_{k\in\mathbb{Z}^d} a(\omega + \mathsf{M}k)e^{-i\xi\cdot(\omega+\mathsf{M}k)}$ is called the *coset* of $\widehat{a}(\xi)$.

The convolution of two sequences u and v is defined to be

$$[u * v](j) := \sum_{k\in\mathbb{Z}^d} u(k)v(j-k), \qquad u \in (\ell_0(\mathbb{Z}^d))^{r\times m}, \quad v \in (\ell_0(\mathbb{Z}^d))^{m\times n},$$

Clearly, $\widehat{u*v} = \widehat{u}\widehat{v}$. For $y \in (\ell_0(\mathbb{Z}^d))^{1\times L}$ and a positive integer κ, we say that a function vector $f = [f_1, \dots, f_L]^T \in (W_p^\kappa(\mathbb{R}^d))^{L\times 1}$ satisfies the *moment conditions* of order $\kappa+1$ with respect to y if

$$\widehat{y}(0)\widehat{f}(0) = 1 \text{ and } \partial^\mu[\widehat{y}(\cdot)\widehat{f}(\cdot)](2\pi\beta) = 0, \ \forall|\mu| \leqslant \kappa; \beta \in \mathbb{Z}^d\backslash\{0\}. \tag{2.2.3}$$

The space $\mathcal{F}_{\kappa,y,p}$ of all *appropriate initial function vectors* in the Sobolev space $(W_p^\kappa(\mathbb{R}))^{L\times 1}$ depending on κ, y, p is defined by

$$\begin{aligned}\mathcal{F}_{\kappa,y,p} :=\{f \in (W_p^\kappa(\mathbb{R}^d))^{L\times 1} \ : \ & f \text{ is compactly supported and} \\ & \text{satisfies the moment conditions of order } \kappa+1 \text{ with} \\ & \text{respect to } y\}.\end{aligned} \tag{2.2.4}$$

We next introduce the quantity $\nu_p(a, \mathsf{M})$. For $y \in (\ell_0(\mathbb{Z}^d))^{1\times L}$ and a positive integer κ, as in [24], we define the space $\mathcal{V}_{\kappa,y}$ by

$$\mathcal{V}_{\kappa,y} := \{v \in (\ell_0(\mathbb{Z}^d))^{L\times 1} \ : \ \partial^\mu[\widehat{y}(\cdot)\widehat{v}(\cdot)](0) = 0 \quad \forall\, |\mu| < \kappa, \ \mu \in \mathbb{N}_0^d\}. \tag{2.2.5}$$

By convention, $\mathcal{V}_{0,y} := (\ell_0(\mathbb{Z}^d))^{L\times 1}$. The above equations in (2.2.1), (2.2.2), and (2.2.5) depend only on the values $\partial^\mu\widehat{y}(0)$, $|\mu| < \kappa$. For a mask a with multiplicity L, a sequence $y \in (\ell_0(\mathbb{Z}^d))^{1\times L}$, a dilation matrix M, and $\kappa \in \mathbb{N}_0$, we define

$$\rho_\kappa(a, \mathsf{M}, y, p) := \sup\left\{ \limsup_{n\to\infty} \|a_n * v\|^{1/n}_{(\ell_p(\mathbb{Z}^d))^{L\times 1}} \ : \ v \in \mathcal{V}_{\kappa,y}\right\}, \tag{2.2.6}$$

where $\widehat{a_n}(\xi) := \widehat{a}((\mathsf{M}^T)^{n-1}\xi)\cdots\widehat{a}(\mathsf{M}^T\xi)\widehat{a}(\xi)$. For $1 \leqslant p \leqslant \infty$, define

$$\begin{aligned}\rho(a, \mathsf{M}, p) := \inf\{\rho_\kappa(a, \mathsf{M}, y, p) \ :\ &(2.2.1) \text{ holds for some } \kappa \in \mathbb{N}_0 \\ &\text{and some } y \in (\ell_0(\mathbb{Z}^d))^{1\times L} \text{ with } \widehat{y}(0) \neq 0\}.\end{aligned} \tag{2.2.7}$$

The quantity $\nu_p(a, \mathsf{M})$ is then defined by:

$$\nu_p(a, \mathsf{M}) := -\log_{\rho(\mathsf{M})}\left[|\det \mathsf{M}|^{1-1/p}\rho(a, \mathsf{M}, p)\right], \qquad 1 \leqslant p \leqslant \infty, \tag{2.2.8}$$

where $\rho(\mathsf{M})$ denotes the spectral radius of the matrix M. Up to a scalar multiplicative constant, the vectors $\partial^\mu \widehat{y}(0),\ \mu \in \mathbb{N}_0^d$ are quite often uniquely determined ([24] and Theorem 2.2 of this chapter).

The quantity $\nu_p(a, \mathsf{M})$ characterizes the convergence of a vector cascade algorithm in a Sobolev space and the L_p smoothness of a refinable function vector. As in [24, Theorem 4.3], the vector cascade algorithm associated with mask a and an isotropic dilation matrix M converges in the Sobolev space $W_p^\kappa(\mathbb{R}^d) := \{f \in L_p(\mathbb{R}^d) : \partial^\mu f \in L_p(\mathbb{R}^d)\ \forall\ |\mu| \leqslant \kappa\}$ for any initial function vector $\phi_0 \in \mathcal{F}_{\kappa,y,p}$ if and only if $\nu_p(a, \mathsf{M}) > \kappa$. Generally, $\nu_p(a, \mathsf{M}) \leqslant \nu_p(\phi)$ always holds, where $\nu_p([f_1, \ldots, f_L]^T) := \min_{1\leqslant \ell \leqslant L} \nu_p(f_\ell)$ and for $f \in L_p(\mathbb{R}^d)$,

$$\nu_p(f) := \sup\{n + \nu : ||\partial^\mu f - \partial^\mu f(\cdot - t)||_{L_p(\mathbb{R}^d)} \leqslant C_f |t|^\nu \quad \forall |\mu| = n; t \in \mathbb{R}^d\}. \tag{2.2.9}$$

Also $\nu_p(\phi) = \nu_p(a, \mathsf{M})$ if the shifts of the refinable function vector ϕ associated with a mask a and an isotropic dilation matrix M are stable in $L_p(\mathbb{R}^d)$ (see (2.5.1) for definiton). Furthermore, we have $\nu_p(a, \mathsf{M}) \geqslant \nu_q(a, \mathsf{M}) \geqslant \nu_p(a, \mathsf{M}) + (1/q - 1/p)\log_{\rho(\mathsf{M})}|\det \mathsf{M}|$ for $1 \leqslant p \leqslant q \leqslant \infty$. In particular, we have $\nu_2(a, \mathsf{M}) \geqslant \nu_\infty(a, \mathsf{M}) \geqslant \nu_2(a, \mathsf{M}) - d/2$ when M is isotropic. For a finitely supported matrix mask a, the quantity $\nu_2(a, \mathsf{M})$ can be numerically computed by finding the spectral radius of certain finite matrix using an algorithm in [45] (also see [25] for computing $\nu_2(a, \mathsf{M})$ using the symmetry of the mask a). For more about vector cascade algorithms and smoothness of refinable function vectors, see [5, 15, 21, 22, 24, 31, 33, 36, 44, 49, 50, 51, 65, 71, 72] and many references therein.

2.3 Analysis of Interpolating Refinable Function Vectors

We have the following result characterizing an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ in terms of its mask.

Theorem 2.1. *Let $h \in \mathbb{N}_0$ and M be a $d \times d$ dilation matrix. When $h > 0$, we further assume that M is isotropic. Let N be a $d\times d$ invertible integer matrix such that $N\mathsf{M}N^{-1}$ is also an integer matrix. Let Γ_N be a given ordered complete set of representatives of $[N^{-1}\mathbb{Z}^d]/\mathbb{Z}^d$ with the first element of Γ_N being $\mathbf{0}$. Let $\phi = (\phi_\gamma)_{\gamma\in\Gamma_N}$ be a $(\#\Gamma_N)(\#\Lambda_h)\times 1$ column vector of compactly supported distributions with each $\phi_\gamma = (\phi_{\gamma,\alpha})_{\alpha\in\Lambda_h}$ being a $\#\Lambda_h \times 1$ column vector and $\widehat{\phi}(\mathsf{M}^T\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$, where $a : \mathbb{Z}^d \to \mathbb{C}^{(\#\Gamma_N)(\#\Lambda_h)\times(\#\Gamma_N)(\#\Lambda_h)}$ is a $(\#\Gamma_N) \times (\#\Gamma_N)$ block matrix mask for ϕ. Then ϕ is an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ if and only if the following statements hold:*

(1) $[(1, 1, \dots, 1)\otimes(1, 0, \dots, 0)]\widehat{\phi}(0) = 1$ (This is a normalization condition on ϕ)*;*

(2) $\nu_\infty(a, \mathsf{M}) > h$*;*

(3) *The mask a is an interpolatory mask of type $(\mathsf{M}, \Gamma_N, h)$; that is,*

(i) *The mask a satisfies the following condition:*

$$[a(\mathsf{M}k + [\mathsf{M}\alpha]_{\Gamma_N})]_{\gamma,\langle \mathsf{M}\alpha\rangle_{\Gamma_N}} = |\det \mathsf{M}|^{-1} S(\mathsf{M}^{-1}, \Lambda_h)\delta(k)\delta(\alpha-\gamma), \quad \forall\, \alpha, \gamma \in \Gamma_N,\ k \in \mathbb{Z}^d, \tag{2.3.1}$$

where $[\mathsf{M}\alpha]_{\Gamma_N} \in \mathbb{Z}^d$ and $\langle \mathsf{M}\alpha\rangle_{\Gamma_N} \in \Gamma_N$ are uniquely determined by the relation $\mathsf{M}\alpha = [\mathsf{M}\alpha]_{\Gamma_N} + \langle \mathsf{M}\alpha\rangle_{\Gamma_N}$.

(ii) *The mask a satisfies the sum rules of order $h+1$ with a $1 \times (\#\Gamma_N)(\#\Lambda_h)$ row vector $y = (y_\gamma)_{\gamma\in\Gamma_N}$ in $(\ell_0(\mathbb{Z}^d))^{1\times(\#\Gamma_N)(\#\Lambda_h)}$ such that*

$$\widehat{y_\gamma}(\xi) = e^{i\gamma\cdot\xi}((i\xi)^\nu)_{\nu\in\Lambda_h} + O(\|\xi\|^{h+1}),\ \xi \to 0,\ \gamma \in \Gamma_N, \tag{2.3.2}$$

or equivalently,

$$\widehat{y}(\xi) = (e^{i\gamma\cdot\xi})_{\gamma\in\Gamma_N} \otimes ((i\xi)^\nu)_{\nu\in\Lambda_h} + O(\|\xi\|^{h+1}),\ \xi \to 0. \tag{2.3.3}$$

To improve the readability, we shall present the proof of Theorem 2.1 in Section 2.5. We mention that the sufficiency part of Theorem 2.1 still holds without assuming that M is isotropic. In general, the conditions in (2.3.1) and (2.2.1) with $\kappa = h+1$ cannot guarantee that up to a scalar multiplicative constant, the vector $\widehat{y}$ in (2.2.1) must satisfy (2.3.3). However, if in addition $\nu_\infty(a, \mathsf{M}) > h$, then up to a scalar multiplicative constant, the vector $\widehat{y}$ in (2.2.1) must be unique and satisfy (2.3.3).

As we discussed before, to design an interpolatory mask of type $(\mathsf{M}, \Gamma_N, h)$ with a preassigned order of sum rules, it is of importance to investigate its sum rule structure; in particular, it is of interest to know the values

$\partial^\mu \widehat{y}(0), |\mu| < \kappa$ in advance so that the nonlinear equations in (2.2.1) will become linear equations. We have the following result concerning the structure of the y vector for an interpolatory mask of type $(\mathsf{M}, \Gamma_N, h)$, whose proof will also be given in Section 2.5.

Theorem 2.2. *Let* M *be a* $d \times d$ *dilation matrix and* N, Γ_N, h *as in Theorem 2.1. Suppose that* a *is an interpolatory mask of type* $(\mathsf{M}, \Gamma_N, h)$ *satisfying the sum rules of order* κ *with* $\kappa > h$ *in* (2.2.1) *with a sequence* $y \in (\ell_0(\mathbb{Z}^d))^{1\times(\#\Gamma_N)(\#\Lambda_h)}$ *satisfying* (2.3.3)*. Let* $\sigma := (\sigma_1, \ldots, \sigma_d)^T$*, where* $\sigma_1, \ldots, \sigma_d$ *are all the eigenvalues of* M*. If*

$$\sigma^\mu \notin \{\sigma^\nu : \nu \in \Lambda_h\}, \qquad \forall\, h < |\mu| < \kappa \tag{2.3.4}$$

((2.3.4) clearly holds when M *is an isotropic dilation matrix), then we must have*

$$\widehat{y}(\xi) = \left(e^{i\gamma\cdot\xi}\right)_{\gamma\in\Gamma_N} \otimes ((i\xi)^\nu)_{\nu\in\Lambda_h} + O(\|\xi\|^\kappa), \qquad \xi \to 0. \tag{2.3.5}$$

In high dimensions, symmetry becomes important for at least two reasons. One is that wavelets and refinable function vectors with symmetry generally provide better results in applications. Another reason is that when designing a matrix mask, symmetry significantly reduces the number of free parameters in the system of linear equations, especially in higher dimensions. In the following, we will discuss symmetry in high dimensions and characterize an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ with symmetry in terms of its mask. With the symmetry condition on the masks and the vector y determined in Theorem 2.2, obtaining interpolatory masks of type $(\mathsf{M}, \Gamma_N, h)$ with high orders of sum rules becomes far more easy (see examples in this section and the following sections).

Unlike the symmetry in dimension one, in which case a function is either symmetric or antisymmetric about some point, symmetry of a function in high dimensions $\mathbb{R}^d$ is closely related to a symmetry group. Let G be a finite set of $d \times d$ integer matrices. We say that G is a *symmetry group with respect to a dilation matrix* M ([25]) if G forms a group under matrix multiplication and

$$|\det E| = 1 \quad \text{and} \quad \mathsf{M}E\mathsf{M}^{-1} \in G \qquad \forall\, E \in G. \tag{2.3.6}$$

For dimension $d = 1$, there is only one nontrivial symmetry group $G = \{-1, 1\}$ with respect to any dilation factor $\mathsf{d} > 1$. In dimension two, two commonly used symmetry groups are D_4 and D_6 for the quadrilateral and triangular meshes, respectively:

$$\begin{aligned} D_4 &:= \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}, \\ D_6 &:= \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \right. \\ &\qquad \left. \pm \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right\}. \end{aligned} \tag{2.3.7}$$

It is easy to verify that D_4 is a symmetry group with respect to the dilation matrix $M_{\sqrt{2}}$ while D_6 is a symmetry group with respect to the dilation matrix $M_{\sqrt{3}} := \left[\begin{smallmatrix} 2 & -1 \\ 1 & -2 \end{smallmatrix}\right]$.

Let G be a symmetry group with respect to a dilation matrix M. Let $\phi = (\phi_\gamma)_{\gamma \in \Gamma_N}$ be an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$.

We say that ϕ is *G-symmetric* if

$$\phi_\beta(E(\cdot - \beta) + \beta) = S(E, \Lambda_h)\phi_\beta \qquad \forall\, E \in G,\ \beta \in \Gamma_N. \tag{2.3.8}$$

For a G-symmetric interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$, we have the following result on characterizing such a refinable function vector in terms of its mask. We shall leave its proof to Section 2.5 as well.

Theorem 2.3. *Let $\mathsf{M}, N, \Gamma_N, h$ and Λ_h be as in Theorem 2.1. Let $\phi = (\phi_\gamma)_{\gamma\in\Gamma_N}$ be an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ with a matrix mask a. Let G be a symmetry group with respect to M. If*

$$E\Gamma_N \subset \Gamma_N + \mathbb{Z}^d \qquad \forall\, E \in G \tag{2.3.9}$$

and ϕ is G-symmetric, then the mask a is (G, M)-symmetric:

$$[a(j)]_{\beta,\alpha} = S(E^{-1}, \Lambda_h)[a(\mathsf{M}E\mathsf{M}^{-1}j + [J_{E,\alpha,\beta}]_{\Gamma_N})]_{\beta,\langle J_{E,\alpha,\beta}\rangle_{\Gamma_N}} S(\mathsf{M}E\mathsf{M}^{-1}, \Lambda_h), \quad \forall j \in \mathbb{Z}^d,\ \alpha, \beta \in \Gamma_N; E \in G, \tag{2.3.10}$$

where

$$J_{E,\alpha,\beta} := \mathsf{M}E\mathsf{M}^{-1}\alpha + \mathsf{M}(I_d - E)\beta, \tag{2.3.11}$$

and $[J_{E,\alpha,\beta}]_{\Gamma_N} \in \mathbb{Z}^d$, $\langle J_{E,\alpha,\beta}\rangle_{\Gamma_N} \in \Gamma_N$ are uniquely determined by the relation $J_{E,\alpha,\beta} = [J_{E,\alpha,\beta}]_{\Gamma_N} + \langle J_{E,\alpha,\beta}\rangle_{\Gamma_N}$.

Conversely, if (2.3.10) holds and

$$(I_d - E)\Gamma_N \subset \mathbb{Z}^d \qquad \forall\, E \in G, \tag{2.3.12}$$

then ϕ is G-symmetric.

Note that (2.3.12) implies (2.3.9). So, if (2.3.12) is satisfied, then an interpolating refinable function vector ϕ of type $(\mathsf{M}, \Gamma_N, h)$ with a mask a and a dilation matrix M is G-symmetric if and only if the mask a is (G, M)-symmetric. In dimension one, it is evident that (2.3.12) is satisfied if $N = 2$ and $G = \{-1, 1\}$ (see Corollary 2.6).

To illustrate the results of this section, we present two examples. Note that D_4 is a symmetry group with respect to $M_{\sqrt{2}}$ and (2.3.12) is satisfied for $G = D_4$ and $N = M_{\sqrt{2}}$.

The example in Goodman [18] is an interpolating refinable function vector of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 0)$, but it is not in $(C^1(\mathbb{R}^2))^{2\times 1}$. In the following, we present a C^1 interpolating refinable function vector of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 0)$.

Example 2.1. Let $\mathsf{M} = N = M_{\sqrt{2}}$, where $M_{\sqrt{2}}$ is defined in (2.1.3). Then $\Gamma_N = \{(0,0), (\frac{1}{2}, \frac{1}{2})\}$. Let a be an interpolatory mask of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 0)$ with support $[-2,1] \times [-2,1]$ and of $(M_{\sqrt{2}}, D_4)$-symmetry (see (2.3.10)). Then a satisfies the sum rules of order 2 and is given by:

$$a(-2,-2) = \begin{bmatrix} 0 & t_3 \\ 0 & 0 \end{bmatrix}, \quad a(-2,-1) = \begin{bmatrix} 0 & t_5 \\ 0 & 0 \end{bmatrix}, \quad a(-2,0) = \begin{bmatrix} 0 & t_5 \\ 0 & 0 \end{bmatrix},$$

$$a(-2,1) = \begin{bmatrix} 0 & t_3 \\ 0 & 0 \end{bmatrix}, \quad a(-1,-2) = \begin{bmatrix} 0 & t_5 \\ 0 & t_4 \end{bmatrix}, \quad a(-1,1) = \begin{bmatrix} 0 & t_5 \\ 0 & t_4 \end{bmatrix},$$

$$a(1,-2) = \begin{bmatrix} 0 & t_3 \\ 0 & t_2 \end{bmatrix}, \quad a(1,-1) = \begin{bmatrix} 0 & t_5 \\ 0 & t_1 \end{bmatrix}, \quad a(1,0) = \begin{bmatrix} 0 & t_5 \\ \frac{1}{2} & t_1 \end{bmatrix},$$

$$a(1,1) = \begin{bmatrix} 0 & t_3 \\ 0 & t_2 \end{bmatrix}, \quad a(0,-2) = \begin{bmatrix} 0 & t_5 \\ 0 & t_2 \end{bmatrix}, \quad a(0,1) = \begin{bmatrix} 0 & t_5 \\ 0 & t_2 \end{bmatrix},$$

$$a(-1,0) = \begin{bmatrix} 0 & \frac{1}{4} - t_3 - 2t_5 - 2t_2 - t_4 - t_1 \\ 0 & t_2 \end{bmatrix},$$
$$a(-1,-1) = \begin{bmatrix} 0 & \frac{1}{4} - t_3 - 2t_5 - 2t_2 - t_4 - t_1 \\ 0 & t_2 \end{bmatrix},$$
$$a(0,-1) = \begin{bmatrix} 0 & \frac{1}{4} - t_3 - 2t_5 - 2t_2 - t_4 - t_1 \\ 0 & t_1 \end{bmatrix},$$
$$a(0,0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} - t_3 - 2t_5 - 2t_2 - t_4 - t_1 \\ 0 & t_1 \end{bmatrix},$$

where $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}$ are free parameters. When $t_2 = t_3 = t_4 = t_5 = 0$, the mask a is supported on $[-1,1] \times [-1,0]$. If in addition $t_1 = 0$, then we have $\nu_2(a, \mathsf{M}) = 1.5$ (and therefore, $\nu_\infty(a, \mathsf{M}) \geqslant \nu_2(a, \mathsf{M}) - 1 \geqslant 0.5$) and this is the mask for the interpolating $M_{\sqrt{2}}$-refinable function vector given in Goodman [18] (See Figure 2.2).

When $t_1 = \frac{1}{4} + 8t_3, t_2 = t_4 = 0, t_5 = -\frac{1}{32} - t_3$, the mask a satisfies the sum rules of order 4. If in addition $t_3 = -\frac{5}{256}$, we have $\nu_2(a, M_{\sqrt{2}}) \approx 2.535219$. Therefore, $\nu_\infty(a, M_{\sqrt{2}}) \geqslant \nu_2(a, M_{\sqrt{2}}) - 1 = 1.535219 > 1$. By Theorem 2.1, its associated $M_{\sqrt{2}}$-refinable function vector $\phi = [\phi_{(0,0)}, \phi_{(\frac{1}{2},\frac{1}{2})}]^T$ is an interpolating refinable function vector in $(C^1(\mathbb{R}^2))^{2\times 1}$ of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 0)$. See Figure 2.3.

Example 2.2. Let $\mathsf{M} = N = M_{\sqrt{2}}$ and $h = 1$. Then

$$\Lambda_h = \{(0,0), (0,1), (1,0)\} \quad \text{and} \quad \Gamma_N = \{(0,0), (\frac{1}{2}, \frac{1}{2})\}.$$

Let a (with multiplicity 6) be an interpolatory mask of type $(M_{\sqrt{2}}, \Gamma_N, h)$. Suppose that a is $(M_{\sqrt{2}}, D_4)$-symmetric and supported on $[-1,1] \times [-1,0]$.

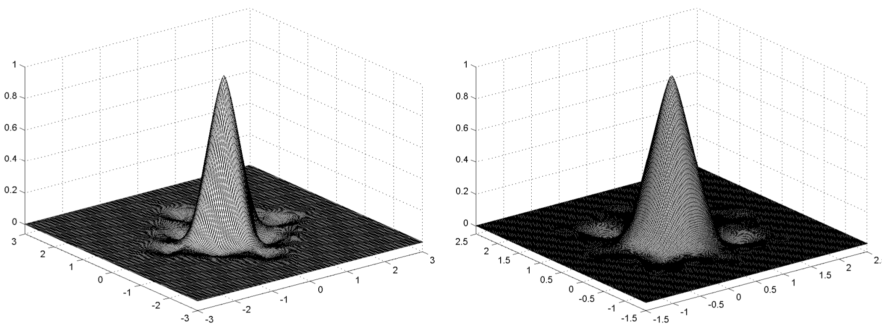

FIGURE 2.3: The graphs of $\phi_{(0,0)}$ (left) and $\phi_{(\frac{1}{2},\frac{1}{2})}$ (right) of the D_4-symmetric interpolating refinable function vector of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 0)$ in Example 2.1 for $t_1 = \frac{3}{32}, t_2 = t_4 = 0, t_3 = -\frac{5}{256}$, and $t_5 = -\frac{3}{256}$.

We obtain an $(M_{\sqrt{2}}, D_4)$-symmetric interpolatory mask a of type $(M_{\sqrt{2}}, \Gamma_N, h)$ which satisfies the sum rules of order 4 and is given by:

$$a(-1,-1) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 64t & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -16t & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad a(-1,0) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 64t & -6 & 6 \\ 0 & 0 & 0 & -16t & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a(0,-1) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 64t & 6 & -6 \\ 0 & 0 & 0 & 16t & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4-64t & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16t-1 & -1 & -1 \end{bmatrix}, \quad a(0,0) = \frac{1}{16}\begin{bmatrix} 8 & 0 & 0 & 64t & -6 & -6 \\ 0 & -4 & 4 & 0 & 0 & 0 \\ 0 & 4 & 4 & 16t & -1 & -1 \\ 0 & 0 & 0 & 4-64t & -6 & 6 \\ 0 & 0 & 0 & 16t-1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a(1,-1) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4-64t & 6 & -6 \\ 0 & 0 & 0 & 1-16t & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad a(1,0) = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 4-64t & -6 & -6 \\ 0 & -4 & 4 & 0 & 0 & 0 \\ 0 & 4 & 4 & 1-16t & -1 & -1 \end{bmatrix},$$

where $t \in \mathbb{R}$ is a free parameter. For $t = \frac{3}{128}$, we have $\nu_2(a, M_{\sqrt{2}}) = 2.5$. Therefore, $\nu_\infty(a, M_{\sqrt{2}}) \geqslant \nu_2(a, M_{\sqrt{2}}) - 1 = 1.5 > 1$. By Theorem 2.1, its associated $M_{\sqrt{2}}$-refinable function vector ϕ is an interpolating refinable function vector of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 1)$. See Figure 2.4 for the graph of ϕ with $t = \frac{3}{128}$.

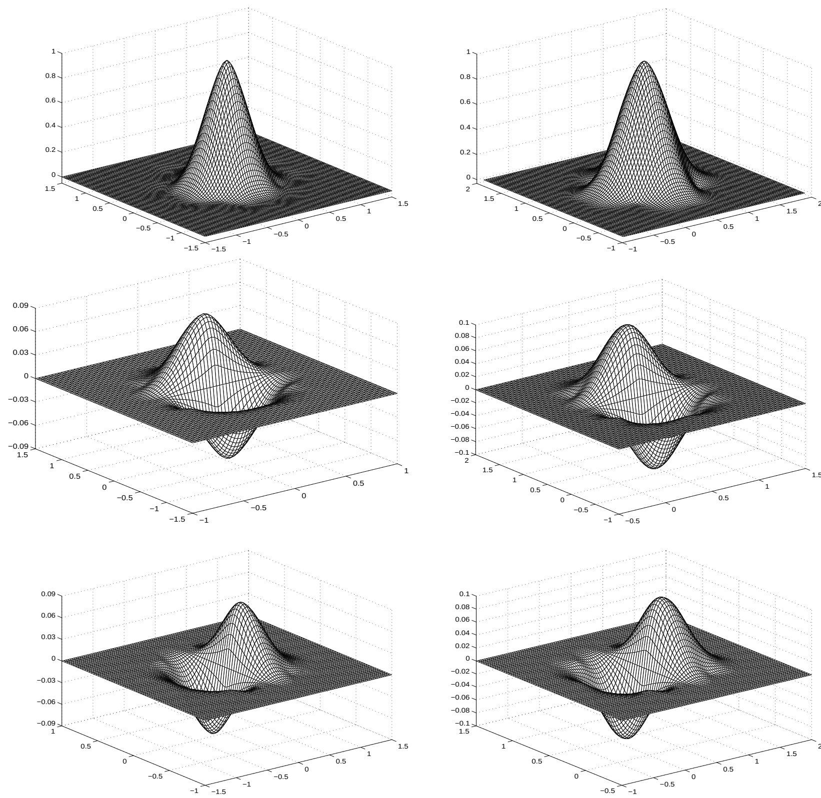

FIGURE 2.4: The graphs of $\phi_{(0,0),\mu}$ (left) and $\phi_{(\frac{1}{2},\frac{1}{2}),\mu}$ (right), $\mu \in \Lambda_h$ in the D_4-symmetric interpolating refinable function vector of type $(M_{\sqrt{2}}, \Gamma_{M_{\sqrt{2}}}, 1)$ in Example 2.2 with $t = \frac{3}{128}$.

2.4 Construction of Univariate Interpolating Refinable Function Vectors

Based on the results in Section 2.3 of this chapter, in this section we shall present a family of interpolatory masks of type $(\mathsf{d}, \Gamma_r, h)$ (more precisely, of type $(\mathsf{d}, \{0, \frac{1}{r}, \ldots, \frac{r-1}{r}\}, h)$ with increasing orders of sum rules in dimension one. Here in dimension one, $\mathsf{d} > 1$ is the dilation factor, $h \in \mathbb{N}_0$ is the order of derivative, and $r \geqslant 1$ is an integer. We follow the idea of CBC similar to [22].

Recall that $[A]_{i,j}$ is the (i,j)-block of a block matrix A and $[A]_{i,j;\ell,k}$ is the (ℓ,k)-entry of the block $[A]_{i,j}$. Let a be an interpolatory mask of type $(\mathsf{d}, \Gamma_r, h)$ in dimension one. a can be viewed as an $r \times r$ block matrix with each block of size $(h+1) \times (h+1)$. For $\ell = 0, 1, \ldots, r-1$, let $E_{\ell+1} := [\mathbf{0}, \ldots, \mathbf{0}, I_{h+1}, \mathbf{0}, \ldots, \mathbf{0}]^T$ be an $r \times 1$ block matrix with each block of size $(h+1) \times (h+1)$ and its nonzero block is located at the $(\ell+1)$-th position. Then (2.3.1) and (2.3.3) in dimension one become

(1) a satisfies the following condition:

$$[a(\mathsf{d}k + R_\ell)]_{:,Q_\ell+1} = \mathsf{d}^{-1}\delta(k)E_{\ell+1}D, \tag{2.4.1}$$

where $D := \operatorname{diag}(1, \mathsf{d}^{-1}, \ldots, \mathsf{d}^{-h})$, $R_\ell := \lfloor \frac{\mathsf{d}\ell}{r} \rfloor$ and $Q_\ell := r(\frac{\mathsf{d}\ell}{r} - R_\ell) = \mathsf{d}\ell \mod r$ for $\ell = 0, \ldots, r-1$;

(2) a satisfies the sum rules of order $h+1$ with a vector y such that

$$\widehat{y}(\xi) = [(1, e^{i\frac{1}{r}\xi}, \ldots, e^{i\frac{r-1}{r}\xi}) \otimes (1, i\xi, \ldots, (i\xi)^h)] + O(|\xi|^{h+1}), \qquad \xi \to 0. \tag{2.4.2}$$

In other words, the $(j, Q_\ell+1)$-block of the mask a for all $j = 1, \ldots, r$ on the coset $R_\ell + \mathsf{d}\mathbb{Z}$, that is, $\{[a(R_\ell + \mathsf{d}k)]_{:,Q_\ell+1}\}_{k\in\mathbb{Z}}$, $\ell = 0, \ldots, r-1$, are completely determined by the condition (2.4.1) for an interpolatory mask of type $(\mathsf{d}, \Gamma_r, h)$. Denote

$$\Gamma_{\mathsf{d},r} := \{(m,n) \ : \ 0 \leqslant m \leqslant \mathsf{d}-1, 1 \leqslant n \leqslant r\} \backslash \{(R_\ell, Q_\ell+1) \ : \ 0 \leqslant \ell \leqslant r-1\}. \tag{2.4.3}$$

Then, in order to construct an interpolatory mask a of type $(\mathsf{d}, \Gamma_r, h)$ with sum rules of order κ, it suffices to construct $\{[a(m+\mathsf{d}k)]_{:,n}\}_{k\in\mathbb{Z}}$ for every $(m,n) \in \Gamma_{\mathsf{d},r}$ such that the sum rule conditions in (2.2.2) are satisfied.

We have the following result on interpolatory masks of type $(\mathsf{d}, \Gamma_r, h)$ with increasing orders of sum rules.

Theorem 2.4. *Let d, r, K be positive integers with $\mathsf{d} > 1$. Let h be a nonnegative integer and $L = r(h+1)$. Suppose that for every $(m,n) \in \Gamma_{\mathsf{d},r}$, $S_{m,n}$ is a subset of $\mathbb{Z}$ such that $\#S_{m,n} = K$. Then there exists a unique finitely supported mask $a : \mathbb{Z} \to \mathbb{C}^{L\times L}$ satisfying the following conditions:*

(1) *a is an interpolatory mask of type $(\mathsf{d}, \Gamma_r, h)$;*

(2) *For every $(m,n) \in \Gamma_{\mathsf{d},r}$, $[a(m+\mathsf{d}k)]_{:,n} = 0$ for all $k \in \mathbb{Z}\backslash S_{m,n}$;*

(3) *a satisfies the sum rules of order LK.*

In fact, the unique mask a must be real-valued, that is, $a : \mathbb{Z} \to \mathbb{R}^{L\times L}$.

Proof. Note that (2.4.1) is equivalent to

$$[\widehat{a}^{R_\ell}(\xi)]_{:,Q_\ell+1} = \mathsf{d}^{-1} e^{-iR_\ell\xi} E_{\ell+1} D, \qquad \ell = 0, \ldots, r-1.$$

Let $\widehat{y}(\xi)=[(1,e^{i\frac{1}{r}\xi},\dots,e^{i\frac{r-1}{r}\xi})\otimes(1,i\xi,\dots,(i\xi)^h)]$. It is easy to see that

$$\begin{aligned}\widehat{y}(\mathsf{d}\xi)[\widehat{a}^{R_\ell}(\xi)]_{:,Q_\ell+1}&=\mathsf{d}^{-1}\widehat{y}(\mathsf{d}\xi)e^{-iR_\ell\xi}E_{\ell+1}D\\&=\mathsf{d}^{-1}e^{i(\frac{\mathsf{d}\ell}{r}-R_\ell)\xi}[1,i\mathsf{d}\xi,\dots,(i\mathsf{d}\xi)^h]D\\&=\mathsf{d}^{-1}e^{i\frac{Q_\ell}{r}\xi}[1,i\xi,\dots,(i\xi)^h]\\&=\mathsf{d}^{-1}\widehat{y_{Q_\ell+1}}(\xi),\quad \xi\to 0,\ \ell=0,\dots,r-1,\end{aligned}$$

where $\widehat{y_n}(\xi)=e^{i\frac{n-1}{r}\xi}[1,i\xi,\dots,(i\xi)^h]$, $1\leqslant n\leqslant r$ and $\widehat{a}^m(\xi):=\sum_{k\in\mathbb{Z}}a(m+\mathsf{d}k)e^{-i(m+\mathsf{d}k)\xi}$, $0\leqslant m\leqslant \mathsf{d}-1$ are the cosets of a. To require that a should satisfy the sum rules of order LK, by Theorem 2.2 and (2.2.2), it is necessary and sufficient to require

$$\widehat{y}(\mathsf{d}\xi)[\widehat{a}^m(\xi)]_{:,n}=\mathsf{d}^{-1}\widehat{y_n}(\xi)+O(|\xi|^{LK}),\qquad \xi\to 0,\ \forall\,(m,n)\in\Gamma_{\mathsf{d},r}.$$

That is, as $\xi\to 0$,

$$\begin{aligned}&\sum_{\ell=0}^{r-1}\sum_{k\in S_{m,n}}\widehat{y_{\ell+1}}(\mathsf{d}\xi)[a(m+\mathsf{d}k)]_{\ell+1,n}e^{-i(m+\mathsf{d}k)\xi}\\&=\frac{1}{\mathsf{d}}e^{i\frac{n-1}{r}\xi}[1,i\xi,\dots,(i\xi)^h]+O(|\xi|^{LK}).\end{aligned}\tag{2.4.4}$$

Now we need to show that for every $(m,n)\in\Gamma_{\mathsf{d},r}$ the above system of linear equations on $\{[a(m+\mathsf{d}k)]_{\ell+1,n}\ :\ \ell=0,\dots,r-1,\ k\in S_{m,n}\}$ has a unique solution.

For $x\in\mathbb{R}$ and $j,s\in\mathbb{N}_0$, denote

$$v_{j,s}(x)=\begin{cases}0, & j<s;\\ \frac{j!}{(j-s)!}x^{j-s}, & j\geqslant s.\end{cases}$$

Note that (2.4.4) is equivalent to: for $t = 0, 1, \dots, h$,

$$\begin{aligned}&\sum_{\ell=0}^{r-1}\sum_{k\in S_{m,n}}\sum_{s=0}^{h}[a(m+\mathsf{d}k)]_{\ell+1,n;\,s+1,t+1}e^{i(\mathsf{d}\ell/r-m-\mathsf{d}k)\xi}(i\mathsf{d}\xi)^s\\ &=\mathsf{d}^{-1}e^{i\frac{n-1}{r}\xi}(i\xi)^t+O(|\xi|^{LK}),\qquad \xi\to 0.\end{aligned}\tag{2.4.5}$$

For each $t = 0, 1, \dots, h$, taking j-th derivative on both side of (2.4.5) and evaluating them at $\xi = 0$, we obtain

$$\begin{aligned}&\sum_{\ell=0}^{r-1}\sum_{k\in S_{m,n}}\sum_{s=0}^{h}[a(m+\mathsf{d}k)]_{\ell+1,n;\,s+1,t+1}\,\mathsf{d}^s\,v_{j,s}(\mathsf{d}\ell/r-m-\mathsf{d}k)\\ &=\mathsf{d}^{-1}v_{j,t}\Big(\frac{n-1}{r}\Big),\qquad j=0,1,\dots,LK-1.\end{aligned}\tag{2.4.6}$$

Since $\#S_{m,n} = K$ for all $(m, n) \in \Gamma_{\mathsf{d},r}$, we see that for each $(m, n) \in \Gamma_{\mathsf{d},r}$, the set $\{\mathsf{d}\ell/r - m - \mathsf{d}k \;:\; k \in S_{m,n},\; \ell = 0, 1, \dots, r-1\}$ consists of rK distinct points in $\mathbb{R}$. The coefficient matrix of (2.4.6) is

$$C=\big(\mathsf{d}^s\,v_{j,s}(\mathsf{d}\ell/r-m-\mathsf{d}k)\big)_{j=0,1,\dots,Lr-1;\,s=0,1,\dots,h,\,k\in S_{m,n},\,\ell=0,1,\dots,r-1},$$

which is an $LK \times LK$ matrix. Notice that

$$V:=\big(v_{j,s}(\mathsf{d}\ell/r-m-\mathsf{d}k)\big)_{j=0,1,\dots,LK-1;\,s=0,1,\dots,h,\,k\in S_{m,n},\,\ell=0,1,\dots,r-1}$$

is a confluent Vandermonde matrix of size $LK \times LK$, which is invertible, and

$$C=V\cdot\mathrm{diag}(\underbrace{1,1,\dots,1}_{rK\text{ times}},\underbrace{\mathsf{d},\mathsf{d},\dots,\mathsf{d}}_{rK\text{ times}},\dots,\underbrace{\mathsf{d}^h,\mathsf{d}^h,\dots,\mathsf{d}^h}_{rK\text{ times}}).$$

Hence C is an invertible matrix of size $LK \times LK$. Moreover, the number of unknowns $\{[a(m + \mathsf{d}k)]_{\ell+1,n;\,s+1,t+1} : s = 0, 1, \dots, h,\ k \in S_{m,n},\ \ell = 0, 1, \dots, r-1\}$ in (2.4.6) is also LK. Consequently, the solution to the system of linear equations in (2.4.6) is unique. Furthermore, it is evident that the solution is real-valued. □

The following result is a direct consequence of Theorem 2.4.

Corollary 2.5. *Let* d, r, K *be positive integers such that* $\mathsf{d} > 1$*. Let* h *be a nonnegative integer and* $L = r(h+1)$*. Let* S *be any subset of* $\mathbb{Z}$ *such that* $\#(S \cap (m + \mathsf{d}\mathbb{Z})) = K$ *for all* $m \in \mathbb{Z}$ *and* $\{R_\ell\}_{\ell=0}^{r-1} \subset S$*, where* $R_\ell := \lfloor \frac{\mathsf{d}\ell}{r} \rfloor$*. Then there exists a unique finitely supported mask* $a : \mathbb{Z} \to \mathbb{R}^{L \times L}$ *satisfying the following conditions:*

(1) *a is an interpolatory mask of type* $(\mathsf{d}, \Gamma_r, h)$*;*

(2) *a is supported on S;*

(3) *a satisfies the sum rules of order LK.*

In particular, if $S = [-N_0, \mathsf{d}K - N_0 - 1] \cap \mathbb{Z}$ *for* $N_0 \in \mathbb{Z}$*, then* $\#(S \cap (m + \mathsf{d}\mathbb{Z})) = K$ *for all* $m \in \mathbb{Z}$*.*

For the case $\mathsf{d} = rr'$ for some $r' \in \mathbb{N}$, we have $Q_\ell = 0$, $R_\ell = r'\ell$ for all $\ell = 0, \dots, r-1$. (2.4.1) is equivalent to

$$[\widehat{a}^{r'\ell}(\xi)]_{:,1} = \mathsf{d}^{-1} e^{-ir'\ell\cdot\xi} E_{\ell+1} D, \quad \ell = 0, \dots, r-1. \tag{2.4.7}$$

In particular, if $\mathsf{d} = r$, i.e., $r' = 1$, then an interpolatory mask of type $(\mathsf{d}, \Gamma_{\mathsf{d}}, h)$ is of the form

$$\widehat{a}(\xi) = \frac{1}{\mathsf{d}} \begin{bmatrix} D & * & \cdots & * \\ De^{-i\xi} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ De^{-i(\mathsf{d}-1)\xi} & * & \cdots & * \end{bmatrix}, \tag{2.4.8}$$

where $D = \mathrm{diag}(1, \mathsf{d}^{-1}, \ldots, \mathsf{d}^{-h})$.

For the case $\mathsf{d} = r = 2$, we have the following result on interpolatory masks of type $(2, \{0, \frac{1}{2}\}, h)$ with symmetry.

Corollary 2.6. *For any positive integer K and any nonnegative integer h, there exists a unique interpolatory mask a of type $(2, \{0, \frac{1}{2}\}, h)$ such that*

(1) *a is supported on $[1-K, K]$;*

(2) *a is real-valued and satisfies the sum rules of order $(h+1)(2K-1)$;*

(3) *The mask a is $(\{-1, 1\}, 2)$-symmetric:*

$$\overline{\widehat{a}(\xi)} = \mathrm{diag}(P, Pe^{2i\xi})\widehat{a}(\xi)\mathrm{diag}(P, Pe^{-i\xi}), \tag{2.4.9}$$

where $P := \mathrm{diag}((-1)^0, (-1)^1, \ldots, (-1)^h)$ is a diagonal matrix of size $(h+1) \times (h+1)$.

Moreover, if $\nu_\infty(a, 2) > h$, then $\phi = [\phi_{0,0}, \ldots, \phi_{0,h}, \phi_{\frac{1}{2},0}, \ldots, \phi_{\frac{1}{2},h}]^T$ is $\{1, -1\}$-symmetric, where ϕ is the 2-refinable function vector associated with mask a. More precisely, $\phi_{0,j}(-\cdot) = (-1)^j \phi_{0,j}$ and $\phi_{\frac{1}{2},j}(1-\cdot) = (-1)^j \phi_{\frac{1}{2},j}$ for $j = 0, 1, \ldots, h$.

Proof. Since $\mathsf{d} = r = 2$, we see that $\{[a(k)]_{:,1}\}_{k\in\mathbb{Z}}$ are completely determined by (2.4.8). By the symmetry condition (2.4.9), $\{[a(2k+1)]_{:,2}\}_{k\in\mathbb{Z}}$ are completely determined by $\{[a(2k)]_{:,2}\}_{k\in\mathbb{Z}}$. Moreover, $[a(k)]_{1,2} = \mathbf{0}$ if k is even, or $[a(1-k)]_{2,2} = \mathbf{0}$ if k is odd due to the symmetry condition (2.4.9). Now the proof is completed by a similar proof of Theorem 2.4. ☐

In the rest of this section, let us present in Tables 2.1 and 2.2 the smoothness exponents of some families of the interpolatory masks constructed in Corollaries 2.5 and 2.6. An example of an interpolatory mask of type $(2, \{0, \frac{1}{2}\}, 1)$ will also be given.

K	1	2	3	4	5	6
$a_{(3,\{0,\frac{1}{2}\},1)}$	0.5	2.557920	2.952713	3.223482	3.425445	3.583893
$a_{(3,\{0,\frac{1}{2}\},2)}$	0.5	3.286249	3.767089	4.065856	4.234592	4.311367

TABLE 2.1: The quantities $\nu_2(a_{(3,\{0,\frac{1}{2}\},1)}, 3)$ and $\nu_2(a_{(3,\{0,\frac{1}{2}\},2)}, 3)$ for the interpolatory masks $a_{(\mathsf{d},\Gamma_r,h)}$ constructed in Corollary 2.5. Here $S := [-N_0, 3K - N_0 - 1]$ with $N_0 := \lfloor 3(K-1)/2 \rfloor$.

K	1	2	3	4	5	6
$a^{sym}_{(2,\{0,\frac{1}{2}\},1)}$	0.5	2.494509	3.051766	3.646481	3.791163	4.000000
$a^{sym}_{(2,\{0,\frac{1}{2}\},2)}$	0.5	2.958569	3.931713	4.471009	4.421853	4.999996
$a^{sym}_{(2,\{0,\frac{1}{2}\},3)}$	0.5	3.351721	4.343120	4.650265	4.890424	5.498152

TABLE 2.2: The quantities $\nu_2(a^{sym}_{(2,\{0,\frac{1}{2}\},h)}, 2)$ for the symmetric interpolatory masks $a^{sym}_{(2,\{0,\frac{1}{2}\},h)}$ constructed in Corollary 2.6 for $h = 1, 2, 3$, respectively.

Example 2.3. Let $\mathsf{d} = r = 2$, $h = 1$ and $K = 2$ in Corollary 2.6. Then we have a symmetric interpolatory mask a of type $(2, \{0, \frac{1}{2}\}, 1)$ satisfying the sum rules of order 6. The mask a is supported on $[-1, 2]$ and $a(0)$, $a(2)$ are given by

$$a(0) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{9}{32} & \frac{-3}{4} \\ 0 & \frac{1}{4} & \frac{9}{128} & \frac{-3}{64} \\ 0 & 0 & \frac{45}{256} & \frac{93}{128} \\ 0 & 0 & \frac{-9}{512} & \frac{-15}{256} \end{bmatrix} \quad a(2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{11}{256} & \frac{3}{128} \\ 0 & 0 & \frac{3}{512} & \frac{1}{256} \end{bmatrix},$$

while $a(-1)$, $a(1)$ can be obtained by symmetry in (2.4.9). Then we have $\nu_2(a, 2) \approx 2.494509$. Therefore, $\nu_\infty(a, 2) \geqslant \nu_2(a, 2) - 1/2 \approx 1.994509 > 1$. By Theorem 2.1, the 2-refinable function vector $\phi = [\phi_{0,0}, \phi_{0,1}, \phi_{\frac{1}{2},0}, \phi_{\frac{1}{2},1}]^T$ with mask a is a symmetric C^1 interpolating refinable function vector of type $(2, \Gamma_2, 1)$. Moreover, $\phi_{0,j}(-\cdot) = (-1)^j \phi_{0,j}$ and $\phi_{\frac{1}{2},j}(1 - \cdot) = (-1)^j \phi_{\frac{1}{2},j}$ for all $j = 0, 1$. See Figure 2.5 for the graph of ϕ.

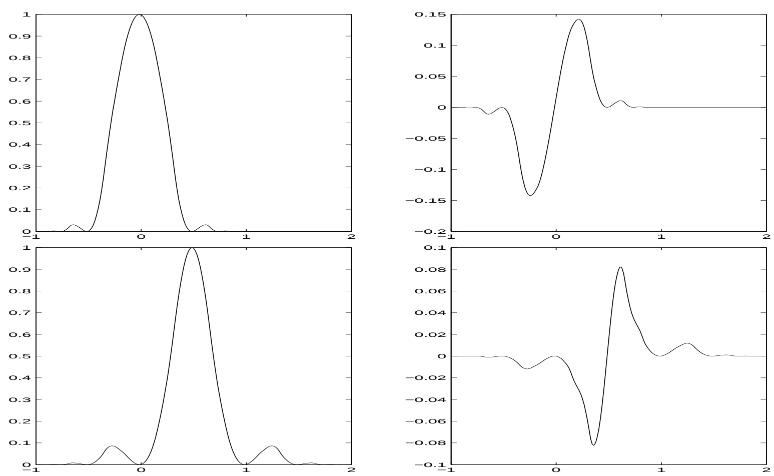

FIGURE 2.5: The graphs of $\phi_{0,j}$, $j = 0, 1$ (top) and $\phi_{\frac{1}{2},j}$, $j = 0, 1$ (bottom) in Example 2.3. $\nu_2(\phi) \approx 2.494509$. $\phi_{0,j}(-\cdot) = (-1)^j \phi_{0,j}$ and $\phi_{\frac{1}{2},j}(1 - \cdot) = (-1)^j \phi_{\frac{1}{2},j}$ for all $j = 0, 1$.

2.5 Proofs of Theorems 2.1, 2.2, and 2.3

In this section, we shall prove Theorems 2.1, 2.2, and 2.3 of Section 2.3. Since stability and linear independence of a refinable function vector will be needed in our proofs, let us recall their definitions here. For an $L \times 1$ vector $\phi = [\phi_1, \ldots, \phi_L]^T$ of compactly supported functions in $L_p(\mathbb{R}^d)$ for $1 \leqslant p \leqslant \infty$, we say that the shifts of ϕ are *stable* in $L_p(\mathbb{R}^d)$ if there exist two positive constants C_1 and C_2 such that

$$C_1 \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} |c_\ell(k)|^p \leqslant \left\| \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} c_\ell(k)\phi_\ell(\cdot - k) \right\|_{L_p(\mathbb{R}^d)}^p \leqslant C_2 \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} |c_\ell(k)|^p \tag{2.5.1}$$

for all finitely supported sequences $c_1, \dots, c_L$ in $\ell_0(\mathbb{Z}^d)$. For a compactly supported function vector $\phi = [\phi_1, \dots, \phi_L]^T$, we say that the shifts of ϕ are *linearly independent* if for any sequences $c_1, \dots, c_L : \mathbb{Z}^d \to \mathbb{C}$ such that

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} c_\ell(k)\phi_\ell(x - k) = 0, \qquad a.e.\ x \in \mathbb{R}^d, \tag{2.5.2}$$

then one must have $c_\ell(k) = 0$ for all $\ell = 1, \dots, L$ and $k \in \mathbb{Z}^d$.

The following lemma is needed latter.

Lemma 2.7. *Let N be a $d \times d$ matrix. Then*

$$\mu! S(N, \Lambda_h)_{\mu,\nu} = \nu! S(N^T, \Lambda_h)_{\nu,\mu} \qquad \forall\ \mu, \nu \in \Lambda_h, h \in \mathbb{N}_0. \tag{2.5.3}$$

Consequently, for a row vector $((i\xi)^\nu)_{\nu\in\Lambda_h}$, we have

$$((i\xi)^\nu)_{\nu\in\Lambda_h} S(N, \Lambda_h) = ((iN^T\xi)^\nu)_{\nu\in\Lambda_h}. \tag{2.5.4}$$

Proof. Let $x, y \in \mathbb{R}^d$. Note that $(x \cdot y)^h = \sum_{\mu\in O_h} \frac{h!}{\mu!} x^\mu y^\mu$. Expanding $e^{x\cdot(Ny)}$ at the origin, we deduce that

$$\begin{aligned} e^{x\cdot(Ny)} &= \sum_{h=0}^{\infty} \frac{(x \cdot (Ny))^h}{h!} = \sum_{h=0}^{\infty} \sum_{\mu\in O_h} x^\mu \frac{(Ny)^\mu}{\mu!} \\ &= \sum_{h=0}^{\infty} \sum_{\mu\in O_h} \sum_{\nu\in O_h} \frac{1}{\nu!} S(N, O_h)_{\mu,\nu} x^\mu y^\nu. \end{aligned}$$

Similarly, we have $e^{y\cdot(N^T x)} = \sum_{h=0}^{\infty}\sum_{\nu\in O_h}\sum_{\mu\in O_h}\frac{1}{\mu!}S(N^T,O_h)_{\nu,\mu}x^\mu y^\nu$. Since $e^{x\cdot(Ny)} = e^{y\cdot(N^T x)}$, comparing the coefficients of $x^\mu y^\nu$ in both expressions, we conclude that $\frac{1}{\nu!}S(N,O_h)_{\mu,\nu} = \frac{1}{\mu!}S(N^T,O_h)_{\nu,\mu}$ for all $\mu,\nu\in O_h$. That is, (2.5.3) holds.

To prove (2.5.4), we have

$$((i\xi)^\nu)_{\nu\in\Lambda_h}S(N,\Lambda_h) = \Big(\sum_{\nu\in\Lambda_h}(i\xi)^\nu S(N,\Lambda_h)_{\nu,\mu}\Big)_{\mu\in\Lambda_h}.$$

By (2.5.3), we deduce that

$$\begin{aligned}\sum_{\nu\in\Lambda_h}(i\xi)^\nu S(N,\Lambda_h)_{\nu,\mu} &= \sum_{\nu\in\Lambda_h}\nu! S(N,\Lambda_h)_{\nu,\mu}\frac{(i\xi)^\nu}{\nu!} = \mu!\sum_{\nu\in\Lambda_h}S(N^T,\Lambda_h)_{\mu,\nu}\frac{(i\xi)^\nu}{\nu!}\\ &= \mu!\frac{(iN^T\xi)^\mu}{\mu!} = (iN^T\xi)^\mu.\end{aligned}$$

So, (2.5.4) is verified. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Necessity. By $\phi(\mathsf{M}^{-1}\cdot) = |\det\mathsf{M}|\sum_{k\in\mathbb{Z}^d}a(k)\phi(\cdot-k)$ and [24, Proposition 2.1], we have

$$\begin{aligned}[\mathscr{D}^{\Lambda_h}\otimes\phi](\mathsf{M}^{-1}\cdot)S(\mathsf{M}^{-1},\Lambda_h) &= \mathscr{D}^{\Lambda_h}\otimes[\phi(\mathsf{M}^{-1}\cdot)]\\ &= |\det\mathsf{M}|\sum_{j\in\mathbb{Z}^d}a(j)[\mathscr{D}^{\Lambda_h}\otimes\phi](\cdot-j).\end{aligned}$$

Hence,

$$[\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\cdot)S(\mathsf{M}^{-1},\Lambda_h) = |\det\mathsf{M}|\sum_{j\in\mathbb{Z}^d}\sum_{\beta\in\Gamma_N}[a(j)]_{\gamma,\beta}[\mathscr{D}^{\Lambda_h}\otimes\phi_\beta](\mathsf{M}\cdot-j).$$

That is, for $\alpha \in \Gamma_N$ and $k \in \mathbb{Z}^d$, we have

$$[\mathscr{D}^{\Lambda_h} \otimes \phi_\gamma](\alpha + k) S(\mathsf{M}^{-1}, \Lambda_h)$$
$$= |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} \sum_{\beta\in\Gamma_N} [a(j)]_{\gamma,\beta} [\mathscr{D}^{\Lambda_h} \otimes \phi_\beta](\mathsf{M}\alpha + \mathsf{M}k - j).$$

Since $N\mathsf{M}N^{-1}$ is an integer matrix, we have $\mathsf{M}N^{-1}\mathbb{Z}^d \subseteq N^{-1}\mathbb{Z}^d$, that is, $\mathsf{M}[\Gamma_N + \mathbb{Z}^d] \subseteq \Gamma_N + \mathbb{Z}^d$. Thus, for each $\alpha \in \Gamma_N$, we can uniquely write $\mathsf{M}\alpha = [\mathsf{M}\alpha]_{\Gamma_N} + \langle \mathsf{M}\alpha \rangle_{\Gamma_N}$ with $[\mathsf{M}\alpha]_{\Gamma_N} \in \mathbb{Z}^d$ and $\langle \mathsf{M}\alpha \rangle_{\Gamma_N} \in \Gamma_N$. Applying (2.1.6) to the above equation, we obtain

$$\delta(k)\delta(\alpha - \gamma) S(\mathsf{M}^{-1}, \Lambda_h)$$
$$= |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} \sum_{\beta\in\Gamma_N} [a(j)]_{\gamma,\beta} [\mathscr{D}^{\Lambda_h} \otimes \phi_\beta](\langle \mathsf{M}\alpha \rangle_{\Gamma_N} + [\mathsf{M}\alpha]_{\Gamma_N} + \mathsf{M}k - j)$$
$$= |\det \mathsf{M}| \sum_{\beta\in\Gamma_N} [a(\mathsf{M}k + [\mathsf{M}\alpha]_{\Gamma_N})]_{\gamma,\beta} \delta(\beta - \langle \mathsf{M}\alpha \rangle_{\Gamma_N})$$
$$= |\det \mathsf{M}| [a(\mathsf{M}k + [\mathsf{M}\alpha]_{\Gamma_N})]_{\gamma, \langle \mathsf{M}\alpha \rangle_{\Gamma_N}}.$$

That is, $[a(\mathsf{M}k + [\mathsf{M}\alpha]_{\Gamma_N})]_{\gamma, \langle \mathsf{M}\alpha \rangle_{\Gamma_N}} = |\det \mathsf{M}|^{-1} S(\mathsf{M}^{-1}, \Lambda_h)\delta(k)\delta(\alpha - \gamma)$ for all $\alpha, \gamma \in \Gamma_N$ and $k \in \mathbb{Z}^d$. So, (2.3.1) holds.

By (2.1.6), it is easy to see that the shifts of ϕ are linearly independent. In fact, suppose $\sum_{\gamma\in\Gamma_N} \sum_{\mu\in\Lambda_h} \sum_{k\in\mathbb{Z}^d} c_{\gamma,\mu}(k)\phi_{\gamma,\mu}(x - k) = 0$. Taking the differentiation operator ∂^ν on both sides and setting $x = \beta + j$, we obtain $c_{\beta,\nu}(j) = 0$ for all $\beta \in \Gamma_N, \nu \in \Lambda_h$ and $j \in \mathbb{Z}^d$. So the shifts of ϕ are linearly independent and therefore stable ([47]). Consequently, by [24, Corollary 5.1] and $\phi \in (C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}$, we must have $\nu_\infty(a, \mathsf{M}) > h$. That is, item (2) holds.

Since $\nu_\infty(a, \mathsf{M}) > h$, by [24, Theorem 4.3], the mask a must satisfy the sum rules of order $h+1$ with a vector $y \in (\ell_0(\mathbb{Z}^d))^{1\times(\#\Gamma_N)(\#\Lambda_h)}$ and $\widehat{y}(0)\widehat{\phi}(0) = 1$.

But this implies ([24]) that $\partial^\mu[\widehat{y}(\cdot)\widehat{\phi}(\cdot)](0) = \delta(\mu)$ and $\partial^\mu[\widehat{y}(\cdot)\widehat{\phi}(\cdot)](2\pi k) = 0$ for all $|\mu| \leqslant h$ and $k \in \mathbb{Z}^d \backslash \{0\}$. By the remark after [24, Proposition 3.2], this is equivalent to $(p * y) * \phi = p$ for all $p \in \Pi_h$, where Π_h denotes the linear space of all polynomials with total degree no greater than h. More precisely, by [24, (2.13)], we have

$$\sum_{j\in\mathbb{Z}^d}\sum_{\mu\in\Lambda_h}\sum_{\gamma\in\Gamma_N}\partial^\mu p(j)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\gamma}(0)\phi_\gamma(\cdot - j) = p, \qquad p \in \Pi_h.$$

Hence,

$$\sum_{j\in\mathbb{Z}^d}\sum_{\mu\in\Lambda_h}\sum_{\gamma\in\Gamma_N}\partial^\mu p(j)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\gamma}(0)[\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\cdot - j) = \mathscr{D}^{\Lambda_h}\otimes p, \qquad p \in \Pi_h.$$

So, for $x = \beta + k,\ \beta \in \Gamma_N$ and $k \in \mathbb{Z}^d$, for any $p \in \Pi_h$, we have

$$\sum_{j\in\mathbb{Z}^d}\sum_{\mu\in\Lambda_h}\sum_{\gamma\in\Gamma_N}\partial^\mu p(j)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\gamma}(0)[\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\beta + k - j) = [\mathscr{D}^{\Lambda_h}\otimes p](\beta + k).$$

By (2.1.6) and the above identity, we obtain

$$\sum_{\mu\in\Lambda_h}\partial^\mu p(k)\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\beta}(0) = [\mathscr{D}^{\Lambda_h}\otimes p](\beta + k), \qquad p \in \Pi_h, k \in \mathbb{Z}^d, \beta \in \Gamma_N.$$

Set $p_\nu(x) := \frac{x^\nu}{\nu!}$. Taking $k = 0$ in the above identity, we get

$$\frac{(-i\partial)^\nu}{\nu!}\widehat{y_\beta}(0) = [\mathscr{D}^{\Lambda_h}\otimes p_\nu](\beta) = ([\partial^\mu p_\nu](\beta))_{\mu\in\Lambda_h}. \qquad (2.5.5)$$

For $\mu = (\mu_1, \ldots, \mu_d)$ and $\nu = (\nu_1, \ldots, \nu_d)$, we say that $\nu \leqslant \mu$ if $\nu_j \leqslant \mu_j$ for all $j = 1, \ldots, d$. Denote $\text{sgn}(\mu) = 1$ if $\mu \geqslant 0$ and 0, otherwise. Now it is easy to see that (2.5.5) is equivalent to

$$\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\beta}(0) = \Big(\frac{\beta^{\mu-\nu}}{(\mu-\nu)!}\operatorname{sgn}(\mu-\nu)\Big)_{\nu\in\Lambda_h}. \tag{2.5.6}$$

Note that (2.5.6) is satisfied by the choice $\widehat{y_\beta}(\xi) := e^{i\beta\cdot\xi}((i\xi)^\eta)_{\eta\in\Lambda_h}$, since

$$\begin{aligned}\frac{(-i\partial)^\mu}{\mu!}\widehat{y_\beta}(0) &= \sum_{o\leqslant\nu\leqslant\mu}\frac{(-i\partial)^{\mu-\nu}}{(\mu-\nu)!}e^{i\beta\cdot\xi}\Big|_{\xi=0}\frac{(-i\partial)^\nu}{\nu!}[(i\xi)^\eta]_{\eta\in\Lambda_h}\Big|_{\xi=0}\\ &= \sum_{0\leqslant\nu\leqslant\mu}\frac{\beta^{\mu-\nu}}{(\mu-\nu)!}[\delta_{\eta-\nu}]_{\eta\in\Lambda_h} = \left(\frac{\beta^{\mu-\nu}}{(\mu-\nu)!}\operatorname{sgn}(\mu-\nu)\right)_{\nu\in\Lambda_h}.\end{aligned}$$

Hence, a is an interpolatory mask of type (M,Γ_N,h). So, item (3) holds.

Since item (3) holds, by (2.3.3), we have $\widehat{y}(0) = [(1,1,\dots,1)\otimes(1,0,\dots,0)]$. By $(p*y)*\phi = p$ with $p=1$, we must have $\widehat{y}(0)*\phi = 1$. Consequently, we have $\widehat{y}(0)\widehat{\phi}(0) = 1$. Thus, item (1) holds.

Sufficiency. Let $g \in (C^h(\mathbb{R}^d))^{(\#\Lambda_h)\times 1}$ be an function vector satisfying (2.1.6) with $N = I_d$ and Hermite order h (see [22] and [24, Corollary 5.2] for the construction of such function vectors) such that

$$[1,0,\dots,0]\widehat{g}(0) = 1 \quad\text{and}\quad ((i\xi)^\nu)_{\nu\in\Lambda_h}\widehat{g}(\xi+2\pi k) = O(\|\xi\|^{h+1}),\quad \xi\to 0 \tag{2.5.7}$$

for $k\in\mathbb{Z}^d\backslash\{0\}$. Define a $(\#\Gamma_N)(\#\Lambda_h)\times 1$ column vector by

$$f := \big(S(N^{-1},\Lambda_h)g(N(\cdot-\gamma))\big)_{\gamma\in\Gamma_N}. \tag{2.5.8}$$

Then we have $\widehat{f}(\xi) = \big(|\det N|^{-1}e^{-i\gamma\cdot\xi}S(N^{-1},\Lambda_h)\widehat{g}((N^T)^{-1}\xi)\big)_{\gamma\in\Gamma_N}$. This can be rewritten as

$$\widehat{f}(\xi) = |\det N|^{-1}[(e^{-i\gamma\cdot\xi})_{\gamma\in\Gamma_N}]^T \otimes [S(N^{-1},\Lambda_h)\widehat{g}((N^T)^{-1}\xi)]. \tag{2.5.9}$$

Note that by (2.5.7), the first component of $\widehat{g}(0)$ is 1. Also, we observe that the first row of $S(N^{-1},\Lambda_h)$ is $[1,0,\dots,0]$. Consequently, the first component of $S(N^{-1},\Lambda_h)\widehat{g}(0)$ is 1. Now by $\widehat{y}(0)=[(1,1,\dots,1)\otimes(1,0,\dots,0)]$, we conclude from (2.5.9) that

$$
\begin{aligned}
&|\det N|\widehat{y}(0)\widehat{f}(0)\\
&=[(1,1,\dots,1)\otimes(1,0,\dots,0)]\times[(1,1,\dots,1)^T\otimes(S(N^{-1},\Lambda_h)\widehat{g}(0))]\\
&=[(1,1,\dots,1)\times(1,1,\dots,1)^T]\otimes[(1,0,\dots,0)\times(1,*,\dots,*)^T]\\
&=|\det N|,
\end{aligned}
$$

where $*$ denotes some number and we used the fact $S(N^{-1},\Lambda_h)\widehat{g}(0)=[1,*,\dots,*]^T$ in the last second identity. Hence $\widehat{y}(0)\widehat{f}(0)=1$.

On the other hand, we deduce from (2.5.9) that as $\xi\to 0$,

$$
\begin{aligned}
&|\det N|\widehat{y}(\xi)\widehat{f}(\xi+2\pi k)=\Big((e^{i\beta\cdot\xi})_{\beta\in\Gamma_N}\otimes((i\xi)^\nu)_{\nu\in\Lambda_h}\Big)\\
&\times\Big([(e^{-i\gamma\cdot(\xi+2\pi k)})_{\gamma\in\Gamma_N}]^T\otimes[S(N^{-1},\Lambda_h)\widehat{g}((N^T)^{-1}(\xi+2\pi k))]\Big)+O(\|\xi\|^{h+1})\\
&=\Big(\sum_{\gamma\in\Gamma_N}e^{-i2\pi k\cdot\gamma}\Big)\Big(((i\xi)^\nu)_{\nu\in\Lambda_h}S(N^{-1},\Lambda_h)\widehat{g}((N^T)^{-1}\xi+2\pi(N^T)^{-1}k)\Big)\\
&\qquad+O(\|\xi\|^{h+1}).
\end{aligned}
$$

By Lemma 2.7, we see that $((i\xi)^\nu)_{\nu\in\Lambda_h}S(N^{-1},\Lambda_h)=((i(N^T)^{-1}\xi)^\nu)_{\nu\in\Lambda_h}$. Consequently, we have

$$
\begin{aligned}
|\det N|\widehat{y}(\xi)\widehat{f}(\xi+2\pi k)&=\Big(\sum_{\gamma\in\Gamma_N}e^{-i2\pi k\cdot\gamma}\Big)\Big(((i(N^T)^{-1}\xi)^\nu)_{\nu\in\Lambda_h}\\
&\times\widehat{g}((N^T)^{-1}\xi+2\pi(N^T)^{-1}k)\Big)+O(\|\xi\|^{h+1}),\qquad \xi\to 0.
\end{aligned}
\tag{2.5.10}
$$

For $k\in\mathbb{Z}^d\backslash[N^T\mathbb{Z}^d]$, we have $\sum_{\gamma\in\Gamma_N}e^{-i2\pi k\cdot\gamma}=0$ and consequently it follows from the above identity that $\widehat{y}(\xi)\widehat{f}(\xi+2\pi k)=O(\|\xi\|^{h+1})$ as $\xi\to 0$ for all $k \in \mathbb{Z}^d \backslash [N^T \mathbb{Z}^d]$. For $k \in [N^T\mathbb{Z}^d]\backslash\{0\}$, we have $k = N^T k'$ for some $k' \in \mathbb{Z}^d\backslash\{0\}$. Therefore, by (2.5.7), we have

$$((i(N^T)^{-1}\xi)^\nu)_{\nu\in\Lambda_h}\widehat{g}((N^T)^{-1}\xi + 2\pi(N^T)^{-1}k) + O(\|\xi\|^{h+1}) = O(\|\xi\|^{h+1}),$$

as $\xi \to 0$. Hence, we conclude that $\widehat{y}(\xi)\widehat{f}(\xi + 2\pi k) = O(\|\xi\|^{h+1})$, $\xi \to 0$ for all $k \in \mathbb{Z}^d\backslash\{0\}$. So, f is a suitable initial function vector with respect to y.

Let $f_0 := f$ and $f_n := Q_{a,\mathsf{M}} f_{n-1}, n \in \mathbb{N}$. Now we prove by induction that all f_n satisfies (2.1.6). When $n = 0$, $f_0 = f$. By the choice of the initial function f in (2.5.8) and by [24, Proposition 2.1], for $\gamma \in \Gamma_N$, we have

$$\begin{aligned}\mathscr{D}^{\Lambda_h} \otimes f_\gamma &= \mathscr{D}^{\Lambda_h} \otimes [S(N^{-1}, \Lambda_h) g(N(\cdot - \gamma))] \\ &= S(N^{-1}, \Lambda_h)[\mathscr{D}^{\Lambda_h} \otimes g](N(\cdot - \gamma))S(N, \Lambda_h).\end{aligned}$$

Since g satisfies (2.1.6) with $N = I_d$, we deduce that for all $\beta \in \Gamma_N$ and $k \in \mathbb{Z}^d$,

$$\begin{aligned}[\mathscr{D}^{\Lambda_h} \otimes f_\gamma](\beta + k) &= S(N^{-1}, \Lambda_h)[\mathscr{D}^{\Lambda_h} \otimes g](Nk + N(\beta - \gamma))S(N, \Lambda_h) \\ &= S(N^{-1}, \Lambda_h)\delta(k)\delta(\beta - \gamma)I_{\#\Lambda_h} S(N, \Lambda_h) \\ &= \delta(k)\delta(\beta - \gamma)I_{\#\Lambda_h}.\end{aligned}$$

So, f satisfies (2.1.6). Suppose that f_{n-1} satisfies (2.1.6). Then by $f_n = Q_{a,\mathsf{M}} f_{n-1} = |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} a(j) f_{n-1}(\mathsf{M} \cdot -j)$, for $\gamma \in \Gamma_N$, we have

$$[f_n]_\gamma = |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} [f_{n-1}(\mathsf{M} \cdot -j)]_\alpha.$$

Hence, by [24, Proposition 2.1], we have

$$\begin{aligned}\mathscr{D}^{\Lambda_h} \otimes [f_n]_\gamma &= |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} \mathscr{D}^{\Lambda_h} \otimes [f_{n-1}(\mathsf{M}\cdot - j)]_\alpha \\ &= |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} [\mathscr{D}^{\Lambda_h} \otimes [f_{n-1}]_\alpha](\mathsf{M}\cdot - j) S(\mathsf{M},\Lambda_h).\end{aligned}$$

So, for $\beta \in \Gamma_N$ and $k \in \mathbb{Z}^d$, we deduce that

$$\begin{aligned}&[\mathscr{D}^{\Lambda_h} \otimes [f_n]_\gamma](\beta + k) \\ &= |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} [a(j)]_{\gamma,\alpha} [\mathscr{D}^{\Lambda_h} \otimes [f_{n-1}]_\alpha](\mathsf{M}\beta + \mathsf{M}k - j) S(\mathsf{M},\Lambda_h).\end{aligned}$$

Now by induction hypothesis, we have

$$\begin{aligned}&[\mathscr{D}^{\Lambda_h} \otimes [f_{n-1}]_\alpha](\mathsf{M}\beta + \mathsf{M}k - j) \\ &= [\mathscr{D}^{\Lambda_h} \otimes [f_{n-1}]_\alpha](\langle \mathsf{M}\beta\rangle_{\Gamma_N} + [\mathsf{M}\beta]_{\Gamma_N} + \mathsf{M}k - j) \\ &= \delta(\langle \mathsf{M}\beta\rangle_{\Gamma_N} - \alpha)\delta([\mathsf{M}\beta]_{\Gamma_N} + \mathsf{M}k - j) I_{\#\Lambda_h}.\end{aligned}$$

Therefore, by (2.3.1), we get

$$\begin{aligned}&[\mathscr{D}^{\Lambda_h} \otimes [f_n]_\gamma](\beta + k) \\ &= |\det \mathsf{M}| \sum_{j\in\mathbb{Z}^d} \sum_{\alpha\in\Gamma_N} \delta(\langle \mathsf{M}\beta\rangle_{\Gamma_N} - \alpha)\delta([\mathsf{M}\beta]_{\Gamma_N} + \mathsf{M}k - j)[a(j)]_{\gamma,\alpha} S(\mathsf{M},\Lambda_h) \\ &= |\det \mathsf{M}| [a(\mathsf{M}k + [\mathsf{M}\beta]_{\Gamma_N})]_{\gamma,\langle \mathsf{M}\beta\rangle_{\Gamma_N}} S(\mathsf{M},\Lambda_h) \\ &= \delta(k)\delta(\beta-\gamma) S(\mathsf{M}^{-1},\Lambda_h) S(\mathsf{M},\Lambda_h) \\ &= \delta(k)\delta(\beta-\gamma) I_{\#\Lambda_h}.\end{aligned}$$

Hence, f_n satisfies (2.1.6). Now by induction, all f_n, $n = 0, 1, \ldots$ satisfy (2.1.6).

Since $\nu_\infty(a, \mathsf{M}) > h$, the cascade algorithm f_n converges in the function space in $(C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}$ ([24, Theorem 4.3]). By (ii) of item (3), we have $\widehat{y}(0) = [(1, 1, \ldots, 1) \otimes (1, 0, \ldots, 0)]$. Now by item (1), we see that $\widehat{y}(0)\widehat{\phi}(0) = 1$. Since $\widehat{y}(0)\widehat{\phi}(0) = \widehat{y}(0)\widehat{f}(0) = 1$, we see that $f_n \to \phi$ in $(C^h(\mathbb{R}^d))^{(\#\Gamma_N)(\#\Lambda_h)\times 1}$ as $n \to \infty$. Consequently, since all f_n satisfies (2.1.6), ϕ is also an interpolating function vector of type $(\mathsf{M}, \Gamma_N, h)$. ☐

Next, we prove Theorem 2.2.

Proof of Theorem 2.2. For simplicity, let us define two operators $R : \Gamma_N + \mathbb{Z}^d \to \mathbb{Z}^d$ and $Q : \Gamma_N + \mathbb{Z}^d \to \Gamma_N$ by $R(\alpha) := [\mathsf{M}\alpha]_{\Gamma_N}$ and $Q(\alpha) = \langle \mathsf{M}\alpha \rangle_{\Gamma_N}$. Let $E_\alpha := [\mathbf{0}, \ldots, \mathbf{0}, I_{\#\Lambda_h}, \mathbf{0}, \ldots, \mathbf{0}]^T$, $\alpha \in \Gamma_N$, be a $(\#\Gamma_N) \times 1$ block matrix with each block of size $(\#\Lambda_h) \times (\#\Lambda_h)$, whose nonzero block is located at the α-th position.

Using the cosets of the mask a, we see that (2.3.1) can be equivalently rewritten as

$$\widehat{a}^{R(\alpha)}(\xi) E_{Q(\alpha)} = |\det \mathsf{M}|^{-1} e^{-iR(\alpha)\cdot\xi} E_\alpha S(\mathsf{M}^{-1}, \Lambda_h) \qquad \forall\, \alpha \in \Gamma_N. \quad (2.5.11)$$

Since a satisfies the sum rules of order κ with the vector $\widehat{y}$, we have (2.2.2). In particular, using (2.2.2) with $\omega = R(\alpha)$, we deduce from (2.5.11) that as $\xi \to 0$,

$$\begin{aligned} |\det \mathsf{M}|^{-1} \widehat{y}(\xi) E_{Q(\alpha)} &= \widehat{y}(\mathsf{M}^T\xi)\widehat{a}^{R(\alpha)}(\xi) E_{Q(\alpha)} + O(\|\xi\|^\kappa) \\ &= |\det \mathsf{M}|^{-1} e^{-iR(\alpha)\cdot\xi} \widehat{y}(\mathsf{M}^T\xi) E_\alpha S(\mathsf{M}^{-1}, \Lambda_h) + O(\|\xi\|^\kappa). \end{aligned}$$

Denote $\widehat{y}(\xi) := (\widehat{y_\alpha}(\xi))_{\alpha\in\Gamma_N}$ with each $\widehat{y_\alpha}$ being a $1\times(\#\Lambda_h)$ row vector. Then the above identity can be rewritten as

$$\widehat{y_{Q(\alpha)}}(\xi) = e^{-iR(\alpha)\cdot\xi}\widehat{y_\alpha}(\mathsf{M}^T\xi)S(\mathsf{M}^{-1},\Lambda_h) + O(\|\xi\|^\kappa), \qquad \xi\to 0.$$

That is, since (2.3.2) is satisfied, for all $\alpha\in\Gamma_N$, as $\xi\to 0$, we have

$$\begin{aligned}\widehat{y_\alpha}(\xi) &= e^{i\alpha\cdot\xi}((i\xi)^\nu)_{\nu\in\Lambda_h} + O(\|\xi\|^{h+1}),\\ \widehat{y_\alpha}(\mathsf{M}^T\xi) &= e^{iR(\alpha)\cdot\xi}\widehat{y_{Q(\alpha)}}(\xi)S(\mathsf{M},\Lambda_h) + O(\|\xi\|^\kappa),\end{aligned} \tag{2.5.12}$$

Note that the above relation is just a system of linear equations on the unknowns $\{\partial^\mu\widehat{y}(0) : h < |\mu| < \kappa\}$. In the following, we shall argue that the above system of linear equations in (2.5.12) has a unique solution for the unknowns $\{\partial^\mu\widehat{y}(0) : h < |\mu| < \kappa\}$. Moreover, we shall prove that the unique solution to (2.5.12) must be given in (2.3.5).

For all $\alpha\in\Gamma_N$ and $n\in\mathbb{N}$, employing (2.5.12) iteratively, we have

$$\begin{aligned}\widehat{y_\alpha}(\xi) &= e^{i\xi\cdot\mathsf{M}^{-1}R(\alpha)}\widehat{y_{Q(\alpha)}}((\mathsf{M}^T)^{-1}\xi)S(\mathsf{M},\Lambda_h) + O(\|\xi\|^\kappa)\\ &= e^{i\xi\cdot(\mathsf{M}^{-2}R(Q(\alpha))+\mathsf{M}^{-1}R(\alpha))}\widehat{y_{Q^2(\alpha)}}((\mathsf{M}^T)^{-2}\xi)S(\mathsf{M}^2,\Lambda_h) + O(\|\xi\|^\kappa)\\ &\;\;\vdots\\ &= e^{i\xi\cdot\left(\sum_{k=1}^n \mathsf{M}^{-k}R(Q^{k-1}(\alpha))\right)}\widehat{y_{Q^n(\alpha)}}((\mathsf{M}^T)^{-n}\xi)S(\mathsf{M}^n,\Lambda_h) + O(\|\xi\|^\kappa),\end{aligned}$$

as $\xi\to 0$. That is, for $\alpha\in\Gamma_N$, as $\xi\to 0$, we have

$$\widehat{y_\alpha}(\xi) = e^{i\xi\cdot\left(\sum_{k=1}^n \mathsf{M}^{-k}R(Q^{k-1}(\alpha))\right)}\widehat{y_{Q^n(\alpha)}}((\mathsf{M}^T)^{-n}\xi)S(\mathsf{M}^n,\Lambda_h) + O(\|\xi\|^\kappa). \tag{2.5.13}$$

Let S denote the set of all $\alpha\in\Gamma_N$ such that $\alpha\in S$ means that there exists $n_\alpha\in\mathbb{N}$ satisfying $Q^{n_\alpha}(\alpha)=\alpha$. For every $\alpha\in S$, since $\{\partial^\mu\widehat{y_\alpha}(0) : |\mu|\leqslant h\}$

is uniquely determined by (2.3.2), by [24, Lemma 2.2], (2.5.13) with $n = n_\alpha$ has a unique solution $\{\partial^\mu \widehat{y_\alpha}(0) \ : \ h < |\mu| < \kappa\}$, which can be obtained recursively. More precisely, since for $\alpha \in S$, we have $Q^{n_\alpha}(\alpha) = \alpha$ for some $n_\alpha \in \mathbb{N}$. Therefore, (2.5.13) becomes

$$\widehat{y_\alpha}(\xi) = X_\alpha((\mathsf{M}^T)^{-n_\alpha}\xi)\widehat{y_\alpha}((\mathsf{M}^T)^{-n_\alpha}\xi)S(\mathsf{M}^{n_\alpha}, \Lambda_h) + O(\|\xi\|^\kappa), \qquad \xi \to 0,$$

where $X_\alpha((\mathsf{M}^T)^{-n_\alpha}\xi) := e^{i\xi\cdot\left(\sum_{k=1}^{n_\alpha} \mathsf{M}^{-k} R(Q^{k-1}(\alpha))\right)}$, or equivalently,

$$\widehat{y_\alpha}((\mathsf{M}^T)^{n_\alpha}\xi)I_{\#\Lambda_h} = X_\alpha(\xi)\widehat{y_\alpha}(\xi)S(\mathsf{M}^{n_\alpha}, \Lambda_h) + O(\|\xi\|^\kappa), \qquad \xi \to 0.$$

Note that $\sigma^{n_\alpha \nu}, \nu \in O_j$ and $\sigma^{-n_\alpha \mu}, \mu \in \Lambda_h$ are eigenvalues of $S(\mathsf{M}^{n_\alpha}, O_j)$ and $S(\mathsf{M}^{-n_\alpha}, \Lambda_h)$, respectively. By our assumption on M, we see that

$$\begin{aligned} & S(\mathsf{M}^{n_\alpha}, O_j) \otimes I_{\#\Lambda_h} - I_{\#O_j} \otimes S(\mathsf{M}^{n_\alpha}, \Lambda_h)^T \\ &= [S(\mathsf{M}^{n_\alpha}, O_j) \otimes S(\mathsf{M}^{-n_\alpha}, \Lambda_h)^T - I_{\#O_j} \otimes I_{\#\Lambda_h}][I_{\#O_j} \otimes S(\mathsf{M}^{n_\alpha}, \Lambda_h)^T] \end{aligned}$$

is invertible for all $j = h+1, \dots, \kappa - 1$. Therefore, by [24, Lemma 2.2],

$$\partial^\mu[\widehat{y_\alpha}((\mathsf{M}^T)^{n_\alpha}\xi)I_{\#\Lambda_h}](0) = \partial^\mu[X_\alpha(\xi)\widehat{y_\alpha}(\xi)S(\mathsf{M}^{n_\alpha}, \Lambda_h)](0), \qquad h < |\mu| < \kappa$$

has a unique solution $\{\partial^\mu \widehat{y_\alpha}(0) \ : \ h < |\mu| < \kappa\}$ for every $\alpha \in S$. Consequently, for every $\alpha \in S$, $\{\partial^\mu \widehat{y_\alpha}(0) \ : \ |\mu| < \kappa\}$ is completely determined by the relation (2.5.12).

For $\alpha \in \Gamma_N \backslash S$, since $Q^n(\alpha) \in \Gamma_N$ for all $n \in \mathbb{N}$, there must exist $N_\alpha \in \mathbb{N}$ such that $Q^{N_\alpha}(\alpha) \in S$. Hence, by (2.5.13) with $n = N_\alpha$, we have

$$\widehat{y_\alpha}(\xi) = e^{i\xi\cdot(\sum_{k=1}^{N_\alpha} \mathsf{M}^{-k} R(Q^{k-1}(\alpha))} \widehat{y_{Q^{N_\alpha}(\alpha)}}((\mathsf{M}^T)^{-N_\alpha}\xi)S(\mathsf{M}^{N_\alpha}, \Lambda_h) + O(\|\xi\|^\kappa). \tag{2.5.14}$$

By what has been proved, all $\{\partial^\mu \widehat{y_{Q^{N_\alpha}(\alpha)}}(0) \ : \ |\mu| < \kappa\}$ is completely determined by (2.5.12). Thus, it follows from (2.5.14) that for every $\alpha \in \Gamma_N \backslash S$, the values $\{\partial^\mu \widehat{y_\alpha}(0) \ : \ |\mu| < \kappa\}$ is completely determined by (2.5.14) and therefore, is uniquely determined by the system of linear equations in (2.5.12).

That is, we proved that if (2.5.12) holds, then all $\partial^\mu \widehat{y_\alpha}(0)$, $h < |\mu| < \kappa, \alpha \in \Gamma_N$ are uniquely determined by (2.5.12). Therefore, if there is a solution to the system of linear equations in (2.5.12), then the solution must be unique according to the above argument.

In the following, let us show that the system of linear equations in (2.5.12) indeed has a solution. Let $Y(\xi) := (Y_\alpha(\xi))_{\alpha \in \Gamma_N}$ with $Y_\alpha(\xi) := e^{i\alpha\cdot\xi}\left((i\xi)^\nu\right)_{\nu\in\Lambda_h}$. Since

$$\mathsf{M}\alpha = [\mathsf{M}\alpha]_{\Gamma_N} + \langle \mathsf{M}\alpha \rangle_{\Gamma_N} = R(\alpha) + Q(\alpha),$$

we have $Y_\alpha(\xi) = e^{i\alpha\cdot\xi}((i\xi)^\nu)_{\nu\in\Lambda_h} + O(\|\xi\|^{h+1})$ as $\xi \to 0$ and $\alpha \in \Gamma_N$, and by Lemma 2.7,

$$\begin{aligned} Y_\alpha(\mathsf{M}^T\xi) &= e^{i\mathsf{M}\alpha\cdot\xi}\left((i\mathsf{M}^T\xi)^\nu\right)_{\nu\in\Lambda_h} = e^{iR(\alpha)\cdot\xi}[e^{iQ(\alpha)\cdot\xi}\left((i\xi)^\nu\right)_{\nu\in\Lambda_h}]S(\mathsf{M},\Lambda_h) \\ &= e^{iR(\alpha)\cdot\xi}Y_{Q(\alpha)}(\xi)S(\mathsf{M},\Lambda_h). \end{aligned}$$

Therefore, if we take $\partial^\mu \widehat{y_\alpha}(0) = \partial^\mu Y_\alpha(0)$ for all $\alpha \in \Gamma_N$ and $|\mu| < \kappa$, then it is a solution to the system of linear equations in (2.5.12). By the uniqueness of the solution to (2.5.12), we must have (2.3.5), which completes the proof. □

Finally, we prove Theorem 2.3.

Proof of Theorem 2.3. Suppose that ϕ is G-symmetric and (2.3.9) holds. Then, by (2.3.8) and the refinement equation (1.1.1), for $\beta \in \Gamma_N$, we deduce that

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(x-k) = |\det \mathsf{M}|^{-1}\phi_\beta(\mathsf{M}^{-1}x)\\
&= |\det \mathsf{M}|^{-1}S(E^{-1},\Lambda_h)\phi_\beta(E(\mathsf{M}^{-1}x-\beta)+\beta)\\
&= S(E^{-1},\Lambda_h)\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(\mathsf{M}E\mathsf{M}^{-1}x-\mathsf{M}(E-I_d)\beta-k)\\
&=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} S(E^{-1},\Lambda_h)[a(k)]_{\beta,\gamma}S(\mathsf{M}E\mathsf{M}^{-1},\Lambda_h)\\
&\qquad\qquad \times \phi_\gamma(x-\mathsf{M}E^{-1}\mathsf{M}^{-1}k-J_{E^{-1},\gamma,\beta}+\gamma).
\end{aligned}
$$

Therefore, for $x=\alpha+j$ with $\alpha\in\Gamma_N$ and $j\in\mathbb{Z}^d$, we deduce that

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}[\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\alpha+j-k)\\
&=\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} S(E^{-1},\Lambda_h)[a(k)]_{\beta,\gamma}S(\mathsf{M}E\mathsf{M}^{-1},\Lambda_h)\\
&\qquad\qquad \times [\mathscr{D}^{\Lambda_h}\otimes\phi_\gamma](\alpha+j-\mathsf{M}E^{-1}\mathsf{M}^{-1}k-J_{E^{-1},\gamma,\beta}+\gamma).
\end{aligned} \tag{2.5.15}
$$

By (2.3.9) and the interpolation property of ϕ in (2.1.6), it is easy to verify that (2.5.15) implies (2.3.10).

Conversely, suppose that (2.3.10) and (2.3.12) are satisfied. By induction on n, we first prove that

$$
\phi_\beta(E(x-\beta)+\beta)=S(E,\Lambda_h)\phi_\beta(x) \tag{2.5.16}
$$

for all $x\in\mathsf{M}^{-n}(\mathbb{Z}^d+\Gamma_N), n\in\mathbb{N}_0, E\in G, \beta\in\Gamma_N$. By $\phi_\gamma(\alpha+j)=\delta(\alpha-\gamma)\delta(j)[1,0,\ldots,0]^T$ for all $\alpha,\gamma\in\Gamma_N$ and $j\in\mathbb{Z}^d$, it is evident that (2.5.16) holds for $n=0$. Suppose that (2.5.16) holds for $n-1$. Then for any $x\in\mathsf{M}^{-n}(\mathbb{Z}^d+\Gamma_N)$, we have $x=\mathsf{M}^{-1}y$ with $y:=\mathsf{M}x\in\mathsf{M}^{-(n-1)}(\mathbb{Z}^d+\Gamma_N)$.

Therefore,

$$|\det \mathsf{M}|^{-1} S(E,\Lambda_h)\phi_\beta(x) = S(E,\Lambda_h)\sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N}[a(k)]_{\beta,\gamma}\phi_\gamma(\mathsf{M}x-k)$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} S(E,\Lambda_h)[a(k)]_{\beta,\gamma}\phi_\gamma(y-k).$$

Since $y-k\in \mathsf{M}^{-(n-1)}(\mathbb{Z}^d+\Gamma_N)$, by our induction hypothesis in (2.5.16),

$$S(\mathsf{M}E\mathsf{M}^{-1},\Lambda_h)\phi_\gamma(y-k) = \phi_\gamma(\mathsf{M}E\mathsf{M}^{-1}(y-k-\gamma)+\gamma)$$
$$= \phi_\gamma(\mathsf{M}Ex - \mathsf{M}E\mathsf{M}^{-1}(k+\gamma)+\gamma).$$

Note that

$$J_{E,\gamma,\beta} = \mathsf{M}E\mathsf{M}^{-1}\gamma + \mathsf{M}(I_d-E)\beta = \gamma - (I_d - \mathsf{M}E\mathsf{M}^{-1})\gamma + \mathsf{M}(I_d-E)\beta.$$

By (2.3.12), we can verify that $\langle J_{E,\gamma,\beta}\rangle_{\Gamma_N} = \gamma$ for all $\gamma,\beta\in\Gamma_N$. Now by (2.3.10) and the above identities, we deduce that for any $x\in \mathsf{M}^{-n}(\mathbb{Z}^d+\Gamma_N)$,

$$|\det \mathsf{M}|^{-1} S(E,\Lambda_h)\phi_\beta(x) = \sum_{k\in\mathbb{Z}^d}\sum_{\beta\in\Gamma_N} S(E,\Lambda_h)[a(k)]_{\beta,\gamma}\phi_\gamma(y-k)$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} S(E,\Lambda_h)[a(k)]_{\beta,\gamma} S(\mathsf{M}E^{-1}\mathsf{M}^{-1},\Lambda_h)\phi_\gamma(\mathsf{M}Ex - \mathsf{M}E\mathsf{M}^{-1}(k+\gamma)+\gamma)$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} [a(\mathsf{M}E\mathsf{M}^{-1}k + [J_{E,\gamma,\beta}]_{\Gamma_N})]_{\beta,\langle J_{E,\gamma,\beta}\rangle_{\Gamma_N}}\phi_\gamma(\mathsf{M}Ex - \mathsf{M}E\mathsf{M}^{-1}(k+\gamma)+\gamma)$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} [a(\mathsf{M}E\mathsf{M}^{-1}k + [J_{E,\gamma,\beta}]_{\Gamma_N})]_{\beta,\gamma}\phi_\gamma(\mathsf{M}Ex - \mathsf{M}E\mathsf{M}^{-1}(k+\gamma)+\gamma)$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} [a(k)]_{\beta,\gamma}\phi_\gamma(\mathsf{M}Ex - k + \gamma - \mathsf{M}E\mathsf{M}^{-1}\gamma + [J_{E,\gamma,\beta}]_{\Gamma_N})$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\in\Gamma_N} [a(k)]_{\beta,\gamma}\phi_\gamma(\mathsf{M}Ex + \mathsf{M}(I_d-E)\beta - k)$$
$$= |\det \mathsf{M}|^{-1}\phi_\beta(Ex+\beta-E\beta) = |\det \mathsf{M}|^{-1}\phi_\beta(E(x-\beta)+\beta).$$

Hence, (2.5.16) holds for n. By induction, (2.5.16) holds for all $n \in \mathbb{N}_0$,

Since ϕ is continuous and $\{\mathsf{M}^{-n}(\mathbb{Z}^d + \Gamma_N) \ : \ n \in \mathbb{N}_0\}$ is dense in $\mathbb{R}^d$, we conclude that (2.3.8) holds. So, ϕ is G-symmetric. $\square$

2.6 Conclusions and Remarks

In this chapter, we present in Theorem 2.1 a complete characterization of a generalized interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ in terms of its mask and study its sum rule structure in Theorem 2.2. We also study symmetry of M-refinable function vectors in high dimensions, which is related to a symmetry group. We give a characterization of an interpolating refinable function vector of type $(\mathsf{M}, \Gamma_N, h)$ to be symmetric with respect to a symmetry group in terms of its mask and present several examples in dimension two. We provide in Section 2.4 a family of one dimensional interpolatory masks of type $(\mathsf{d}, \Gamma_r, h)$ with arbitrarily high orders of sum rules.

When dimension d is higher than 2, for example, in dimension three, there is a great number of symmetry groups. However, other than the trivial dilation matrix kI_d for some integer $|k| > 1$, we do not know whether there exists a nontrivial dilation matrix M for which there is a symmetry group G satisfying (2.3.6); that is, G is a symmetry group with respect to M.

When $L = 1$ (scalar function), it is well known that a scalar mask a satisfying the sum rules of order $\kappa + 1$ with respect to a dilation matrix M as in (2.2.1) is equivalent to (see [43])

$$\sum_{\beta \in \mathsf{M}\mathbb{Z}^d} a(\gamma+\beta)p(\gamma+\beta) = \sum_{\beta \in \mathsf{M}\mathbb{Z}^d} a(\beta)p(\beta), \quad \forall \gamma \in \mathbb{Z}^d, p \in \Pi_\kappa. \qquad (2.6.1)$$

Note that (2.6.1) depends only on the lattice $\mathsf{M}\mathbb{Z}^d$.

The lattice generated by the dilation matrix $\mathsf{M} = M_{\sqrt{2}}$ in dimension two is

$$M_{\sqrt{2}}\mathbb{Z}^2 = \{(\beta_1, \beta_2) \in \mathbb{Z}^2 \,:\, \beta_1 + \beta_2 \text{ is an even number}\},$$

which is called the *quincunx lattice*. An interpolatory mask a of type $(M_{\sqrt{2}}, \Gamma_{I_2}, 0)$ is called a *quincunx interpolatory mask*; that is, the mask a satisfies

$$a(0) = \frac{1}{2} \quad \text{and} \quad a(\beta) = 0 \qquad \forall \beta \in M_{\sqrt{2}}\mathbb{Z}^2 \backslash \{0\}.$$

In [32, Theorem 3.3], Han and Jia proved the following result:

Theorem 2.8. *Given a pair (m, n) of nonnegative integers with $m+n$ being an odd integer, there exists a unique quincunx interpolatory mask $a_{m,n}$ such that $a_{m,n}$ is supported on*

$$\{(\beta_1, \beta_2) \in \mathbb{Z}^2 \,:\, |\beta_1| \leqslant m, |\beta_2| \leqslant n\},$$

and $a_{m,n}$ satisfies the sum rules of order $m + n + 1$ with respect to the quincunx lattice: $\{(\beta_1, \beta_2) \in \mathbb{Z}^2 \,:\, \beta_1 + \beta_2$ is an even integer$\}$.

A natural question is whether the above result is still true in high dimensions. The answer is *yes*. In high dimensions, the lattice corresponding the quincunx lattice is called the *checkerboard lattice* given by

$$\{(\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d \, : \, \beta_1 + \cdots + \beta_d \text{ is an even integer}\}.$$

There are many dilation matrices that can generate the above lattice, for example, a dilation matrix M in dimension three given by

$$\mathsf{M} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

generates the checkerboard lattice in dimension three. We have the following theorem that generalizes Theorem 2.8 to any dimension (see Appendix A for its proof).

Theorem 2.9. *Let $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$ be such that $m_1 + \cdots + m_d$ is an odd integer. Let $\Gamma := \{(\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d \, : \, \beta_1 + \cdots + \beta_d$ is an even integer$\}$ be the checkerboard lattice. Then there exists a unique interpolatory mask a_m such that*

(1) *$a_m(0) = \frac{1}{2}$ and $a_m(\beta) = 0$ for all $\beta \in \Gamma \backslash \{0\}$;*

(2) *a_m is supported on $S := \{(\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d \, : \, |\beta_1| \leqslant m_1, \ldots, |\beta_d| \leqslant m_d\}$;*

(3) *a_m satisfies the sum rules of order $|m| + 1$ with respect to the lattice Γ:*

$$\sum_{\beta \in \Gamma} a_m(\gamma + \beta) p(\gamma + \beta) = \sum_{\beta \in \Gamma} a_m(\beta) p(\beta), \ \forall \gamma \in \mathbb{Z}^d; p \in \Pi_{|m|}. \quad (2.6.2)$$

Chapter 3

Matrix Extension with Symmetry

3.1 Introduction and Main Results

In Chapters 1 and 2, we discussed the characterizations and construction of interpolating refinable function vectors with orthogonality, compact support, and symmetry. After we obtained such refinable function vectors with those nice properties, a natural question is: How to derive the corresponding multiwavelets? More importantly, how to derive the multiwavelets with symmetry when the given refinable function vectors have symmetry? In this chapter, we shall study this problem, which is the so-called *matrix extension* problem (we shall discuss in Section 3.4 for the connection of multiwavelets to matrix extension).

The matrix extension problem plays a fundamental role in many areas such as electronic engineering, system sciences, mathematics, and etc. To mention only a few references here on this topic, see [7, 8, 11, 16, 20, 27, 47, 52, 54, 59, 64, 67, 69]. For example, matrix extension is an indispensable tool in the design of filter banks in electronic engineering ([52, 67, 69]) and in the construction of multiwavelets in wavelet analysis ([7, 8, 11, 14, 16, 20, 27, 35, 42, 54, 59]). In order to state the matrix extension problem and our main results on this topic, let us introduce some notation and definitions first.

Let $\mathsf{p}(z) = \sum_{k\in\mathbb{Z}} p_k z^k, z \in \mathbb{C}\backslash\{0\}$ be a Laurent polynomial with complex coefficients $p_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$. We say that p has *symmetry* if its coefficient sequence $\{p_k\}_{k\in\mathbb{Z}}$ has symmetry; more precisely, there exist $\varepsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ such that

$$p_{c-k} = \varepsilon p_k, \qquad \forall\, k \in \mathbb{Z}. \tag{3.1.1}$$

If $\varepsilon = 1$, then p is symmetric about the point $c/2$; if $\varepsilon = -1$, then p is antisymmetric about the point $c/2$. Symmetry of a Laurent polynomial can be conveniently expressed using a symmetry operator $\mathcal{S}$ defined by

$$\mathcal{S}\mathsf{p}(z) := \frac{\mathsf{p}(z)}{\mathsf{p}(1/z)}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{3.1.2}$$

When p is not identically zero, it is evident that (3.1.1) holds if and only if $\mathcal{S}\mathsf{p}(z) = \varepsilon z^c$. For the zero polynomial, it is very natural that $\mathcal{S}0$ can be assigned any symmetry pattern; that is, for every occurrence of $\mathcal{S}0$ appearing in an identity in this paper, $\mathcal{S}0$ is understood to take an appropriate choice of εz^c for some $\varepsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ so that the identity holds. If P is an $r \times s$ matrix of Laurent polynomials with symmetry, then we can apply the operator $\mathcal{S}$ to each entry of P, that is, $\mathcal{S}\mathsf{P}$ is an $r \times s$ matrix such that $[\mathcal{S}\mathsf{P}]_{j,k} := \mathcal{S}([\mathsf{P}]_{j,k})$, where $[\mathsf{P}]_{j,k}$ is the (j, k)-entry of the matrix P.

For two matrices P and Q of Laurent polynomials with symmetry, even though all the entries in P and Q have symmetry, their sum $\mathsf{P}+\mathsf{Q}$, difference $\mathsf{P}-\mathsf{Q}$, or product PQ, if well defined, generally may not have symmetry any more. This is one of the difficulties for matrix extension with symmetry. In order for $\mathsf{P}\pm\mathsf{Q}$ or PQ to possess some symmetry, the symmetry patterns of P and Q should be compatible. For example, if $\mathcal{S}\mathsf{P} = \mathcal{S}\mathsf{Q}$, that is, both P and Q have the same symmetry pattern, then indeed $\mathsf{P}\pm\mathsf{Q}$ has symmetry and $\mathcal{S}(\mathsf{P}\pm\mathsf{Q}) = \mathcal{S}\mathsf{P} = \mathcal{S}\mathsf{Q}$. In the following, we discuss the compatibility of symmetry patterns of matrices of Laurent polynomials. For an $r \times s$ matrix $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}} P_k z^k$, we denote

$$\mathsf{P}^*(z) := \sum_{k\in\mathbb{Z}} P_k^* z^{-k} \quad \text{with} \quad P_k^* := \overline{P_k}^T, \qquad k \in \mathbb{Z}, \tag{3.1.3}$$

where $\overline{P_k}^T$ denotes the transpose of the complex conjugate of the constant matrix P_k in $\mathbb{C}$. We say that *the symmetry of* P *is compatible* or P *has compatible symmetry*, if

$$\mathcal{S}\mathsf{P}(z) = (\mathcal{S}\theta_1)^*(z)\mathcal{S}\theta_2(z), \tag{3.1.4}$$

for some $1 \times r$ and $1 \times s$ row vectors θ_1 and θ_2 of Laurent polynomials with symmetry. For an $r \times s$ matrix P and an $s \times t$ matrix Q of Laurent polynomials, we say that (P, Q) *has mutually compatible symmetry* if

$$\mathcal{S}\mathsf{P}(z) = (\mathcal{S}\theta_1)^*(z)\mathcal{S}\theta(z) \quad \text{and} \quad \mathcal{S}\mathsf{Q}(z) = (\mathcal{S}\theta)^*(z)\mathcal{S}\theta_2(z) \tag{3.1.5}$$

for some $1\times r$, $1\times s$, $1\times t$ row vectors $\theta_1, \theta, \theta_2$ of Laurent polynomials with symmetry. If (P, Q) has mutually compatible symmetry as in (3.1.5), then their product PQ has compatible symmetry and in fact $\mathcal{S}(\mathsf{PQ}) = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$.

For a matrix of Laurent polynomials, another important property is the support of its coefficient sequence. For $\mathsf{P} = \sum_{k\in\mathbb{Z}} P_k z^k$ such that $P_k = \mathbf{0}$ for all $k \in \mathbb{Z}\backslash[m, n]$ with $P_m \neq \mathbf{0}$ and $P_n \neq \mathbf{0}$, we define its coefficient support to be $\mathrm{coeffsupp}(\mathsf{P}) := [m, n]$ and the length of its coefficient support to be $|\mathrm{coeffsupp}(\mathsf{P})| := n - m$. In particular, we define $\mathrm{coeffsupp}(\mathbf{0}) := \emptyset$, the empty set, and $|\mathrm{coeffsupp}(\mathbf{0})| := -\infty$. Also, we use $\mathrm{coeff}(\mathsf{P}, k) := P_k$ to denote the coefficient matrix (vector) P_k of z^k in P. In this thesis, $\mathbf{0}$ always denotes a general zero matrix whose size can be determined in the context.

The Laurent polynomials that we shall consider have their coefficients in a subfield $\mathbb{F}$ of the complex field $\mathbb{C}$. Let $\mathbb{F}$ denote a subfield of $\mathbb{C}$ such that $\mathbb{F}$ is closed under the operations of complex conjugate of $\mathbb{F}$ and square roots of positive numbers in $\mathbb{F}$. In other words, the subfield $\mathbb{F}$ of $\mathbb{C}$ satisfies the following properties:

$$\bar{x} \in \mathbb{F} \quad \text{and} \quad \sqrt{y} \in \mathbb{F}, \qquad \forall\, x, y \in \mathbb{F} \quad \text{with} \quad y > 0. \tag{3.1.6}$$

Two particular examples of such subfields $\mathbb{F}$ are $\mathbb{F} = \mathbb{R}$ (the field of real numbers) and $\mathbb{F} = \mathbb{C}$ (the field of complex numbers). A nontrivial example is the field of all algebraic number, i.e., the algebraic closure $\overline{\mathbb{Q}}$ of the rational number $\mathbb{Q}$. A subfield of $\mathbb{R}$ given by $\overline{\mathbb{Q}} \cap \mathbb{R}$ also satisfies (3.1.6).

Now, we introduce the general matrix extension problem with symmetry. We shall use r and s to denote two positive integers such that $1 \leqslant r \leqslant s$. Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in $\mathbb{F}$ such that $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible, where I_r denotes the $r \times r$ identity matrix. The matrix extension problem with symmetry is to find an $s \times s$ square matrix P_e of Laurent polynomials with coefficients in $\mathbb{F}$ and with symmetry such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$ (that is, the submatrix of the first r rows of P_e is the given matrix P), the symmetry of P_e is compatible, and $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$ (that is, P_e is paraunitary). Moreover, in many applications, it is often highly desirable that the coefficient support of P_e can be controlled by that of P in some way.

In this chapter, we study this general matrix extension problem with symmetry and we completely solve this problem as follows:

Theorem 3.1. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ such that (3.1.6) holds. Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in $\mathbb{F}$ such that the symmetry of P is compatible and $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$. Then there exists an $s \times s$ square matrix P_e, which can be constructed by Algorithm 3.1 in Section 3.3 from the given matrix P, of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(i) $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$, *that is, the submatrix of the first r rows of P_e is P;*

(ii) P_e *is paraunitary:* $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ *for all $z \in \mathbb{C}\backslash\{0\}$;*

(iii) *The symmetry of P_e is compatible;*

(iv) *The coefficient support of P_e is controlled by that of P in the following sense:*

$$|\mathrm{coeffsupp}([\mathsf{P}_e]_{j,k})| \leqslant \max_{1\leqslant n\leqslant r} |\mathrm{coeffsupp}([\mathsf{P}]_{n,k})|, \qquad 1 \leqslant j, k \leqslant s. \tag{3.1.7}$$

i.e., the length of the coefficient support of any entry in the k-th column of P_e is controlled by that of the entry in the k-th column of P with maximal length of coefficient support.

Theorem 3.1 on matrix extension with symmetry is built on a stronger result which represents any given paraunitary matrix having compatible symmetry by a simple cascade structure. The following result leads to a proof of Theorem 3.1 and completely characterizes any paraunitary matrix P in Theorem 3.1.

Theorem 3.2. *Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in a subfield $\mathbb{F}$ of $\mathbb{C}$ such that (3.1.6) holds. Then $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible as in (3.1.4), if and only if, there exist $s \times s$ matrices $\mathsf{P}_0, \ldots, \mathsf{P}_{J+1}$ of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(1) P *can be represented as a product of* $\mathsf{P}_0, \ldots, \mathsf{P}_{J+1}$*:*

$$\mathsf{P}(z) = [I_r, \mathbf{0}]\mathsf{P}_{J+1}(z)\mathsf{P}_J(z)\cdots\mathsf{P}_1(z)\mathsf{P}_0(z); \tag{3.1.8}$$

(2) $\mathsf{P}_j, 1 \leqslant j \leqslant J$, *are elementary:* $\mathsf{P}_j(z)\mathsf{P}_j^*(z) = I_s$ *and* $\operatorname{coeffsupp}(\mathsf{P}_j) \subseteq [-1, 1]$*;*

(3) $(\mathsf{P}_{j+1}, \mathsf{P}_j)$ *has mutually compatible symmetry for all* $0 \leqslant j \leqslant J$*;*

(4) $\mathsf{P}_0 = \mathsf{U}_{\mathcal{S}\theta_2}^*$ *and* $\mathsf{P}_{J+1} = diag(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$*, where* $\mathsf{U}_{\mathcal{S}\theta_1}$*,* $\mathsf{U}_{\mathcal{S}\theta_2}$ *are products of a permutation matrix with a diagonal matrix of monomials, as defined in (3.3.2);*

(5) $J \leqslant \max_{1 \leqslant m \leqslant r, 1 \leqslant n \leqslant s} \lceil |\operatorname{coeffsupp}([\mathsf{P}]_{m,n})|/2 \rceil$*, where* $\lceil \cdot \rceil$ *is the ceiling function.*

The representation in (3.1.8) (without symmetry) is often called the cascade structure in the literature of engineering, see [52, 67]. In the context of wavelet analysis, matrix extension without symmetry has been discussed by Lawton, Lee, and Shen in their paper [54]. In electronic engineering, an algorithm using the cascade structure for matrix extension without symmetry has been given in [67] for filter banks with the perfect reconstruction property. The algorithms in [54, 67] mainly deal with the special case that P is a row vector (that is, $r = 1$ in our case) without symmetry and the coefficient support of the derived matrix P_e indeed can be controlled by that of P. The algorithms in [54, 67] for the special case $r = 1$ can be employed to handle a general $r \times s$ matrix P without symmetry, see [54, 67] for detail. However, for the general case $r > 1$, it is no longer clear whether the coefficient support of the derived matrix P_e obtained by the algorithms in [54, 67] can still be controlled by that of P.

Several special cases of matrix extension with symmetry have been considered in the literature. For $\mathbb{F} = \mathbb{R}$ and $r = 1$, matrix extension with symmetry has been considered in [59]. For $r = 1$, matrix extension with symmetry has been studied in [27] and a simple algorithm is given there. In the context of wavelet analysis, several particular cases of matrix extension with symmetry related to the construction of wavelets and multiwavelets have been investigated in [8, 20, 27, 52, 59]. However, for the general case of an $r \times s$ matrix, the approaches on matrix extension with symmetry in [27, 59] for the particular case $r = 1$ cannot be employed to handle the general case. The algorithms in [27, 59] are very difficult to be generalized to the general case $r > 1$, partially due to the complicated relations of the symmetry patterns between different rows of P. For the general case of matrix extension with symmetry, it becomes much harder to control the coefficient support of the derived matrix P_e, comparing with the special case $r = 1$. Extra effort is needed in any algorithm of deriving P_e so that its coefficient support can be controlled by that of P.

The structure of this chapter is as follows. In Section 3.2, we shall introduce some auxiliary results. In Section 3.3, we shall present a step-by-step algorithm which leads to constructive proofs of Theorems 3.1 and 3.2. In Section 3.4, we shall discuss an application of our main results on matrix extension with symmetry to the construction of symmetric orthonormal multiwavelets in wavelet analysis. Examples will be provided to illustrate our algorithms. We shall prove Theorems 3.1 and 3.2 in Section 3.5. Conclusions and remarks shall be given in the last section. Most of the results in this chapter have been accepted for publication in [38].

3.2 Auxiliary Results

In this section, we shall introduce some auxiliary results.

First, let us introduce a unitary matrix $U_{\mathtt{f}}$ constructed with respect to a given row vector $\mathtt{f}$ and a unitary matrix U_G constructed with respect to a given matrix G, which shall be used in Algorithm 3.1 and the proofs of Theorems 3.1 and 3.2. For a $1 \times n$ row vector $\mathtt{f}$ in $\mathbb{F}$ such that $\|\mathtt{f}\| \neq 0$, where $\|\mathtt{f}\|^2 := \mathtt{f}\mathtt{f}^*$, we define $n_{\mathtt{f}}$ to be the number of nonzero entries in $\mathtt{f}$ and $\mathbf{e}_j := [0, \dots, 0, 1, 0, \dots, 0]$ to be the j-th unit coordinate row vector in $\mathbb{R}^n$. Let $E_{\mathtt{f}}$ be a permutation matrix such that $\mathtt{f}E_{\mathtt{f}} = [f_1, \dots, f_{n_{\mathtt{f}}}, 0, \dots, 0]$ with $f_j \neq 0$ for $j = 1, \dots, n_{\mathtt{f}}$. We define

$$V_{\mathtt{f}} := \begin{cases} \frac{\bar{f}_1}{|f_1|}, & \text{if } n_{\mathtt{f}} = 1; \\ \frac{\bar{f}_1}{|f_1|}\left(I_n - \frac{2}{\|v_{\mathtt{f}}\|^2} v_{\mathtt{f}}^* v_{\mathtt{f}}\right), & \text{if } n_{\mathtt{f}} > 1, \end{cases} \tag{3.2.1}$$

where $v_{\mathtt{f}} := \mathtt{f} - \frac{f_1}{|f_1|}\|\mathtt{f}\|\mathbf{e}_1$. Observing that $\|v_{\mathtt{f}}\|^2 = 2\|\mathtt{f}\|(\|\mathtt{f}\| - |f_1|)$, we can verify that $V_{\mathtt{f}}V_{\mathtt{f}}^* = I_n$ and $\mathtt{f}E_{\mathtt{f}}V_{\mathtt{f}} = \|\mathtt{f}\|\mathbf{e}_1$. Let $U_{\mathtt{f}} := E_{\mathtt{f}}V_{\mathtt{f}}$. Then $U_{\mathtt{f}}$ is unitary and satisfies $U_{\mathtt{f}} = [\frac{\mathtt{f}^*}{\|\mathtt{f}\|}, F^*]$ for some $(n-1) \times n$ matrix F in $\mathbb{F}$ such that $\mathtt{f}U_{\mathtt{f}} = [\|\mathtt{f}\|, 0, \ldots, 0]$. We also define $U_{\mathtt{f}} := I_n$ if $\mathtt{f} = \mathbf{0}$ and $U_{\mathtt{f}} := \emptyset$ if $\mathtt{f} = \emptyset$. Here, $U_{\mathtt{f}}$ plays the role of reducing the number of nonzero entries in $\mathtt{f}$. More generally, for an $r \times n$ nonzero matrix G of rank m in $\mathbb{F}$, employing the above procedure recursively to each row of G, we can obtain an $n \times n$ unitary matrix U_G such that $GU_G = [R, \mathbf{0}]$ for some $r \times m$ lower triangular matrix R of rank m (This is in fact the QR decomposition). If $G_1G_1^* = G_2G_2^*$, then the above procedure produces two matrices U_{G_1}, U_{G_2} such that $G_1U_{G_1} = [R, \mathbf{0}]$ and $G_2U_{G_2} = [R, \mathbf{0}]$ for some lower triangular matrix R of full rank. It is important to notice that the constructions of $U_{\mathtt{f}}$ and U_G only involve the nonzero entries of $\mathtt{f}$ and nonzero columns of G, respectively. In other words, up to a permutation matrix, we have

$$\begin{aligned} [U_{\mathtt{f}}]_{j,:} &= ([U_{\mathtt{f}}]_{:,j})^T = \mathbf{e}_j, \quad \text{if } [\mathtt{f}]_j = 0, \\ [U_G]_{j,:} &= ([U_G]_{:,j})^T = \mathbf{e}_j, \quad \text{if } [G]_{:,j} = \mathbf{0}. \end{aligned} \tag{3.2.2}$$

Next, we shall construct a paraunitary matrix B_{q} with respect to a row vector q of Laurent polynomial such that B_{q} reduces the length of the coefficient support of q by 2 and keeps its symmetry pattern.

Let q be a $1 \times s$ row vector of Laurent polynomials satisfying $\mathsf{q}\mathsf{q}^* = 1$ and $\mathcal{S}\mathsf{q} = \varepsilon z^c[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some $\varepsilon \in \{-1, 1\}$, $c \in \{0, 1\}$ and nonnegative integers $s_1, \ldots, s_4$ such that $s_1 + s_2 + s_3 + s_4 = s$. For $\varepsilon = -1$, there is a permutation matrix E_ε such that

$$\mathcal{S}(\mathsf{q}E_\varepsilon) = z^c[\mathbf{1}_{s_2}, -\mathbf{1}_{s_1}, z^{-1}\mathbf{1}_{s_4}, -z^{-1}\mathbf{1}_{s_3}].$$

For $\varepsilon = 1$, we let $E_\varepsilon := I_s$. Then, $\mathsf{q}E_\varepsilon$ must take the form in either (3.2.3) or (3.2.4) with $\mathtt{f}_1 \neq \mathbf{0}$ as follows:

$$\begin{aligned} \mathsf{q}E_\varepsilon = {} & [\mathtt{f}_1, -\mathtt{f}_2, \mathtt{g}_1, -\mathtt{g}_2]z^{\ell_1} + [\mathtt{f}_3, -\mathtt{f}_4, \mathtt{g}_3, -\mathtt{g}_4]z^{\ell_1+1} + \cdots \\ & + [\mathtt{f}_3, \mathtt{f}_4, \mathtt{g}_1, \mathtt{g}_2]z^{\ell_2-1} + [\mathtt{f}_1, \mathtt{f}_2, \mathbf{0}, \mathbf{0}]z^{\ell_2}; \end{aligned} \tag{3.2.3}$$

$$\begin{aligned} \mathsf{q}E_\varepsilon = {} & [\mathbf{0}, \mathbf{0}, \mathtt{f}_1, -\mathtt{f}_2]z^{\ell_1} + [\mathtt{g}_1, -\mathtt{g}_2, \mathtt{f}_3, -\mathtt{f}_4]z^{\ell_1+1} + \cdots \\ & + [\mathtt{g}_3, \mathtt{g}_4, \mathtt{f}_3, \mathtt{f}_4]z^{\ell_2-1} + [\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]z^{\ell_2}. \end{aligned} \tag{3.2.4}$$

If $\mathsf{q}E_\varepsilon$ takes the form in (3.2.4), we construct a permutation matrix E_q so that $[\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]E_\mathsf{q} = [\mathtt{f}_1, \mathtt{f}_2, \mathtt{g}_1, \mathtt{g}_2]$ and let $\mathsf{U}_{\mathsf{q},\varepsilon} := E_\varepsilon E_\mathsf{q} \operatorname{diag}(I_{s-s_\mathtt{g}}, z^{-1}I_{s_\mathtt{g}})$, where $s_\mathtt{g}$ is the size of the row vector $[\mathtt{g}_1, \mathtt{g}_2]$. Then $\mathsf{q}\mathsf{U}_{\mathsf{q},\varepsilon}$ takes the form in (3.2.3). For $\mathsf{q}E_\varepsilon$ of form (3.2.3), we simply let $\mathsf{U}_{\mathsf{q},\varepsilon} := E_\varepsilon$. In this way, $\mathsf{q}_0 := \mathsf{q}\mathsf{U}_{\mathsf{q},\varepsilon}$ always takes the form in (3.2.3) with $\mathtt{f}_1 \neq \mathbf{0}$.

Note that $\mathsf{U}_{\mathsf{q},\varepsilon}\mathsf{U}_{\mathsf{q},\varepsilon}^* = I_s$ and $\|\mathtt{f}_1\| = \|\mathtt{f}_2\|$ if $\mathsf{q}_0\mathsf{q}_0^* = 1$. Now an $s \times s$ paraunitary matrix $\mathsf{B}_{\mathsf{q}_0}$ to reduce the coefficient support of q_0 as in (3.2.3) with $\mathtt{f}_1 \neq \mathbf{0}$ from $[\ell_1, \ell_2]$ to $[\ell_1 + 1, \ell_2 - 1]$ is given by:

$$\mathsf{B}_{\mathsf{q}_0}^* := \frac{1}{c} \left[\begin{array}{c|c|c|c} \mathtt{f}_1(z + \frac{c_0}{c_{\mathtt{f}_1}} + \frac{1}{z}) & \mathtt{f}_2(z - \frac{1}{z}) & \mathtt{g}_1(1 + \frac{1}{z}) & \mathtt{g}_2(1 - \frac{1}{z}) \\ cF_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline -\mathtt{f}_1(z - \frac{1}{z}) & -\mathtt{f}_2(z - \frac{c_0}{c_{\mathtt{f}_1}} + \frac{1}{z}) & -\mathtt{g}_1(1 - \frac{1}{z}) & -\mathtt{g}_2(1 + \frac{1}{z}) \\ \mathbf{0} & cF_2 & \mathbf{0} & \mathbf{0} \\ \hline \frac{c_{\mathtt{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1 + z) & -\frac{c_{\mathtt{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1 - z) & c_{\mathtt{g}_1'}\mathtt{g}_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & cG_1 & \mathbf{0} \\ \hline \frac{c_{\mathtt{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1 - z) & -\frac{c_{\mathtt{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1 + z) & \mathbf{0} & c_{\mathtt{g}_2'}\mathtt{g}_2' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & cG_2 \end{array} \right], \tag{3.2.5}$$

where $[\frac{\mathtt{f}_j^*}{\|\mathtt{f}_j\|}, F_j^*] = U_{\mathtt{f}_j}$, $[\mathtt{g}_j'^*, G_j^*] = U_{\mathtt{g}_j}$ are unitary constant extension matrices in $\mathbb{F}$ for vectors $\mathtt{f}_j, \mathtt{g}_j$ in $\mathbb{F}$, for $j = 1, 2$, respectively. And the constant

$c, c_0, c_{\mathtt{f}_1}, c_{\mathtt{g}_1}, c_{\mathtt{g}_2}, c_{\mathtt{g}'_1}, c_{\mathtt{g}'_2}$ are: $c_{\mathtt{f}_1} := \|\mathtt{f}_1\|$, $c_{\mathtt{g}_1} := \|\mathtt{g}_1\|$, $c_{\mathtt{g}_2} := \|\mathtt{g}_2\|$,

$$\begin{aligned} & c_{\mathtt{g}'_1} := \begin{cases} \frac{-2c_{\mathtt{f}_1} - \overline{c_0}}{c_{\mathtt{g}_1}} & \text{if } \mathtt{g}_1 \neq \mathbf{0}; \\ c & \text{otherwise,} \end{cases} \qquad c_{\mathtt{g}'_2} := \begin{cases} \frac{2c_{\mathtt{f}_1} - \overline{c_0}}{c_{\mathtt{g}_2}} & \text{if } \mathtt{g}_2 \neq \mathbf{0}; \\ c & \text{otherwise,} \end{cases} \\ & c := (4c_{\mathtt{f}_1}^2 + 2c_{\mathtt{g}_1}^2 + 2c_{\mathtt{g}_2}^2 + |c_0|^2)^{1/2}, \\ & c_0 := \text{coeff}(\mathsf{q}_0, \ell_1 + 1)\text{coeff}(\mathsf{q}_0^*, -\ell_2)/c_{\mathtt{f}_1}. \end{aligned} \tag{3.2.6}$$

The operations for the emptyset $\emptyset$ are defined by $\|\emptyset\| = \emptyset$, $\emptyset + A = A$ and $\emptyset \cdot A = \emptyset$ for any object A.

In fact, the matrix $\mathsf{B}_{\mathsf{q}_0}$ is obtained as follows.

Suppose, $\mathcal{S}\mathsf{q}_0 = [\mathbf{1}_{s'_1}, -\mathbf{1}_{s'_2}, z^{-1}\mathbf{1}_{s'_3}, -z^{-1}\mathbf{1}_{s'_4}]$. Then due to $\|\mathtt{f}_1\| = \|\mathtt{f}_2\| > 0$, we have $s'_1, s'_2 > 0$. Let $\mathcal{I} := \{1, 1+s'_1, (1-\delta(s'_3))(1+s'_1+s'_2), (1-\delta(s'_4))(1+s'_1+s'_2+s'_3)\}\backslash\{0\}$ be an index set and s_I be its number of entries, which is at most 4. Let $U := \text{diag}(U_{\mathtt{f}_1}, U_{\mathtt{f}_2}, U_{\mathtt{g}_1}, U_{\mathtt{g}_2})$. Then by our construction of $U_{\mathtt{f}}$, it is easy to verify that $\text{coeffsupp}([\mathsf{q}U]_j) \subseteq [\ell_1 + 1, \ell_2 - 1]$ for all $j \notin \mathcal{I}$. Hence, we only need to find a paraunitary matrix that reduces the lengths of the coefficient support of those entries $j \in \mathcal{I}$ for $\mathsf{q}_0 U$. Let q_1 be a $1 \times s_I$ row vector such that $[\mathsf{q}_1]_j = [\mathsf{q}_0 U]_{\mathcal{I}_j}$ for all $j = 1, \ldots, s_I$. Then q_1 is of the form:

$$\begin{aligned} \mathsf{q}_1 = {} & [c_{\mathtt{f}_1}, -c_{\mathtt{f}_1}, c_{\mathtt{g}_1}, -c_{\mathtt{g}_2}]z^{\ell_1} + [c_{\mathtt{f}_3}, -c_{\mathtt{f}_4}, *, -*]z^{\ell_1+1} + \cdots \\ & + [c_{\mathtt{f}_3}, c_{\mathtt{f}_4}, c_{\mathtt{g}_1}, c_{\mathtt{g}_2}]z^{\ell_2-1} + [c_{\mathtt{f}_1}, c_{\mathtt{f}_1}, 0, 0]z^{\ell_2}, \end{aligned}$$

where $c_{\mathtt{f}_3} := \mathtt{f}_3\mathtt{f}_1^*/c_{\mathtt{f}_1}$ and $c_{\mathtt{f}_4} := \mathtt{f}_4\mathtt{f}_2^*/c_{\mathtt{f}_1}$. Define an $s_I \times s_I$ matrix U_1:

$$\mathsf{U}_1 := \frac{1}{c}\begin{bmatrix} c_{\mathtt{f}_1}(z+\frac{1}{z}) + \overline{c_0} & c_{\mathtt{f}_1}(z-\frac{1}{z}) & c_{\mathtt{g}_1}(1+\frac{1}{z}) & c_{\mathtt{g}_2}(1-\frac{1}{z}) \\ -c_{\mathtt{f}_1}(z-\frac{1}{z}) & -c_{\mathtt{f}_1}(z+\frac{1}{z}) + \overline{c_0} & -c_{\mathtt{g}_1}(z+\frac{1}{z}) & -c_{\mathtt{g}_2}(z+\frac{1}{z}) \\ c_{\mathtt{g}_1}(1+z) & -c_{\mathtt{g}_1}(1-z) & c_3 & 0 \\ c_{\mathtt{g}_2}(1-z) & -c_{\mathtt{g}_2}(1+z) & 0 & c_4 \end{bmatrix},$$

where

$$c_3 := \begin{cases} -2c_{\mathtt{f}_1} - \overline{c_0} & \text{if } c_{\mathtt{g}_1} > 0; \\ \emptyset & \text{if } \mathtt{g}_1 = \emptyset; \\ c & \text{otherwise,} \end{cases} \quad \text{and} \quad c_4 := \begin{cases} 2c_{\mathtt{f}_1} - \overline{c_0} & \text{if } c_{\mathtt{g}_2} > 0; \\ \emptyset & \text{if } \mathtt{g}_2 = \emptyset; \\ c & \text{otherwise.} \end{cases}$$

Direct computations show that U_1 is paraunitary and U_1 reduces the length of the coefficient support of q_1 exactly by 2. Moreover, U_1 does not change the symmetry pattern of q_1. Let U_0 be the paraunitary matrix of size $s \times s$ that extends U_1, i.e., $[\mathsf{U}_0]_{\mathcal{I}_j,\mathcal{I}_k} := [\mathsf{U}_1]_{j,k}$, for $1 \leqslant j,k \leqslant s_I$ and $[\mathsf{U}_0]_{j,k} = \delta(j-k)$ for all $j,k \notin \mathcal{I}$. Then, one can easily check that $\mathsf{B}_{\mathsf{q}_0} = U\mathsf{U}_0$.

Define $\mathsf{B}_{\mathsf{q}} := \mathsf{U}_{\mathsf{q},\varepsilon}\mathsf{B}_{\mathsf{q}_0}\mathsf{U}^*_{\mathsf{q},\varepsilon}$. Then B_{q} is paraunitary. Due to the particular form of $\mathsf{B}_{\mathsf{q}_0}$ as in (3.2.5), we have the following lemma regarding the properties of the paraunitary matrix B_{q}, which plays an important role in our matrix extension with symmetry.

Lemma 3.3. *Let* q *be a* $1 \times s$ *row vector of Laurent polynomial satisfying* $\mathsf{q}\mathsf{q}^* = 1$ *and* $\mathcal{S}\mathsf{q} = \varepsilon z^c \mathcal{S}\theta$ *with* $\mathcal{S}\theta := [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$*, for some* $\varepsilon \in \{-1,1\}$*,* $c \in \{0,1\}$ *and nonnegative integers* $s_1,\ldots,s_4$ *such that* $s_1 + s_2 + s_3 + s_4 = s$*. Let* $\mathsf{B}_{\mathsf{q}} := \mathsf{U}_{\mathsf{q},\varepsilon}\mathsf{B}_{\mathsf{q}_0}\mathsf{U}^*_{\mathsf{q},\varepsilon}$ *be constructed as above. Then,*

(P1) $\mathcal{S}\mathsf{B}_{\mathsf{q}} = (\mathcal{S}\theta)^*\mathcal{S}\theta$, coeffsupp$(\mathsf{B}_{\mathsf{q}}) = [-1,1]$, *and* coeffsupp$(\mathsf{q}\mathsf{B}_{\mathsf{q}}) = [\ell_1 + 1, \ell_2 - 1]$*. That is,* B_{q} *has compatible symmetry with coefficient support on* $[-1,1]$ *and* B_{q} *reduces the length of the coefficient support of* q *exactly by* 2*. Moreover,* $\mathcal{S}(\mathsf{q}\mathsf{B}_{\mathsf{q}}) = \mathcal{S}\mathsf{q}$*.*

(P2) *if* $(\mathsf{p},\mathsf{q}^*)$ *has mutually compatible symmetry such that* $\mathsf{p}\mathsf{q}^* = 0$*, then* $\mathcal{S}(\mathsf{p}\mathsf{B}_{\mathsf{q}}) = \mathcal{S}(\mathsf{p})$ *and* coeffsupp$(\mathsf{p}\mathsf{B}_{\mathsf{q}}) \subseteq$ coeffsupp(p)*. That is,* B_{q} *keeps* *the symmetry pattern of* p *and does not increase the length of the coefficient support of* p.

Proof. Property (P1) follows directly from our construction. Due to $(\mathsf{p}, \mathsf{q}^*)$ has mutually compatible symmetry, up to a permutation matrix $\mathsf{U}_{\mathsf{p},\varepsilon}$, p takes the form in (3.2.3). Then, Property (P2) can be checked by directly computation using the orthogonality of p, q ($\mathsf{p}\mathsf{q}^* = 0$) and the definition of B_{q} from (3.2.5). ☐

Finally, we introduce a paraunitary matrix $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ constructed with respect to a pair $(\mathsf{q}_1, \mathsf{q}_2)$ from two rows of P so that $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ reduces the length of the coefficient support of of the pair $(\mathsf{q}_1, \mathsf{q}_2)$ by 2 and keeps their symmetry patterns.

Let $\mathsf{q}_1, \mathsf{q}_2$ be two $1 \times s$ row vectors of Laurent polynomials with symmetry such that $\mathsf{q}_{j_1}\mathsf{q}_{j_2}^* = \delta(j_1 - j_2)$ for $j_1, j_2 = 1, 2$, $\mathcal{S}\mathsf{q}_1 = \varepsilon_1 \mathcal{S}\theta$ and $\mathcal{S}\mathsf{q}_2 = \varepsilon_2 z \mathcal{S}\theta$ with $\mathcal{S}\theta := [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Suppose $\operatorname{coeffsupp}(\mathsf{q}_1) = [-k, k-1]$ and $\operatorname{coeffsupp}(\mathsf{q}_2) = [-k+1, k]$ with $k \geqslant 1$. Then, similar to the discussion before (3.2.3), there is a permutation matrix $E_{(\mathsf{q}_1,\mathsf{q}_2)}$ such that $\widetilde{\mathsf{q}}_1 := \mathsf{q}_1 E_{(\mathsf{q}_1,\mathsf{q}_2)}$ and $\widetilde{\mathsf{q}}_2 := \mathsf{q}_2 E_{(\mathsf{q}_1,\mathsf{q}_2)}$ take the following form:

$$\begin{bmatrix} \widetilde{\mathsf{q}}_1 \\ \widetilde{\mathsf{q}}_2 \end{bmatrix} := \begin{bmatrix} \mathsf{q}_1 \\ \mathsf{q}_2 \end{bmatrix} E_{(\mathsf{q}_1,\mathsf{q}_2)} = \left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \widetilde{g}_3 & -\widetilde{g}_4 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{-k} + \left[\begin{array}{cccc} \widetilde{f}_5 & -\widetilde{f}_6 & \widetilde{g}_7 & -\widetilde{g}_8 \\ \hline \varepsilon\widetilde{g}_1 & -\varepsilon\widetilde{g}_2 & \varepsilon\widetilde{f}_7 & -\varepsilon\widetilde{f}_8 \end{array}\right] z^{-k+1}$$
$$+ \sum_{n=2-k}^{k-2} \operatorname{coeff}\left(\begin{bmatrix} \widetilde{\mathsf{q}}_1 \\ \widetilde{\mathsf{q}}_2 \end{bmatrix}, n\right) + \left[\begin{array}{cccc} \widetilde{f}_5 & \widetilde{f}_6 & \widetilde{g}_3 & \widetilde{g}_4 \\ \hline \widetilde{g}_5 & \widetilde{g}_6 & \widetilde{f}_7 & \widetilde{f}_8 \end{array}\right] z^{k-1} + \left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \widetilde{g}_1 & \widetilde{g}_2 & \mathbf{0} & \mathbf{0} \end{array}\right] z^{k}, \tag{3.2.7}$$

where $\varepsilon \in \{-1, 1\}$ and all $\widetilde{\mathsf{g}}_j$'s are nonzero row vectors. Note that $\|\widetilde{\mathsf{g}}_1\| = \|\widetilde{\mathsf{g}}_2\| =: c_{\widetilde{\mathsf{g}}_1}$ and $\|\widetilde{\mathsf{g}}_3\| = \|\widetilde{\mathsf{g}}_4\| =: c_{\widetilde{\mathsf{g}}_3}$. Similar to the construction of $\mathsf{B}_{\mathsf{q}_0}$, we can build an $s \times s$ paraunitary matrix $\mathsf{B}_{(\widetilde{\mathsf{q}}_1, \widetilde{\mathsf{q}}_2)}$ that reduces the length of the coefficient support of both $\widetilde{\mathsf{q}}_1$ and $\widetilde{\mathsf{q}}_2$ as follows:

$$\mathsf{B}^*_{(\widetilde{\mathsf{q}}_1, \widetilde{\mathsf{q}}_2)} := \frac{1}{c} \left[\begin{array}{c|c|c|c} \frac{c_0}{c_{\widetilde{\mathsf{g}}_1}} \widetilde{\mathsf{g}}_1 & \mathbf{0} & \widetilde{\mathsf{g}}_3(1+\frac{1}{z}) & \widetilde{\mathsf{g}}_4(1-\frac{1}{z}) \\ c\widetilde{G}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \frac{c_0}{c_{\widetilde{\mathsf{g}}_1}} \widetilde{\mathsf{g}}_2 & -\widetilde{\mathsf{g}}_3(1-\frac{1}{z}) & -\widetilde{\mathsf{g}}_4(1+\frac{1}{z}) \\ \mathbf{0} & c\widetilde{G}_2 & \mathbf{0} & \mathbf{0} \\ \hline \frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}} \widetilde{\mathsf{g}}_1(1+z) & -\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}} \widetilde{\mathsf{g}}_2(1-z) & -\frac{\overline{c_0}}{c_{\widetilde{\mathsf{g}}_3}} \widetilde{\mathsf{g}}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c\widetilde{G}_3 & \mathbf{0} \\ \hline \frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}} \widetilde{\mathsf{g}}_1(1-z) & -\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}} \widetilde{\mathsf{g}}_2(1+z) & \mathbf{0} & -\frac{\overline{c_0}}{c_{\widetilde{\mathsf{g}}_3}} \widetilde{\mathsf{g}}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & c\widetilde{G}_4 \end{array} \right], \tag{3.2.8}$$

where $c_0 := \operatorname{coeff}(\widetilde{\mathsf{q}}_1, -k+1)\operatorname{coeff}(\widetilde{\mathsf{q}}_2^*, -k)/c_{\widetilde{\mathsf{g}}_1}$, $c := (|c_0|^2 + 4c^2_{\widetilde{\mathsf{g}}_3})^{1/2}$, and $[\frac{\widetilde{\mathsf{g}}_j^*}{\|\widetilde{\mathsf{g}}_j\|}, \widetilde{G}_j^*] = U_{\widetilde{\mathsf{g}}_j}$ are unitary constant extension matrices in $\mathbb{F}$ for vectors $\widetilde{\mathsf{g}}_j$ in $\mathbb{F}$, $j = 1, \ldots, 4$, respectively. Let $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)} := E_{(\mathsf{q}_1,\mathsf{q}_2)} \mathsf{B}_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)} E^T_{(\mathsf{q}_1,\mathsf{q}_2)}$. Then $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ is paraunitary and also has properties similar to (P1) and (P2) of B_{q}, which are summerized by the following lemma.

Lemma 3.4. *Let* $\mathsf{q}_1, \mathsf{q}_2$ *be two* $1 \times s$ *row vectors of Laurent polynomials such that* $\mathsf{q}_{j_1}\mathsf{q}^*_{j_2} = \delta(j_1 - j_2)$ *for* $j_1, j_2 = 1, 2$, $\mathcal{S}\mathsf{q}_1 = \varepsilon_1[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ *and* $\mathcal{S}\mathsf{q}_2 = \varepsilon_2 z[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ *for some* $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. *Suppose* $\operatorname{coeffsupp}(\mathsf{q}_1) = [-k, k-1]$ *and* $\operatorname{coeffsupp}(\mathsf{q}_2) = [-k+1, k]$ *with* $k \geqslant 1$. *Let* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ *be constructed as above. Then,*

(P3) $\mathcal{S}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$, *the coefficient support of* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ *is on* $[-1, 1]$, $\operatorname{coeffsupp}(\mathsf{q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-k+1, k-1]$ *and* $\operatorname{coeffsupp}(\mathsf{q}_2\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-k+1, k-1]$. *That is,* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ *has compatible symmetry with coefficient support on* $[-1, 1]$ *and* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$

reduces the length of both the coefficient supports of q_1 *and* q_2 *by* 2. *Moreover,* $\mathcal{S}(\mathsf{q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{q}_1$ *and* $\mathcal{S}(\mathsf{q}_2\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{q}_2$.

(P4) *if both* $(\mathsf{p}, \mathsf{q}_1^*)$ *and* $(\mathsf{p}, \mathsf{q}_2^*)$ *have mutually compatible symmetry and* $\mathsf{p}\mathsf{q}_1^* = \mathsf{p}\mathsf{q}_2^* = 0$, *then* $\mathcal{S}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{p}$ *and* $\operatorname{coeffsupp}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq \operatorname{coeffsupp}(\mathsf{p})$. *That is,* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ *keeps the symmetry pattern of* p *and does not increase the length of the coefficient support of* p.

Proof. Direct computations yield the results. □

3.3 An Algorithm for Matrix Extension with Symmetry

In this section, we present a step-by-step algorithm on matrix extension with symmetry to derive a desired matrix P_e in Theorem 3.2 from a given matrix P. Our algorithm has three steps: initialization, support reduction, and finalization. The step of initialization reduces the symmetry pattern of P to a standard form. The step of support reduction is the main body of the algorithm, producing a sequence of elementary matrices $\mathsf{A}_1, \ldots, \mathsf{A}_J$ that reduce the length of the coefficient support of P to 0. The step of finalization generates the desired matrix P_e as in Theorem 3.2. More precisely, our algorithm written in the form of *pseudo-code* for Theorem 3.2 is as follows:

Algorithm 3.1. *Input* P *as in Theorem 3.2 with* $\mathcal{S}\mathsf{P} = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$ *for some* $1\times r$ *and* $1 \times s$ *row vectors* θ_1 *and* θ_2 *of Laurant polynomials with symmetry.*
1. Initialization: *Let* $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^*\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$. *Then the symmetry pattern of* Q *is*

$$\mathcal{S}\mathsf{Q} = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}], \tag{3.3.1}$$

where all nonnegative integers $r_1, \dots, r_4, s_1, \dots, s_4$ are uniquely determined by $\mathcal{S}\mathsf{P}$.

2. Support Reduction: *Let* $\mathsf{P}_0 := \mathsf{U}^*_{\mathcal{S}\theta_2}$ *and* $J := 1$.

```
while (|coeffsupp(Q)| > 0) do        %% outer while loop
  Let Q_0 := Q, [k_1, k_2] := coeffsupp(Q), and A_J := I_s.
  if k_2 = -k_1 then
    for j from 1 to r do
      Let q := [Q_0]_{j,:}, p := [Q]_{j,:}, the j-th row of Q_0, Q, respectively.
      Let [ℓ_1, ℓ_2] := coeffsupp(q), ℓ := ℓ_2 - ℓ_1, and B_j := I_s.
      if coeffsupp(q) = coeffsupp(p) and ℓ ⩾ 2 and (ℓ_1 = k_1 or ℓ_2 = k_2)
      then
        B_j := B_q. A_J := A_J B_j. Q_0 := Q_0 B_j.
      end if
    end for
    Q_0 takes the form in (3.3.4).
    Let B_(-k_2,k_2) := I_s, Q_1 := Q_0, j_1 := 1 and j_2 := r_3 + r_4 + 1.
    while j_1 ⩽ r_1 + r_2 and j_2 ⩽ r do        %% inner while loop
      Let q_1 := [Q_1]_{j_1,:} and q_2 := [Q_1]_{j_2,:}.
      if coeff(q_1, k_1) = 0 then j_1 := j_1 + 1. end if
      if coeff(q_2, k_2) = 0 then j_2 := j_2 + 1. end if
      if coeff(q_1, k_1) ≠ 0 and coeff(q_2, k_2) ≠ 0 then
        B_(-k_2,k_2) := B_(-k_2,k_2) B_(q_1,q_2).
        Q_1 := Q_1 B_(q_1,q_2). A_J := A_J B_(q_1,q_2).
        j_1 := j_1 + 1. j_2 := j_2 + 1.
      end if
    end while        %% end inner while loop
  end if
```

Q_1 *takes the form in* (3.3.4) *with either* $\operatorname{coeff}(\mathsf{Q}_1, -k) = \mathbf{0}$

or $\operatorname{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$.

Let $\mathsf{A}_J := \mathsf{A}_J \mathsf{B}_{\mathsf{Q}_1}$ *and* $\mathsf{Q} := \mathsf{Q}\mathsf{A}_J$.

Then $\mathcal{S}\mathsf{Q} = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T [\mathbf{1}_{s'_1}, -\mathbf{1}_{s'_2}, z^{-1}\mathbf{1}_{s'_3}, -z^{-1}\mathbf{1}_{s'_4}]$.

Replace $s_1, \ldots, s_4$ *by* $s'_1, \ldots, s'_4$, *respectively.*

Let $\mathsf{P}_J := \mathsf{A}_J^*$ *and* $J := J + 1$.

`end while` %% *end outer while loop*

3. Finalization: $\mathsf{Q} = \operatorname{diag}(F_1, F_2, F_3, F_4)$ *for some* $r_j \times s_j$ *constant matrices* F_j *in* $\mathbb{F}$, $j = 1, \ldots, 4$. *Let* $U := \operatorname{diag}(U_{F_1}, U_{F_2}, U_{F_3}, U_{F_4})$ *so that* $\mathsf{Q}U = [I_r, \mathbf{0}]$. *Define* $\mathsf{P}_J := U^*$ *and* $\mathsf{P}_{J+1} := \operatorname{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$.

Output a desired matrix P_e *satisfying all the properties in Theorem 3.2.*

In the following subsections, we shall give a detailed explanation of each step of Algorithm 3.1.

3.3.1 Initialization

Let θ be a $1 \times n$ row vector of Laurent polynomials with symmetry such that $\mathcal{S}\theta = [\varepsilon_1 z^{c_1}, \ldots, \varepsilon_n z^{c_n}]$ for some $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and $c_1, \ldots, c_n \in \mathbb{Z}$. Then, the symmetry of any entry in the vector $\theta \operatorname{diag}(z^{-\lceil c_1/2 \rceil}, \ldots, z^{-\lceil c_n/2 \rceil})$ belongs to $\{\pm 1, \pm z^{-1}\}$. Thus, there is a permutation matrix E_θ to regroup these four types of symmetries together so that

$$\mathcal{S}(\theta \mathsf{U}_{\mathcal{S}\theta}) = [\mathbf{1}_{n_1}, -\mathbf{1}_{n_2}, z^{-1}\mathbf{1}_{n_3}, -z^{-1}\mathbf{1}_{n_4}], \tag{3.3.2}$$

where $\mathsf{U}_{\mathcal{S}\theta} := \operatorname{diag}(z^{-\lceil c_1/2 \rceil}, \ldots, z^{-\lceil c_n/2 \rceil}) E_\theta$, $\mathbf{1}_m$ denotes the $1 \times m$ row vector $[1, \ldots, 1]$, and $n_1, \ldots, n_4$ are nonnegative integers uniquely determined by

$\mathcal{S}\theta$. Since P satisfies (3.1.4), it is easy to see that $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^* \mathsf{P} \mathsf{U}_{\mathcal{S}\theta_2}$ has the symmetry pattern as in (3.3.1). Note that $\mathsf{U}_{\mathcal{S}\theta_1}$ and $\mathsf{U}_{\mathcal{S}\theta_2}$ do not increase the length of the coefficient support of P.

3.3.2 Support Reduction

Denote $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^* \mathsf{P} \mathsf{U}_{\mathcal{S}\theta_2}$ as in Algorithm 3.1. The outer `while` loop in the step of support reduction produces a sequence of elementary paraunitary matrices $\mathsf{A}_1, \ldots, \mathsf{A}_J$ that reduce the length of the coefficient support of Q gradually to 0. The construction of each A_j has three parts: $\{\mathsf{B}_1, \ldots, \mathsf{B}_r\}$, $\mathsf{B}_{(-k,k)}$, and $\mathsf{B}_{\mathsf{Q}_1}$. We next explain the construction and purpose of each part.

The first part $\{\mathsf{B}_1, \ldots, \mathsf{B}_r\}$ (see the `for` loop) is constructed recursively for each of the r rows of Q so that $\mathsf{Q}_0 := \mathsf{Q}\mathsf{B}_1 \cdots \mathsf{B}_r$ has a special form as in (3.3.4). In fact, suppose $\operatorname{coeffsupp}(\mathsf{Q}) = [-k, k]$ with $k \geqslant 1$. Then, Q is of the form as follows:

$$\mathsf{Q} = \left[\begin{array}{cccc} F_{11} & -F_{21} & G_{31} & -G_{41} \\ -F_{12} & F_{22} & -G_{32} & G_{42} \\ \hline \mathbf{0} & \mathbf{0} & F_{31} & -F_{41} \\ \mathbf{0} & \mathbf{0} & -F_{32} & F_{42} \end{array}\right] z^{-k} + \left[\begin{array}{cccc} F_{51} & -F_{61} & G_{71} & -G_{81} \\ -F_{52} & F_{61} & -G_{72} & G_{82} \\ \hline G_{11} & -G_{21} & F_{71} & -F_{81} \\ -G_{12} & G_{22} & -F_{72} & F_{82} \end{array}\right] z^{-k+1}$$
$$+ \sum_{n=2-k}^{k-2} \operatorname{coeff}(\mathsf{Q}, n) + \left[\begin{array}{cccc} F_{51} & F_{61} & G_{31} & G_{41} \\ F_{52} & F_{61} & G_{32} & G_{42} \\ \hline G_{51} & G_{61} & F_{71} & F_{81} \\ G_{52} & G_{62} & F_{72} & F_{82} \end{array}\right] z^{k-1} + \left[\begin{array}{cccc} F_{11} & F_{21} & \mathbf{0} & \mathbf{0} \\ F_{12} & F_{22} & \mathbf{0} & \mathbf{0} \\ \hline G_{11} & G_{21} & F_{31} & F_{41} \\ G_{12} & G_{22} & F_{32} & F_{42} \end{array}\right] z^{k} \tag{3.3.3}$$

with all F_{jk}'s and G_{jk}'s being constant matrices in $\mathbb{F}$ and $F_{11}, F_{22}, F_{31}, F_{42}$ being of size $r_1 \times s_1$, $r_2 \times s_2$, $r_3 \times s_3$, $r_4 \times s_4$, respectively. In the `for` loop, B_j is simply B_{q} with q being the current j-th row of $\mathsf{Q}\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$ and with $\mathsf{B}_0 := I_s$. Due to properties (P1) and (P2) of B_{q} (see Lemma 3.3), each B_j does not increase the lengths of the coefficient support of any other rows and also keeps the symmetry patterns of any rows. More importantly, B_j reduces the length of the coefficient support of current j-th row by 2 if the current j-th row satisfies conditions of the `if` sentence in the `for` loop. Consequently, the `for` loop in Algorithm 3.1 reduces Q in (3.3.3) to $\mathsf{Q}_0 := \mathsf{Q}\mathsf{B}_1 \cdots \mathsf{B}_r$ as follows:

$$\left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \widetilde{G}_{31} & -\widetilde{G}_{41} \\ \mathbf{0} & \mathbf{0} & -\widetilde{G}_{32} & \widetilde{G}_{42} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{-k} + \cdots + \left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \widetilde{G}_{11} & \widetilde{G}_{21} & \mathbf{0} & \mathbf{0} \\ \widetilde{G}_{12} & \widetilde{G}_{22} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{k}. \tag{3.3.4}$$

The second part $\mathsf{B}_{(-k,k)}$ is constructed recursively from pairs $(\mathsf{q}_1, \mathsf{q}_2)$ with $\mathsf{q}_1, \mathsf{q}_2$ being two rows of Q_0 satisfying $\text{coeff}(\mathsf{q}_1, -k) \neq \mathbf{0}$ and $\text{coeff}(\mathsf{q}_2, k) \neq \mathbf{0}$. In fact, if both $\text{coeff}(\mathsf{Q}_0, -k) \neq \mathbf{0}$ and $\text{coeff}(\mathsf{Q}_0, k) \neq \mathbf{0}$, then the inner `while` loop chooses a pair $(\mathsf{q}_1, \mathsf{q}_2)$ satisfies conditions in Lemma 3.4 and constructs the corresponding paraunitary matrix $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$. By properties (P3) and (P4) of each $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, i.e., $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ does not increase the lengths of the coefficient support of any other rows, keeps the symmetry patterns of any rows, and reduces the lengths of both q_1 and q_2, the matrix $\mathsf{B}_{(-k,k)}$ constructed in the inner `while` loop reduces Q_0 of the form in (3.3.4) to $\mathsf{Q}_1 := \mathsf{Q}_0\mathsf{B}_{(-k,k)}$ of the form in (3.3.4) with at least one of $\text{coeff}(\mathsf{Q}_1, -k)$ and $\text{coeff}(\mathsf{Q}_1, k)$ being $\mathbf{0}$.

The last part $\mathsf{B}_{\mathsf{Q}_1}$ is constructed to handle the case that $\text{coeffsupp}(\mathsf{Q}_1) = [-k, k-1]$ or $\text{coeffsupp}(\mathsf{Q}_1) = [-k+1, k]$ so that $\text{coeffsupp}(\mathsf{Q}_1\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-k+1, k-1]$. In fact, if $\text{coeffsupp}(\mathsf{Q}_1) = [-k+1, k-1]$ after the `for` loop and inner `while` loop, then we simply define $\mathsf{B}_{\mathsf{Q}_1} := I_s$. If one of $\text{coeff}(\mathsf{Q}_1, -k)$ and $\text{coeff}(\mathsf{Q}_1, k)$ is nonzero, then $\mathsf{B}_{\mathsf{Q}_1} := \text{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ with matrices U_1, W_1 constructed with respect to $\text{coeff}(\mathsf{Q}_1, k) \neq \mathbf{0}$ or $\mathsf{B}_{\mathsf{Q}_1} :=$

$\operatorname{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ with U_3, W_3 constructed with respect to $\operatorname{coeff}(\mathsf{Q}_1, -k) \neq \mathbf{0}$, where E is a permutation matrix. Let Q_1 take form in (3.3.4). The matrices U_1, W_1 or U_3, W_3, and E are constructed as follows.

Let $U_1 := \operatorname{diag}(U_{\widetilde{G}_1}, U_{\widetilde{G}_2})$ and $U_3 := \operatorname{diag}(U_{\widetilde{G}_3}, U_{\widetilde{G}_4})$ with

$$\widetilde{G}_1 := \begin{bmatrix} \widetilde{G}_{11} \\ \widetilde{G}_{12} \end{bmatrix}, \widetilde{G}_2 := \begin{bmatrix} \widetilde{G}_{21} \\ \widetilde{G}_{22} \end{bmatrix}, \widetilde{G}_3 := \begin{bmatrix} \widetilde{G}_{31} \\ \widetilde{G}_{32} \end{bmatrix}, \widetilde{G}_4 := \begin{bmatrix} \widetilde{G}_{41} \\ \widetilde{G}_{42} \end{bmatrix}. \tag{3.3.5}$$

Here, for a nonzero matrix G with rank m, U_G is a unitary matrix such that $GU_G = [R, \mathbf{0}]$ for some matrix R of rank m. For $G = \mathbf{0}, U_G := I$ and for $G = \emptyset, U_G := \emptyset$. When $G_1G_1^* = G_2G_2^*$, U_{G_1} and U_{G_2} can be constructed such that $G_1U_{G_1} = [R, \mathbf{0}]$ and $G_2U_{G_2} = [R, \mathbf{0}]$.

Let m_1, m_3 be the ranks of $\widetilde{G}_1, \widetilde{G}_3$, respectively ($m_1 = 0$ when $\operatorname{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$ and $m_3 = 0$ when $\operatorname{coeff}(\mathsf{Q}_1, -k) = \mathbf{0}$). Note that $\widetilde{G}_1\widetilde{G}_1^* = \widetilde{G}_2\widetilde{G}_2^*$ or $\widetilde{G}_3\widetilde{G}_3^* = \widetilde{G}_4\widetilde{G}_4^*$ due to $\mathsf{Q}_1\mathsf{Q}_1^* = I_r$. The matrices $\mathsf{W}_1, \mathsf{W}_3$ are then constructed by:

$$\mathsf{W}_1 := \left[\begin{array}{c|c|c|c} \mathsf{U}_1 & & \mathsf{U}_2 & \\ \hline & I_{s_1-m_1} & & \\ \hline \mathsf{U}_2 & & \mathsf{U}_1 & \\ \hline & & & I_{s_2-m_1} \end{array}\right], \mathsf{W}_3 := \left[\begin{array}{c|c|c|c} \mathsf{U}_3 & & \mathsf{U}_4 & \\ \hline & I_{s_3-m_3} & & \\ \hline \mathsf{U}_4 & & \mathsf{U}_3 & \\ \hline & & & I_{s_4-m_3} \end{array}\right], \tag{3.3.6}$$

where $\mathsf{U}_1(z) = -\mathsf{U}_2(-z) := \frac{1+z^{-1}}{2} I_{m_1}$ and $\mathsf{U}_3(z) = \mathsf{U}_4(-z) := \frac{1+z}{2} I_{m_3}$.

Let $\mathsf{W}_{\mathsf{Q}_1} := \operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})$ for the case that $\operatorname{coeff}(\mathsf{Q}_1, k) \neq \mathbf{0}$ or $\mathsf{W}_{\mathsf{Q}_1} := \operatorname{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)$ for the case that $\operatorname{coeff}(\mathsf{Q}_1, -k) \neq \mathbf{0}$. Then $\mathsf{W}_{\mathsf{Q}_1}$ is paraunitary. By the symmetry pattern and orthogonality of Q_1, $\mathsf{W}_{\mathsf{Q}_1}$ reduces the coefficient support of Q_1 to $[-k+1, k-1]$, i.e., $\operatorname{coeffsupp}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [-k+1, k-1]$. Moreover, $\mathsf{W}_{\mathsf{Q}_1}$ changes the symmetry pattern of Q_1 such that $\mathcal{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T\mathcal{S}\theta_1$ with

$$\mathcal{S}\theta_1 = [z^{-1}\mathbf{1}_{m_1}, \mathbf{1}_{s_1-m_1}, -z^{-1}\mathbf{1}_{m_1}, -\mathbf{1}_{s_2-m_1}, \mathbf{1}_{m_3}, z^{-1}\mathbf{1}_{s_3-m_3}, -\mathbf{1}_{m_3}, -z^{-1}\mathbf{1}_{s_4-m_3}].$$

E is then the permutation matrix such that

$$\mathcal{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1})E = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4},]^T\mathcal{S}\theta,$$

with $\mathcal{S}\theta = [\mathbf{1}_{s_1-m_1+m_3},, -\mathbf{1}_{s_2-m_1+m_3}, z^{-1}\mathbf{1}_{s_3-m_3+m_1}, -z^{-1}\mathbf{1}_{s_4-m_3+m_1}]$.

3.3.3 Finalization

After the second step of support reduction, we must have $|\mathrm{coeffsupp}(\mathsf{Q})| = 0$ and $\mathcal{S}\mathsf{Q} = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some nonnegative integers $s_1', \ldots, s_4'$ such that $s_1 + \cdots + s_4 = s$. Thus, $\mathsf{Q} = \mathrm{diag}(F_1, F_2, F_3, F_4)$ for some $r_j \times s_j$ constant matrices F_j in $\mathbb{F}$, $j = 1, \ldots, 4$. Then, due to $\mathsf{Q}\mathsf{Q}^* = I_r$, $\mathsf{A}_J = U := \mathrm{diag}(U_{F_1}, U_{F_2}, U_{F_3}, U_{F_4})$ normalizes Q to be $\mathsf{Q}U = [I_r, \mathbf{0}]$. Consequently, $\mathsf{U}^*_{\mathcal{S}\theta_1}\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}\mathsf{A}_1 \ldots \mathsf{A}_J = [I_r, \mathbf{0}]$. Let $\mathsf{P}_j := \mathsf{A}_j^*$ for $j = 1, \ldots, J$, $\mathsf{P}_0 := \mathsf{U}_{\mathcal{S}\theta_2}$, and $\mathsf{P}_{J+1} := \mathrm{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$. Then items (1), (3) and (4) of Theorem 3.2 can be easily checked by our construction.

3.4 Applications to Orthonormal Multiwavelets and Filter Banks with Symmetry

In this section, we shall discuss the application of our results on matrix extension with symmetry to d-band symmetric paraunitary filter banks in electronic engineering and to orthonormal multiwavelets with symmetry in wavelet analysis. In order to do so, let us introduce some definitions first.

Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ such that (3.1.6) holds. Let $a_0 : \mathbb{Z} \to \mathbb{F}^{r\times r}$ be a mask with multiplicity r (a low-pass filter in electronic engineering) . The *symbol* of the filter a_0 is defined to be $\mathsf{a}_0(z) := \sum_{k\in\mathbb{Z}} a_0(k)z^k$, which is a matrix of Laurent polynomials with coefficients in $\mathbb{F}$. Let d be a dilation factor. The d*-band subsymbols (polyphase)* of a_0 are defined by $\mathsf{a}_{0;\gamma}(z) := \sqrt{\mathsf{d}}\sum_{k\in\mathbb{Z}} a_0(\gamma+\mathsf{d}k)z^k$, $\gamma \in \mathbb{Z}$. It is easily seen that a_0 is a d-band orthogonal mask (i.e., (1.2.4) holds for a_0) if

$$\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\mathsf{a}_{0;\gamma}^*(z) = I_r, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{3.4.1}$$

To construct orthogonal multiwavelets (or an orthogonal filter bank with the perfect reconstruction property in electronic engineering), one has to design masks (high-pass filters) $a_1, \ldots, a_{\mathsf{d}-1} : \mathbb{Z} \to \mathbb{F}^{r\times r}$ such that the polyphase matrix

$$\mathbf{P}(z) = \begin{bmatrix} \mathsf{a}_{0;0}(z) & \cdots & \mathsf{a}_{0;\mathsf{d}-1}(z) \\ \mathsf{a}_{1;0}(z) & \cdots & \mathsf{a}_{1;\mathsf{d}-1}(z) \\ \vdots & \vdots & \vdots \\ \mathsf{a}_{\mathsf{d}-1;0}(z) & \cdots & \mathsf{a}_{\mathsf{d}-1;\mathsf{d}-1}(z) \end{bmatrix} \tag{3.4.2}$$

is paraunitary, that is, $\mathbf{P}(z)\mathbf{P}^*(z) = I_{\mathsf{d}r}$, where each $\mathsf{a}_{m;\gamma}$ is a subsymbol of a_m for $m, \gamma = 0, \ldots, \mathsf{d}-1$, respectively. We say that the mask (low-pass filter) a_0 (or a_0) has symmetry if

$$\mathsf{a}_0(z) = \operatorname{diag}(\varepsilon_1 z^{\mathsf{d}c_1}, \ldots, \varepsilon_r z^{\mathsf{d}c_r})\mathsf{a}_0(1/z)\operatorname{diag}(\varepsilon_1 z^{-c_1}, \ldots, \varepsilon_r z^{-c_r}) \tag{3.4.3}$$

for some $\varepsilon_1, \dots, \varepsilon_r \in \{-1, 1\}$ and $c_1, \dots, c_r \in \mathbb{R}$ such that $\mathsf{d}c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$. To construct orthogonal multiwavelets from an orthogonal mask, one has to construct masks (high-pass filters) $a_1, \dots, a_{\mathsf{d}-1} : \mathbb{Z} \to \mathbb{F}^{r\times r}$ such that all of them have symmetry that is compatible with the symmetry of a_0 in (3.4.3) and the polyphase matrix $\mathbf{P}$ in (3.4.2) is paraunitary. Let $\phi = [\phi_1, \dots, \phi_r]^T$ be an orthogonal compactly supported d-refinable function vector in $L_2(\mathbb{R})$ associated with an orthogonal mask (low-pass filter) a_0. Define multiwavelet function vectors $\psi^m = [\psi_1^m, \dots, \psi_r^m]^T$ associated with the masks (high-pass filters) a_m, $m = 1, \dots, \mathsf{d}-1$, by

$$\widehat{\psi^m}(\mathsf{d}\xi) := \mathsf{a}_m(e^{-i\xi})\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R},\ m = 1, \dots, \mathsf{d}-1. \tag{3.4.4}$$

It is well known that $\{\psi^1, \dots, \psi^{\mathsf{d}-1}\}$ generates an orthonormal multiwavelet basis in $L_2(\mathbb{R})$; that is, $\{\mathsf{d}^{j/2}\psi_\ell^m(\mathsf{d}^j \cdot -k)\ :\ j, k \in \mathbb{Z}; m = 1, \dots, \mathsf{d}-1; \ell = 1, \dots, r\}$ is an orthonormal basis of $L_2(\mathbb{R})$, for example, see [11, 30, 37, 64] and references therein.

In what follows, to distinguish mask a_0 for the d-refinable function vector ϕ and mask a_m for the multiwavelet function vector ψ^m, $m = 1, \dots, \mathsf{d}-1$, we shall refer a_0 as *the low-pass filter* for ϕ and a_m as *the high-pass filter* for ψ^m, $m = 1, \dots, \mathsf{d}-1$. $\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}\}$ denotes *an orthogonal filter bank with the perfect reconstruction property* if its corresponding polyphase matrix $\mathbf{P}$ in (3.4.2) is paraunitary.

If a_0 has symmetry as in (3.4.3) and if 1 is a simple eigenvalue of $\mathsf{a}_0(1)$, then it is well known that the d-refinable function vector ϕ in (1.1.1) associated with the low-pass filter a_0 has the following symmetry:

$$\phi_1(c_1 - \cdot) = \varepsilon_1\phi_1, \quad \phi_2(c_2 - \cdot) = \varepsilon_2\phi_2, \quad \dots, \quad \phi_r(c_r - \cdot) = \varepsilon_r\phi_r. \tag{3.4.5}$$

Under the symmetry condition in (3.4.3), to apply Theorem 3.1, we first show that there exists a suitable paraunitary matrix U acting on $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$ so that $\mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry. Note that $\mathsf{P}_{\mathsf{a}_0}$ itself may not have any symmetry.

Lemma 3.5. *Let* $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$*, where* $\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}$ *are* d*-band subsymbols of a* d*-band orthogonal low-pass filter* a_0 *satisfying* (3.4.3)*. Then there exists a* $\mathsf{d}r \times \mathsf{d}r$ *paraunitary matrix* U *such that* $\mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ *has compatible symmetry.*

Proof. From (3.4.3), we have $[a_0(k)]_{\ell,j} = \varepsilon_\ell \varepsilon_j [a_0(\mathsf{d}c_\ell - c_j - k)]_{\ell,j}$, which implies that for $\gamma = 0, \dots, \mathsf{d}-1$ and $\ell, j = 1, \dots, r$,

$$[\mathsf{a}_{0;\gamma}(z)]_{\ell,j} = \varepsilon_\ell \varepsilon_j z^{R^\gamma_{\ell,j}} [\mathsf{a}_{0;Q^\gamma_{\ell,j}}(z^{-1})]_{\ell,j}, \tag{3.4.6}$$

where $Q^\gamma_{\ell,j} \in \Gamma := \{0, \dots, \mathsf{d}-1\}$ and $R^\gamma_{\ell,j}$, $Q^\gamma_{\ell,j}$ are uniquely determined by

$$\mathsf{d}c_\ell - c_j - \gamma = \mathsf{d}R^\gamma_{\ell,j} + Q^\gamma_{\ell,j} \quad \text{with} \quad R^\gamma_{\ell,j} \in \mathbb{Z},\ Q^\gamma_{\ell,j} \in \Gamma. \tag{3.4.7}$$

Since $\mathsf{d}c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$, we have $c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$ and therefore, $Q^\gamma_{\ell,j}$ is independent of ℓ. Consequently, by (3.4.6), for every $1 \leqslant j \leqslant r$, the j-th column of the matrix $\mathsf{a}_{0;\gamma}$ is a flipped version of the j-th column of the matrix $\mathsf{a}_{0;Q^\gamma_{\ell,j}}$. Let $\kappa_{j,\gamma} \in \mathbb{Z}$ be an integer such that $|\mathrm{coeffsupp}([\mathsf{a}_{0;\gamma}]_{:,j} + z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j})|$ is as small as possible. Define $\mathsf{P} := [\mathsf{b}_{0;0}, \dots, \mathsf{b}_{0;\mathsf{d}-1}]$ as follows:

$$[\mathsf{b}_{0;\gamma}]_{:,j} := \begin{cases} [\mathsf{a}_{0;\gamma}]_{:,j}, & \gamma = Q^\gamma_{\ell,j}; \\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j} + z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma < Q^\gamma_{\ell,j}; \\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j} - z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma > Q^\gamma_{\ell,j}, \end{cases} \tag{3.4.8}$$

where $[\mathsf{a}_{0;\gamma}]_{:,j}$ denotes the j-th column of $\mathsf{a}_{0;\gamma}$. Let U denote the unique transform matrix corresponding to (3.4.8) such that $\mathsf{P} := [\mathsf{b}_{0;0}, \ldots, \mathsf{b}_{0;\mathsf{d}-1}] = [\mathsf{a}_{0;0}, \ldots, \mathsf{a}_{0;\mathsf{d}-1}]\mathsf{U}$. It is evident that U is paraunitary and $\mathsf{P} = \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$. We now show that P has compatible symmetry. Indeed, by (3.4.6) and (3.4.8),

$$[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,j} = \operatorname{sgn}(Q^{\gamma}_{\ell,j} - \gamma)\varepsilon_\ell\varepsilon_j z^{R^{\gamma}_{\ell,j}+\kappa_{j,\gamma}}, \tag{3.4.9}$$

where $\operatorname{sgn}(x) = 1$ for $x \geqslant 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$. By (3.4.7) and noting that $Q^{\gamma}_{\ell,j}$ is independent of ℓ, we have

$$\frac{[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,j}}{[\mathcal{S}\mathsf{b}_{0;\gamma}]_{n,j}} = \varepsilon_\ell\varepsilon_n z^{R^{\gamma}_{\ell,j}-R^{\gamma}_{n,j}} = \varepsilon_\ell\varepsilon_n z^{c_\ell - c_n},$$

for all $1 \leqslant \ell, n \leqslant r$, which is equivalent to saying that P has compatible symmetry. □

Now, for a d-band orthogonal low-pass filter a_0 satisfying (3.4.3), we have the following algorithm to construct high-pass filters $\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{d}-1}$ such that $\{\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{d}-1}\}$ forms a symmetric paraunitary filter bank with the perfect reconstruction property.

Algorithm 3.2. *Input an orthogonal d-band filter a_0 with symmetry in (3.4.3).*

(1) *Construct U as in (3.4.8) such that $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry: $\mathcal{S}\mathsf{P} = [\varepsilon_1 z^{k_1}, \ldots, \varepsilon_r z^{k_r}]^T \mathcal{S}\theta$ for some $k_1, \ldots, k_r \in \mathbb{Z}$ and some $1 \times \mathsf{d}r$ row vector θ of Laurent polynomials with symmetry.*

(2) *Derive P_e with all the properties as in Theorem 3.1 from P by Algorithm 3.1.*

(3) *Let* $\mathbf{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant \mathsf{d}-1}$ *as in* (3.4.2). *Define high-pass filters*

$$\mathsf{a}_m(z) := \frac{1}{\sqrt{\mathsf{d}}}\sum_{\gamma=0}^{\mathsf{d}-1}\mathsf{a}_{m;\gamma}(z^{\mathsf{d}})z^{\gamma}, \qquad m = 1,\dots,\mathsf{d}-1. \tag{3.4.10}$$

Output a symmetric filter bank $\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}\}$ *with the perfect reconstruction property, i.e.* $\mathbf{P}$ *in* (3.4.2) *is paraunitary and all filters* a_m, $m = 1,\dots,\mathsf{d}-1$, *have symmetry:*

$$\mathsf{a}_m(z) = \operatorname{diag}(\varepsilon_1^m z^{\mathsf{d}c_1^m},\dots,\varepsilon_r^m z^{\mathsf{d}c_r^m})\mathsf{a}_m(1/z)\operatorname{diag}(\varepsilon_1 z^{-c_1},\dots,\varepsilon_r z^{-c_r}), \tag{3.4.11}$$

where $c_\ell^m := (k_\ell^m - k_\ell) + c_\ell \in \mathbb{R}$ *and all* $\varepsilon_\ell^m \in \{-1,1\}$, $k_\ell^m \in \mathbb{Z}$, *for* $\ell = 1,\dots,r$ *and* $m = 1,\dots,\mathsf{d}-1$, *are determined by the symmetry pattern of* P_e *as follows:*

$$[\varepsilon_1 z^{k_1},\dots,\varepsilon_r z^{k_r},\varepsilon_1^1 z^{k_1^1},\dots,\varepsilon_r^1 z^{k_r^1},\dots,\varepsilon_1^{\mathsf{d}-1} z^{k_1^{\mathsf{d}-1}},\dots,\varepsilon_r^{\mathsf{d}-1} z^{k_r^{\mathsf{d}-1}}]^T\mathcal{S}\theta := \mathcal{S}\mathsf{P}_e. \tag{3.4.12}$$

Proof. Rewrite $\mathsf{P}_e = (\mathsf{b}_{m;\gamma})_{0\leqslant m,\gamma\leqslant \mathsf{d}-1}$ as a $\mathsf{d}\times\mathsf{d}$ block matrix with $r\times r$ blocks $\mathsf{b}_{m;\gamma}$. Since P_e has compatible symmetry as in (3.4.12), we have $[\mathcal{S}\mathsf{b}_{m;\gamma}]_{\ell,:} = \varepsilon_\ell^m\varepsilon_\ell z^{k_\ell^m-k_\ell}[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,:}$ for $\ell = 1,\dots,r$ and $m = 1,\dots,\mathsf{d}-1$. By (3.4.9), we have

$$[\mathcal{S}\mathsf{b}_{m;\gamma}]_{\ell,j} = \operatorname{sgn}(Q_{\ell,j}^{\gamma}-\gamma)\varepsilon_\ell^m\varepsilon_j z^{R_{\ell,j}^{\gamma}+\kappa_{j,\gamma}+k_\ell^m-k_\ell}, \qquad \ell,j = 1,\dots,r. \tag{3.4.13}$$

By (3.4.13) and the definition of U^* in (3.4.8), we deduce that

$$[\mathsf{a}_{m;\gamma}]_{\ell,j} = \varepsilon_\ell^m\varepsilon_j z^{R_{\ell,j}^{\gamma}+k_\ell^m-k_\ell}[\mathsf{a}_{m;Q_{\ell,j}^{\gamma}}(z^{-1})]_{\ell,j}. \tag{3.4.14}$$

This implies that $[\mathcal{S}\mathsf{a}_m]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{\mathsf{d}(k_\ell^m - k_\ell + c_\ell) - c_j}$, which is equivalent to (3.4.11) with $c_\ell^m := k_\ell^m - k_\ell + c_\ell$ for $m = 1, \dots, \mathsf{d}-1$ and $\ell = 1, \dots, r$. ☐

Since the high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}$ satisfy (3.4.11), it is easy to verify that each $\psi^m = [\psi_1^m, \dots, \psi_r^m]^T$ defined in (3.4.4) also has the following symmetry:

$$\psi_1^m(c_1^m - \cdot) = \varepsilon_1^m \psi_1^m, \quad \psi_2^m(c_2^m - \cdot) = \varepsilon_2^m \psi_2^m, \quad \dots, \quad \psi_r^m(c_r^m - \cdot) = \varepsilon_r^m \psi_r^m. \tag{3.4.15}$$

In the following, let us present several examples to demonstrate our results and illustrate our algorithms.

Example 3.1. Let $\mathsf{d} = 2$ and $r = 2$. A 2-band orthogonal low-pass filter a_0 with multiplicity 2 in [16] is given by

$$\mathsf{a}_0(z) = \frac{1}{40} \begin{bmatrix} 12(1+z^{-1}) & 16\sqrt{2}z^{-1} \\ -\sqrt{2}(z^2 - 9z - 9 + z^{-1}) & -2(3z - 10 + 3z^{-1}) \end{bmatrix}.$$

The low-pass filter a_0 satisfies (3.4.3) with $c_1 = -1, c_2 = 0$ and $\varepsilon_1 = \varepsilon_2 = 1$. Using Lemma 3.5, we obtain $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}]$ and U as follows:

$$\mathsf{P}_{\mathsf{a}_0} = \frac{1}{20} \left[\begin{array}{cc|cc} 6\sqrt{2} & 0 & \frac{6\sqrt{2}}{z} & \frac{16}{z} \\ 9 - z & 10\sqrt{2} & 9 - \frac{1}{z} & -3\sqrt{2}(1 + \frac{1}{z}) \end{array}\right], \mathsf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ z & 0 & -z & 0 \\ 0 & 0 & 0 & \sqrt{2}z \end{bmatrix}.$$

Then $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [1, z]^T[1, z^{-1}, -1, 1]$ and is given by

$$\mathsf{P} = \frac{\sqrt{2}}{20} \begin{bmatrix} 6\sqrt{2} & 0 & 0 & 8\sqrt{2} \\ 4(1+z) & 10 & 5(1-z) & -3(1+z) \end{bmatrix}.$$

Applying Algorithm 3.1, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = \frac{\sqrt{2}}{20}\begin{bmatrix} 6\sqrt{2} & 0 & 0 & 8\sqrt{2} \\ 4(1+z) & 10 & 5(1-z) & -3(1+z) \\ 4(1+z) & -10 & 5(1-z) & -3(1+z) \\ 4\sqrt{2}(1-z) & 0 & 5\sqrt{2}(z+1) & 3\sqrt{2}(z-1) \end{bmatrix}.$$

We have $\mathcal{S}\mathsf{P}_e = [1, z, z, -z]^T[1, z^{-1}, -1, 1]$ and the coefficient supports of P_e satisfies $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 4$. Now, from the polyphase matrix $\mathbf{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant 1}$, we derive a high-pass filter a_1 as follows:

$$\mathsf{a}_1(z) = \frac{1}{40}\begin{bmatrix} -\sqrt{2}(z^2 - 9z - 9 + z^{-1}) & -2(3z + 10 + 3z^{-1}) \\ 2(z^2 - 9z + 9 - z^{-1}) & 6\sqrt{2}(z - z^{-1}) \end{bmatrix}.$$

Then the high-pass filter a_1 satisfies (3.4.11) with $c_1^1 = c_2^1 = 0$ and $\varepsilon_1^1 = 1, \varepsilon_2^1 = -1$. See Figure 3.1 for the graphs of the 2-refinable function vector ϕ associated with the low-pass filter a_0 and the multiwavelet function vector ψ associated with the high-pass filter a_1.

Example 3.2. Let $\mathsf{d} = 3$ and $r = 2$. Let a_0 be the 3-band orthogonal low-pass filter with multiplicity 2 obtained in Example 1.5. Then

$$\mathsf{a}_0(z) = \frac{1}{540}\begin{bmatrix} a_{11}(z) + a_{11}(z^{-1}) & a_{12}(z) + z^{-1}a_{12}(z^{-1}) \\ a_{21}(z) + z^3 a_{21}(z^{-1}) & a_{22}(z) + z^2 a_{22}(z^{-1}) \end{bmatrix},$$

where

$$a_{11}(z) = 90 + (55 - 5\sqrt{41})z - (8 + 2\sqrt{41})z^2 + (7\sqrt{41} - 47)z^4;$$
$$a_{12}(z) = 145 + 5\sqrt{41} + (1 - \sqrt{41})z^2 + (34 - 4\sqrt{41})z^3;$$

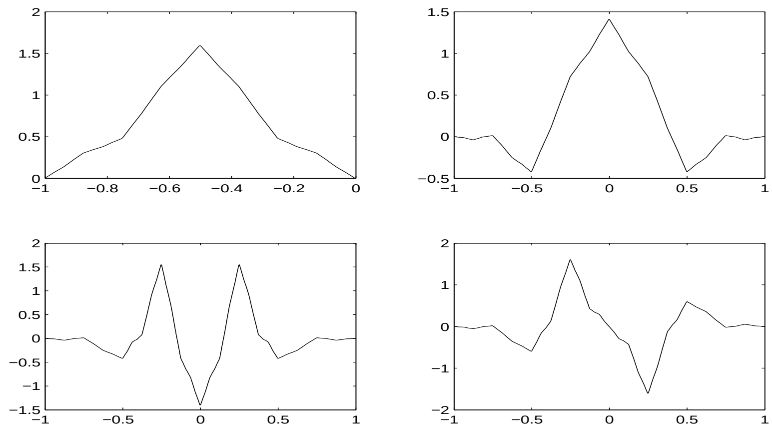

FIGURE 3.1: The graphs of the 2-refinable function vector $\phi = [\phi_1, \phi_2]^T$ associated with a_0 (top row) and the multiwavelet function vector $\psi = [\psi_1, \psi_2]^T$ associated with a_1 (bottom 2) in Example 3.1.

$$a_{21}(z) = (111 + 9\sqrt{41})z^2 + (69 - 9\sqrt{41})z^4;$$
$$a_{22}(z) = 90z + (63 - 3\sqrt{41})z^2 + (3\sqrt{41} - 63)z^3.$$

The low-pass filter a_0 satisfies (3.4.3) with $c_1 = 0, c_2 = 1$ and $\varepsilon_1 = \varepsilon_2 = 1$. From $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}, \mathsf{a}_{0;2}]$, the matrix U constructed by Lemma 3.5 is given by

$$\mathsf{U} := \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & z & 0 & -z & 0 \\ 0 & z & 0 & 0 & 0 & -z \end{bmatrix}.$$

Let

$$c_0 = 11 - \sqrt{41}; \quad t_{12} = 5(7 - \sqrt{41}); \quad c_{12} = 10(29 + \sqrt{41}); \quad t_{13} = -5c_0;$$
$$t_{16} = 3c_0; \quad t_{15} = 3(3\sqrt{41} - 13); \quad t_{25} = 6(7 + 3\sqrt{41}); \quad t_{26} = 6(21 - \sqrt{41});$$
$$t_{53} = 400\sqrt{6}/c_0; \quad t_{55} = 12\sqrt{6}(\sqrt{41} - 1); \quad t_{56} = 6\sqrt{6}(4 + \sqrt{41}); \quad c_{66} = 3\sqrt{6}(3 + 7\sqrt{41}).$$

Then $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [1, z]^T[1, 1, 1, z^{-1}, -1, -1]$ and is given by

$$\mathsf{P} = \frac{\sqrt{6}}{1080}\begin{bmatrix} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z - z^{-1}) & t_{16}(z - z^{-1}) \\ 0 & 0 & 180(1+z) & 180\sqrt{2} & t_{25}(1-z) & t_{26}(1-z) \end{bmatrix},$$

where $b_{12}(z) = t_{12}(z + z^{-1}) + c_{12}$ and $b_{13}(z) = t_{13}(z - 2 + z^{-1})$. Applying Algorithm 3.1, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = \frac{\sqrt{6}}{1080}\left[\begin{array}{cccccc} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z - \frac{1}{z}) & t_{16}(z - \frac{1}{z}) \\ 0 & 0 & 180(1+z) & 180\sqrt{2} & t_{25}(1-z) & t_{26}(1-z) \\ \hline 360 & -\frac{b_{12}(z)}{\sqrt{2}} & -\frac{b_{13}(z)}{\sqrt{2}} & 0 & \frac{t_{15}}{\sqrt{2}}(\frac{1}{z} - z) & \frac{t_{16}}{\sqrt{2}}(\frac{1}{z} - z) \\ 0 & 0 & 90\sqrt{2}(1+z) & -360 & \frac{t_{25}}{\sqrt{2}}(1-z) & \frac{t_{26}}{\sqrt{2}}(1-z) \\ \hline 0 & \sqrt{6}t_{13}(1-z) & t_{53}(1-z) & 0 & t_{55}(1+z) & t_{56}(1+z) \\ 0 & \frac{\sqrt{6}t_{12}}{2}(\frac{1}{z} - z) & \frac{\sqrt{6}t_{13}}{2}(\frac{1}{z} - z) & 0 & b_{65}(z) & b_{66}(z) \end{array}\right],$$

where $b_{65}(z) = -\sqrt{6}(5t_{15}(z + z^{-1}) + 3c_{12})/10$ and $b_{66}(z) = -\sqrt{6}t_{16}(z + z^{-1})/2 + c_{66}$. Note that $\mathcal{S}\mathsf{P}_e = [1, z, 1, z, -z, -1]^T[1, 1, 1, z^{-1}, -1, -1]$ and the coefficient support of P_e satisfies $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 6$. From the polyphase matrix $\mathbf{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0 \leqslant m,\gamma \leqslant 2}$, we derive two high-pass filters $\mathsf{a}_1, \mathsf{a}_2$ as follows:

$$\mathsf{a}_1(z) = \frac{\sqrt{2}}{1080}\begin{bmatrix} a^1_{11}(z) + a^1_{11}(z^{-1}) & a^1_{12}(z) + z^{-1}a^1_{12}(z^{-1}) \\ a^1_{21}(z) + z^3 a^1_{21}(z^{-1}) & a^1_{22}(z) + z^2 a^1_{22}(z^{-1}) \end{bmatrix},$$

$$\mathsf{a}_2(z) = \frac{\sqrt{6}}{1080}\begin{bmatrix} a^2_{11}(z) - z^3 a^2_{11}(z^{-1}) & a^2_{12}(z) - z^2 a^2_{12}(z^{-1}) \\ a^2_{21}(z) - a^2_{21}(z^{-1}) & a^2_{22}(z) - z^{-1}a^2_{22}(z^{-1}) \end{bmatrix},$$

where

$$a^1_{11}(z) = (47 - 7\sqrt{41})z^4 + 2(4 + \sqrt{41})z^2 + 5(\sqrt{41} - 11)z + 180;$$

$$a^1_{12}(z) = 2(2\sqrt{41} - 17)z^3 + (\sqrt{41} - 1)z^2 - 5(29 + \sqrt{41});$$

$$a_{21}^1(z) = 3(37 + 3\sqrt{41})z + 3(23 - 3\sqrt{41})z^{-1};$$

$$a_{22}^1(z) = -180z + 3(21 - \sqrt{41}) - 3(21 - \sqrt{41})z^{-1};$$

$$a_{11}^2(z) = (43 + 17\sqrt{41})z + (67 - 7\sqrt{41})z^{-1};$$

$$a_{12}^2(z) = 11\sqrt{41} - 31 - (79 + \sqrt{41})z^{-1};$$

$$a_{21}^2(z) = (47 - 7\sqrt{41})z^4 + 2(4 + \sqrt{41})z^2 - 3(29 + \sqrt{41})z;$$

$$a_{22}^2(z) = 2(2\sqrt{41} - 17)z^3 + (\sqrt{41} - 1)z^2 + 3(3 + 7\sqrt{41}).$$

Then the high-pass filters a_1, a_2 satisfy (3.4.11) with $c_1^1 = 0, c_2^1 = 1$, $\varepsilon_1^1 = \varepsilon_2^1 = 1$ and $c_1^2 = 1, c_2^2 = 0$, $\varepsilon_1^2 = \varepsilon_2^2 = -1$. See Figure 3.2 for the graphs of the 3-refinable function vector ϕ associated with the low-pass filter a_0 and the multiwavelet function vectors ψ^1, ψ^2 associated with the high-pass filters $\mathsf{a}_1, \mathsf{a}_2$, respectively.

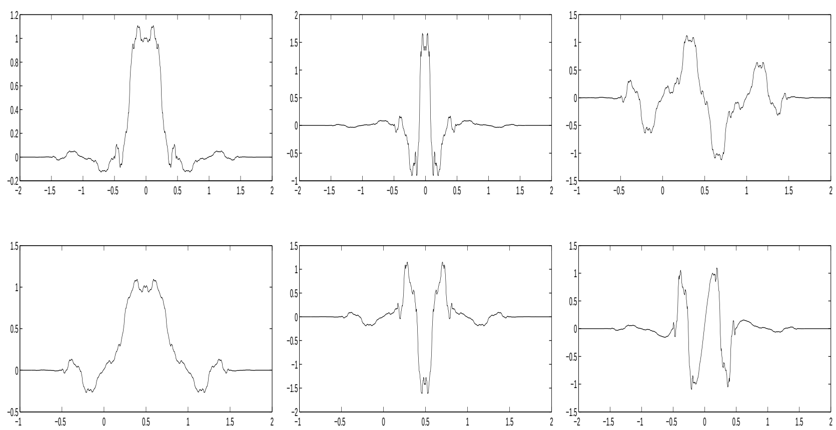

FIGURE 3.2: The graphs of the 3-refinable function vector $\phi = [\phi_1, \phi_2]^T$ associated with a_0 (left column), multiwavelet function vector $\psi^1 = [\psi_1^1, \psi_2^1]^T$ associated with a_1 (middle column), and multiwavelet function vector $\psi^2 = [\psi_1^2, \psi_2^2]^T$ associated with a_2 (right column) in Example 3.2.

As demonstrated by the following example, our Algorithm 3.2 also applies to low-pass filters with symmetry patterns other than those in (3.4.3).

Example 3.3. Let $\mathsf{d} = 3$ and $r = 2$. Let a_0 be the 3-band orthogonal low-pass filter with multiplicity 2 obtained in Example 1.4. Then

$$\mathsf{a}_0(z) = \frac{1}{702}\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix},$$

where

$$a_{11}(z) = (11-14\sqrt{17})z^2 + (29+8\sqrt{17})z + 234 + (85-16\sqrt{17})z^{-1} - (17+2\sqrt{17})z^{-2};$$
$$a_{12}(z) = (5\sqrt{17}-16)z^3 + (2+\sqrt{17})z^2 + 238 - 11\sqrt{17} + (136+29\sqrt{17})z^{-1};$$
$$a_{21}(z) = (136+29\sqrt{17})z^2 + (238-11\sqrt{17})z + (2+\sqrt{17})z^{-1} + (5\sqrt{17}-16)z^{-2};$$
$$a_{22}(z) = (-17-2\sqrt{17})z^3 + (85-16\sqrt{17})z^2 + 234z + 29 + 8\sqrt{17} + (11-14\sqrt{17})z^{-1}.$$

This low-pass filter a_0 does not satisfy (3.4.3). However, we can employ a very simple orthogonal transform $E := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ to a_0 so that the symmetry in (3.4.3) holds. That is, for $\widetilde{\mathsf{a}}_0(z) := E\mathsf{a}_0(z)E$, it is easy to verify that $\widetilde{\mathsf{a}}_0$ satisfies (3.4.3) with $c_1 = c_2 = 1/2$ and $\varepsilon_1 = 1, \varepsilon_2 = -1$. Construct $\mathsf{P}_{\widetilde{\mathsf{a}}_0} := [\widetilde{\mathsf{a}}_{0;0}, \widetilde{\mathsf{a}}_{0;1}, \widetilde{\mathsf{a}}_{0;2}]$ from $\widetilde{\mathsf{a}}_0$. The matrix U constructed by Lemma 3.5 from $\mathsf{P}_{\widetilde{\mathsf{a}}_0}$ is given by:

$$\mathsf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} z^{-1} & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & z^{-1} & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Then $\mathsf{P} := \mathsf{P}_{\widetilde{a}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [z^{-1}, -z^{-1}]^T[1,-1,-1,1,1,-1]$ and

$$\mathsf{P} = c\begin{bmatrix} 234(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & 0 & 117\sqrt{2}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \end{bmatrix},$$

where $c = \frac{\sqrt{6}}{1404}$ and t_{jk}'s are constants defined as follows:

$$
\begin{array}{lll}
t_{12} = 3(11-\sqrt{17}); & t_{13} = 3(\sqrt{17}-89); & t_{16} = 15\sqrt{2}(2+\sqrt{17}); \\
t_{21} = 13(\sqrt{17}-17); & t_{22} = 6(2+\sqrt{17}); & t_{23} = 6(37-\sqrt{17}); \\
t_{24} = -13(1+\sqrt{17}); & t_{25} = -13\sqrt{2}(8+\sqrt{17}); & t_{26} = -3\sqrt{2}(7+10\sqrt{17}).
\end{array}
$$

Applying Algorithm 3.1 to P, we obtain a desired paraunitary matrix P_e as follows:

$$
\mathsf{P}_e = c \left[\begin{array}{cccccc}
234(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & 0 & 117\sqrt{2}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\
t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \\
\hline
t_{31}(1-\frac{1}{z}) & t_{32}(1+\frac{1}{z}) & t_{33}(1+\frac{1}{z}) & t_{34}(1-\frac{1}{z}) & t_{35}(1-\frac{1}{z}) & t_{36}(1+\frac{1}{z}) \\
t_{41}(1+\frac{1}{z}) & t_{42}(1-\frac{1}{z}) & t_{43}(1-\frac{1}{z}) & t_{44}(1+\frac{1}{z}) & -\sqrt{2}t_{41}(1+\frac{1}{z}) & t_{46}(1-\frac{1}{z}) \\
\hline
\frac{2}{\sqrt{3}}t_{44} & 0 & 0 & -2\sqrt{3}t_{41} & -\frac{4}{\sqrt{6}}t_{44} & 0 \\
0 & t_{62} & t_{63} & 0 & 0 & t_{66}
\end{array}\right],
$$

where all t_{jk}'s are constants given by:

$$
\begin{array}{ll}
t_{31} = -\sqrt{26}(61+25\sqrt{17})/4; & t_{32} = -3\sqrt{26}(397+23\sqrt{17})/52; \\
t_{33} = 3\sqrt{26}(553+23\sqrt{17})/52; & t_{34} = 25\sqrt{26}(1+\sqrt{17})/4; \\
t_{35} = \sqrt{13}(25\sqrt{17}-43)/2; & t_{36} = 15\sqrt{13}(23\sqrt{17}-19)/26 \\
t_{41} = 9\sqrt{26}(1-3\sqrt{17})/4; & t_{42} = -3\sqrt{26}(383+29\sqrt{17})/52; \\
t_{43} = 3\sqrt{26}(29\sqrt{17}+227)/52; & t_{44} = 27\sqrt{26}(1+\sqrt{17})/4; \\
t_{46} = 3\sqrt{13}(145\sqrt{17}-61)/26; & t_{62} = 9\sqrt{78}(41\sqrt{17}-9)/26; \\
t_{63} = 9\sqrt{78}(11\sqrt{17}+9)/26; & t_{66} = 27\sqrt{3}(\sqrt{17}+15)/\sqrt{13}.
\end{array}
$$

Note P_e satisfies $\mathcal{S}\mathsf{P}_e = [z^{-1}, -z^{-1}, -z^{-1}, z^{-1}, 1, -1]^T[1, -1, -1, 1, 1, -1]$ and we have $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 6$. From the polyphase matrix $\mathbf{P} := \mathsf{P}_e\mathsf{U}^*$, we derive two high-pass filters $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ as follows:

$$\widetilde{\mathsf{a}}_1(z) = \frac{\sqrt{26}}{36504}\begin{bmatrix} a^1_{11}(z) - za^1_{11}(z^{-1}) & a^1_{12}(z) + za^1_{12}(z^{-1}) \\ a^1_{21}(z) + za^1_{21}(z^{-1}) & a^1_{22}(z) - za^1_{22}(z^{-1}) \end{bmatrix},$$

$$\widetilde{\mathsf{a}}_2(z) = \frac{\sqrt{78}}{4056}\begin{bmatrix} a^2_{11}(z) & a^2_{12}(z) \\ a^2_{21}(z) & a^2_{22}(z) \end{bmatrix},$$

where

$$\begin{aligned}
a^1_{11}(z) &= (433 - 128\sqrt{17})z^3 + 13(25\sqrt{17} - 43)z^2 - (1226 + 197\sqrt{17})z;\\
a^1_{12}(z) &= (128\sqrt{17} - 433)z^3 + 15(23\sqrt{17} - 19)z^2 - (758 + 197\sqrt{17})z;\\
a^1_{21}(z) &= 3(133 - 44\sqrt{17})z^3 + 117(3\sqrt{17} - 1)z^2 - 3(73\sqrt{17} + 94)z;\\
a^1_{22}(z) &= 3(44\sqrt{17} - 133)z^3 + 3(145\sqrt{17} - 61)z^2 - 3(250 + 73\sqrt{17})z;\\
a^2_{11}(z) &= 13(1 + \sqrt{17})(z^3 - 2z^2 + z);\\
a^2_{12}(z) &= 13(3\sqrt{17} - 1)(z^3 - z);\\
a^2_{21}(z) &= (9 + 11\sqrt{17})(z^3 - z);\\
a^2_{22}(z) &= (41\sqrt{17} - 9)(z^3 + 24z^2/137 + 18\sqrt{17}z^2/137 + z).
\end{aligned}$$

Then the high-pass filters $\widetilde{\mathsf{a}}_1$ and $\widetilde{\mathsf{a}}_2$ satisfy (3.4.11) with $c^1_1 = c^1_2 = 1/2$, $\varepsilon^1_1 = -1, \varepsilon^1_2 = 1$ and $c^2_1 = c^2_2 = 3/2$, $\varepsilon^1_1 = 1, \varepsilon^1_2 = -1$, respectively.

Let $\mathsf{a}_1, \mathsf{a}_2$ be high-pass filters constructed from $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ by $\mathsf{a}_1(z) := E\widetilde{\mathsf{a}}_1(z)E$ and $\mathsf{a}_2(z) := E\widetilde{\mathsf{a}}_2(z)E$. Then due to the orthogonality of E, $\{\mathsf{a}_0, \mathsf{a}_1, \mathsf{a}_2\}$ still forms a d-band filter bank with the perfect reconstruction property but their symmetry patterns are different to those of $\widetilde{\mathsf{a}}_0, \widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$. See Figure 3.3 for the graphs of the 3-refinable function vector ϕ associated with the low-pass filter a_0, the multiwavelet function vectors ψ^1, ψ^2 associated with the high-pass filters $\mathsf{a}_1, \mathsf{a}_2$, respectively. Also, see Figure 3.3 for the graphs of the 3-refinable function vector η associated with the low-pass filter $\widetilde{\mathsf{a}}_0$, the multiwavelet function vectors ζ^1, ζ^2 associated with the high-pass filters $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$, respectively.

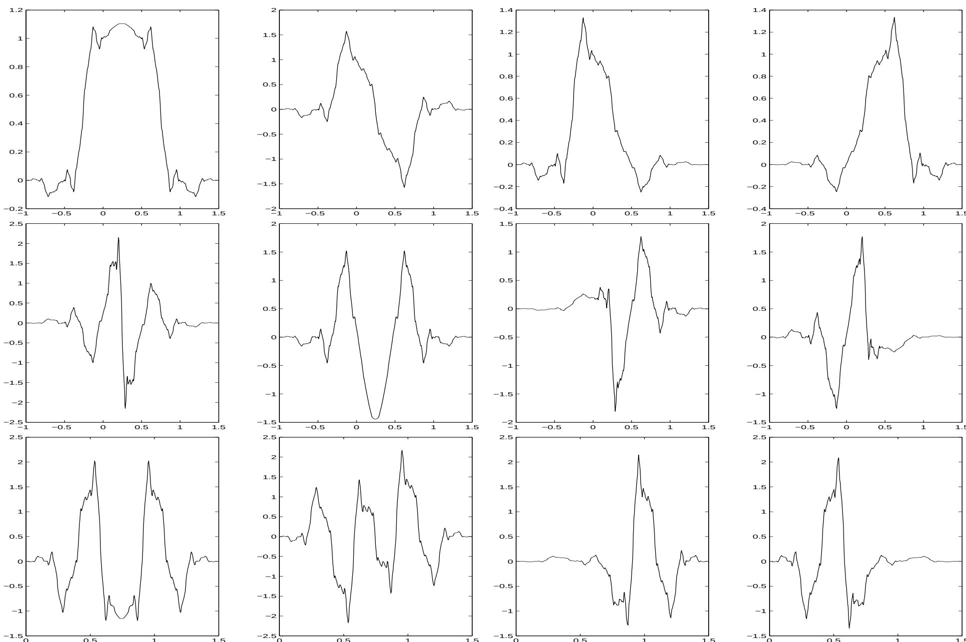

FIGURE 3.3: The graphs of the function vectors $\eta = [\eta_1, \eta_2]^T$, $\zeta^1 = [\zeta_1^1, \zeta_2^1]^T, \zeta^2 = [\zeta_1^2, \zeta_2^2]^T$ (left two columns, from top to bottom) associated with $\widetilde{\mathsf{a}}_0, \widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ in Example 3.3. And the graphs of function vector $\phi = [\phi_1, \phi_2]^T, \psi^1 = [\psi_1^1, \psi_2^1]^T, \psi^2 = [\psi_1^2, \psi_2^2]^T$ (right two columns, from top to bottom) associated with $\mathsf{a}_0, \mathsf{a}_1, \mathsf{a}_2$. Note that $[\eta, \zeta^1, \zeta^2] = E[\phi, \psi^1, \psi^2]$.

3.5 Proofs of Theorems 3.1 and 3.2

In this section, we shall prove Theorems 3.1 and 3.2. The key ingredient is to prove that the coefficient supports of $\mathsf{A}_1, \ldots, \mathsf{A}_J$ constructed in Algorithm 3.1 are all contained inside $[-1, 1]$. Note that each A_j takes the form $\mathsf{A}_j = (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}\mathsf{B}_{\mathsf{Q}_1}$. We first show that the coefficient support of $\mathsf{B} := (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1, 1]$ and then show that the coefficient support of $\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}$ is also contained inside $[-1, 1]$.

We establish the following lemma, which is needed later to show that the coefficient support of $(\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1,1]$.

Lemma 3.6. *Let* B *be an* $s \times s$ *paraunitary matrix such that* $\operatorname{coeffsupp}(\mathsf{B}) \subseteq [-1,1]$ *and* $\mathcal{S}\mathsf{B} = (\mathcal{S}\theta)^*\mathcal{S}\theta$ *with* $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ *for some nonnegative integers* $s_1, \ldots, s_4$ *such that* $s_1 + s_2 + s_3 + s_4 = s$*. Then the following statements hold.*

(1) *Let* p *be a* $1 \times s$ *row vector of Laurent polynomials with symmetry such that* $\mathsf{p}\mathsf{p}^* = 1$, $\operatorname{coeffsupp}(\mathsf{p}) = [k_1, k_2]$ *with* $k_2 - k_1 \geqslant 2$*, and* $\mathcal{S}\mathsf{p} = \varepsilon z^c \mathcal{S}\theta$ *for some* $\varepsilon \in \{-1,1\}$ *and* $c \in \{0,1\}$*. Let* $\mathsf{q} := \mathsf{p}\mathsf{B}$*. If* $\operatorname{coeffsupp}(\mathsf{q}) = \operatorname{coeffsupp}(\mathsf{p})$*, then* $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_\mathsf{q}) \subseteq [-1,1]$*, where* B_q *is constructed with respect to* q *as in Section 3.2.*

(2) *Let* $\mathsf{p}_1, \mathsf{p}_2$ *be two* $1 \times s$ *row vectors of Laurent polynomials with symmetry such that* $\mathsf{p}_{j_1}\mathsf{p}_{j_2}^* = \delta(j_1 - j_2)$ *for* $j_1, j_2 = 1, 2$*,* $\mathcal{S}\mathsf{p}_1 = \varepsilon_1 \mathcal{S}\theta$ *and* $\mathcal{S}\mathsf{p}_2 = \varepsilon_2 z \mathcal{S}\theta$ *for some* $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$*, and* $\operatorname{coeffsupp}(\mathsf{p}_1) = \operatorname{coeffsupp}(\mathsf{p}_2) \subseteq [-k,k]$ *with* $k \geqslant 1$*. Let* $\mathsf{q}_1 := \mathsf{p}_1\mathsf{B}$ *and* $\mathsf{q}_2 := \mathsf{p}_2\mathsf{B}$*. If* $\operatorname{coeffsupp}(\mathsf{q}_1) = [-k, k-1]$ *and* $\operatorname{coeffsupp}(\mathsf{q}_2) = [-k+1, k]$*, then* $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-1,1]$*, where* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ *is constructed with respect to the pair* $(\mathsf{q}_1, \mathsf{q}_2)$ *as in Section 3.2.*

Proof. Due to $\mathcal{S}\mathsf{p} = \varepsilon z^c \mathcal{S}\theta$, as we discussed in Section 3.2, there is an $\mathsf{U}_{\mathsf{p},\varepsilon}$ such that $\mathsf{p}\mathsf{U}_{\mathsf{p},\varepsilon}$ takes the form in (3.2.3). Since $\mathsf{U}_{\mathsf{p},\varepsilon}$ is a product of a permutation matrix and a diagonal matrix of monomials, we shall consider the case that $\mathsf{U}_{\mathsf{p},\varepsilon} = I_s$, while the proofs for other cases of $\mathsf{U}_{\mathsf{p},\varepsilon}$ can be obtained accordingly. Then p takes the standard form in (3.2.3) with $\mathtt{f}_1 \neq \mathbf{0}$. In this case, $s_1 > 0$ and $s_2 > 0$ due to $\|\mathtt{f}_1\| = \|\mathtt{f}_2\| \neq 0$. By our assumptions, $\mathsf{q} := \mathsf{pB}$ must take the following form:

$$
\begin{aligned}
\mathsf{q} := \mathsf{pB} =& [\widetilde{\mathtt{f}}_1, -\widetilde{\mathtt{f}}_2, \widetilde{\mathtt{g}}_1, -\widetilde{\mathtt{g}}_2] z^{k_1} + [\widetilde{\mathtt{f}}_3, -\widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_3, -\widetilde{\mathtt{g}}_4] z^{k_1+1} + \sum_{n=k_1+2}^{k_2-2} \operatorname{coeff}(\mathsf{pB}, n) z^n \\
&+ [\widetilde{\mathtt{f}}_3, \widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2] z^{k_2-1} + [\widetilde{\mathtt{f}}_1, \widetilde{\mathtt{f}}_2, \mathbf{0}, \mathbf{0}] z^{k_2}
\end{aligned}
$$

with $\widetilde{\mathtt{f}}_1 \neq \mathbf{0}$. Then B_{q} is given by (3.2.5) with $\mathtt{f}_1$, $\mathtt{f}_2$, $\mathtt{g}_1$, $\mathtt{g}_2$, F_1, F_2, G_1, G_2 being replaced by $\widetilde{\mathtt{f}}_1$, $\widetilde{\mathtt{f}}_2$, $\widetilde{\mathtt{g}}_1$, $\widetilde{\mathtt{g}}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{G}_1$, $\widetilde{G}_2$ respectively and all constants $c_{\widetilde{\mathtt{f}}_1}, c_{\widetilde{\mathtt{g}}_1}, c_{\widetilde{\mathtt{g}}_2}, c_0, c, c_{\widetilde{\mathtt{g}}'_1}, c_{\widetilde{\mathtt{g}}'_2}$ being defined accordingly.

Also, due to the symmetry pattern and $\operatorname{coeffsupp}(\mathsf{B}) \subseteq [-1, 1]$, B is of the form:

$$
\mathsf{B} = \left[\begin{array}{c|c|c|c}
A_1(z+\frac{1}{z}) + D_1 & A_3(z-\frac{1}{z}) & B_3(1+\frac{1}{z}) & B_4(1-\frac{1}{z}) \\
\hline
A_2(z-\frac{1}{z}) & A_4(z+\frac{1}{z}) + D_2 & C_3(1-\frac{1}{z}) & C_4(1+\frac{1}{z}) \\
\hline
B_1(1+z) & C_1(1-z) & A_5(z+\frac{1}{z}) + D_3 & A_7(z-\frac{1}{z}) \\
\hline
B_2(1-z) & C_2(1+z) & A_6(z-\frac{1}{z}) & A_8(z+\frac{1}{z}) + D_4
\end{array}\right], \tag{3.5.1}
$$

where A_j's, B_j's, C_j's and D_j's are all constant matrices in $\mathbb{F}$ and D_j is of size $s_j \times s_j$ for $j = 1, \ldots, 4$.

Let $\mathcal{I} := \{1, s_1+1, (1-\delta(s_3))(s_1+s_2+1), (1-\delta(s_4))(s_1+s_2+s_3+1)\}$ be an index set. It is easy to verify that $\operatorname{coeffsupp}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}) \subseteq [-1, 1]$ for all $j \notin \mathcal{I}$. Hence, by $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{q}}) \subseteq [-2, 2]$, we only need to compute $\operatorname{coeff}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}, 2)$ and $\operatorname{coeff}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}, -2)$ for those $j \in \mathcal{I}$. Let us show that $\operatorname{coeff}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}, 2) = \mathbf{0}$ for $j = 1$, i.e., the coefficient vector of z^2 for the first column of $\mathsf{B}\mathsf{B}_{\mathsf{q}}$ is $\mathbf{0}$.

Since $\mathrm{coeff}(\mathsf{pB}, k_1) = \mathrm{coeff}(\mathsf{p}, k_1+1)\mathrm{coeff}(\mathsf{B}, -1) + \mathrm{coeff}(\mathsf{p}, k_1)\mathrm{coeff}(\mathsf{B}, 0)$, we have

$$\begin{aligned}
\widetilde{\mathtt{f}}_1 &= \mathtt{f}_3 A_1 + \mathtt{f}_4 A_2 + \mathtt{f}_1 D_1 + \mathtt{g}_1 B_1 - \mathtt{g}_2 B_2;\\
\widetilde{\mathtt{f}}_2 &= \mathtt{f}_3 A_3 + \mathtt{f}_4 A_4 + \mathtt{f}_2 D_2 - \mathtt{g}_1 C_1 + \mathtt{g}_2 C_2;\\
\widetilde{\mathtt{g}}_1 &= \mathtt{f}_3 B_3 + \mathtt{f}_4 C_3 + \mathtt{g}_3 A_5 + \mathtt{g}_4 A_6 + \mathtt{f}_1 B_3 - \mathtt{f}_2 C_3 + \mathtt{g}_1 D_3;\\
\widetilde{\mathtt{g}}_2 &= \mathtt{f}_3 B_4 + \mathtt{f}_4 C_4 + \mathtt{g}_3 A_7 + \mathtt{g}_4 A_8 - \mathtt{f}_1 B_4 + \mathtt{f}_2 C_4 + \mathtt{g}_2 D_4.
\end{aligned} \tag{3.5.2}$$

Similarly, by $\mathrm{coeff}(\mathsf{BB_q}, 2) = \mathrm{coeff}(\mathsf{B}, 1)\mathrm{coeff}(\mathsf{B_q}, 1)$, we have

$$\mathrm{coeff}([\mathsf{BB_q}]_{:,1}, 2) = \frac{1}{c}\begin{bmatrix} A_1 & A_3 & \mathbf{0} & \mathbf{0}\\ A_2 & A_4 & \mathbf{0} & \mathbf{0}\\ B_1 & -C_1 & A_5 & A_7\\ -B_2 & C_2 & A_6 & A_8 \end{bmatrix}\begin{bmatrix} \widetilde{\mathtt{f}}_1^*\\ -\widetilde{\mathtt{f}}_2^*\\ \widetilde{\mathtt{g}}_1^*\\ -\widetilde{\mathtt{g}}_2^* \end{bmatrix} = \frac{1}{c}\begin{bmatrix} A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^*\\ A_2\widetilde{\mathtt{f}}_1^* - A_4\widetilde{\mathtt{f}}_2^*\\ B_1\widetilde{\mathtt{f}}_1^* + C_1\widetilde{\mathtt{f}}_2^* + A_5\widetilde{\mathtt{g}}_1^* - A_7\widetilde{\mathtt{g}}_2^*\\ -B_2\widetilde{\mathtt{f}}_1^* - C_1\widetilde{\mathtt{f}}_2^* + A_6\widetilde{\mathtt{g}}_1^* - A_8\widetilde{\mathtt{g}}_2^* \end{bmatrix}.$$

By the paraunitary property of B, i.e., $\mathsf{BB}^* = I_s$, we have

$$\begin{cases} A_1A_1^* - A_3A_3^* = \mathbf{0},\ A_1A_2^* - A_3A_4^* = \mathbf{0};\\ A_1D_1^* + D_1A_1^* + B_3B_3^* - B_4B_4^* = \mathbf{0};\\ D_1A_2^* - A_3D_2^* + B_3C_3^* - B_4C_4^* = \mathbf{0};\\ A_1B_1^* + A_3C_1^* + B_3A_5^* - B_4A_7^* = \mathbf{0};\\ -A_1B_2^* - A_3C_2^* + B_3A_6^* - B_4A_8^* = \mathbf{0}. \end{cases}$$

Applying the above identities to $A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^*$ and using (3.5.2), we get

$$\begin{aligned}
A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^* &= A_1(\mathtt{f}_3A_1 + \mathtt{f}_4A_2 + \mathtt{f}_1D_1 + \mathtt{g}_1B_1 - \mathtt{g}_2B_2)^*\\
&\quad - A_3(\mathtt{f}_3A_3 + \mathtt{f}_4A_4 + \mathtt{f}_2D_2 - \mathtt{g}_1C_1 + \mathtt{g}_2C_2)^*\\
&= (A_1A_1^* - A_3A_3^*)\mathtt{f}_3^* + (A_1A_2^* - A_3A_4^*)\mathtt{f}_4^* + (A_1D_1^*)\mathtt{f}_1^*\\
&\quad + (-A_3D_2^*)\mathtt{f}_2^* + (A_1B_1^* + A_3C_1^*)\mathtt{g}_1^* - (A_1B_2^* + A_3C_2^*)\mathtt{g}_2^*\\
&= -(D_1A_1^* + B_3B_3^* - B_4B_4^*)\mathtt{f}_1^* - (D_1A_2^* + B_3C_3^* - B_4C_4^*)\mathtt{f}_2^*\\
&\quad - (B_3A_5^* - B_4A_7^*)\mathtt{g}_1^* - (B_3A_6^* - B_4A_8^*)\mathtt{g}_2^*
\end{aligned}$$

$$
\begin{aligned}
&= -D_1(\mathtt{f}_1A_1 + \mathtt{f}_2A_2)^* - B_3(\mathtt{f}_1B_3 + \mathtt{f}_2C_3 + \mathtt{g}_1A_5 + \mathtt{g}_2A_6)^* \\
&\quad + B_4(\mathtt{f}_1B_4 + \mathtt{f}_2C_4 + \mathtt{g}_1A_7 + \mathtt{g}_2A_8)^* \\
&= \mathbf{0},
\end{aligned}
$$

where the last identity follows from $\mathrm{coeff}(\mathsf{pB}, k_2+1) = \mathrm{coeff}(\mathsf{pB}, k_1-1) = \mathbf{0}$. Similarly, we can show that $A_2\widetilde{\mathtt{f}}_1^* - A_4\widetilde{\mathtt{f}}_2^* = \mathbf{0}$, $B_1\widetilde{\mathtt{f}}_1^* + C_1\widetilde{\mathtt{f}}_2^* + A_5\widetilde{\mathtt{g}}_1^* - A_7\widetilde{\mathtt{g}}_2^* = \mathbf{0}$, and $-B_2\widetilde{\mathtt{f}}_1^* - C_1\widetilde{\mathtt{f}}_2^* + A_6\widetilde{\mathtt{g}}_1^* - A_8\widetilde{\mathtt{g}}_2^* = \mathbf{0}$. Hence, $\mathrm{coeff}([\mathsf{BB}_{\mathsf{q}}]_{:,1}, 2) = \mathbf{0}$. By similar computations as above and using the paraunitary property of B, we have $\mathrm{coeff}([\mathsf{BB}_{\mathsf{q}}]_{:,j}, \pm 2) = \mathbf{0}$ for all $j \in \mathcal{I}$. Therefore, we conclude that $\mathrm{coeffsupp}(\mathsf{BB}_{\mathsf{q}}) \subseteq [-1, 1]$. Item (1) holds.

For item (2), up to a permutation matrix $E_{(\mathsf{q}_1,\mathsf{q}_2)}$ as in Section 3.2, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ takes the form in (3.2.8). Since B takes the form in (3.5.1), to show that the coefficient support of $\mathsf{BB}_{(\mathsf{q}_1,\mathsf{q}_2)}$ is contained inside $[-1,1]$, we only need to show that $\mathrm{coeff}([\mathsf{BB}_{(\mathsf{q}_1,\mathsf{q}_2)}]_{:,j}, \pm 2) = \mathbf{0}$ for all $j \in \mathcal{I}$, which is to show that all the coefficient vectors $A_1\widetilde{\mathtt{g}}_1^* - A_3\widetilde{\mathtt{g}}_2^*$, $A_2\widetilde{\mathtt{g}}_1^* - A_4\widetilde{\mathtt{g}}_2^*$, $A_5\widetilde{\mathtt{g}}_3^* - A_7\widetilde{\mathtt{g}}_4^*$, and $A_6\widetilde{\mathtt{g}}_3^* - A_8\widetilde{\mathtt{g}}_4^*$ are zero. Again, using the paraunitary property of B and expressing $\widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2, \widetilde{\mathtt{g}}_3, \widetilde{\mathtt{g}}_4$ in terms of the original vectors from $\mathsf{p}_1, \mathsf{p}_2$ similar to (3.5.2), we conclude that $\mathrm{coeffsupp}(\mathsf{BB}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-1, 1]$. ☐

With the results of Lemma 3.6, the next lemma shows that the coefficient support of $\mathsf{B} := (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1,1]$. Moreover, the next lemma shows that the coefficient support of $\mathsf{A} := \mathsf{BB}_{\mathsf{Q}_1}$ is also contained inside $[-1, 1]$.

Lemma 3.7. *Suppose* Q *is an* $r \times s$ *matrix of Laurent polynomials such that* $\mathsf{QQ}^* = I_r$, $\mathcal{S}\mathsf{Q}$ *satisfies* (3.3.1), *and* $\mathrm{coeffsupp}(\mathsf{Q}) = [k_1, k_2]$ *with* $k_2 - k_1 \geqslant 1$. *Then there exists an* $s \times s$ *paraunitary matrix* A *of Laurent polynomials with symmetry such that*

(1) *the coefficient supports of* A *and* Q *satisfy* $\operatorname{coeffsupp}(\mathsf{A}) \subseteq [-1,1]$ *and* $|\operatorname{coeffsupp}(\mathsf{QA})| \leqslant |\operatorname{coeffsupp}(\mathsf{Q})| - |\operatorname{coeffsupp}(\mathsf{A})|$*;*

(2) *if the j-th column* $\mathsf{p} := [\mathsf{Q}]_{:,j}$ *of* Q *satisfies* $\operatorname{coeff}(\mathsf{p}, k_1) = \operatorname{coeff}(\mathsf{p}, k_2) = \mathbf{0}$*, then, up to a permutation matrix,* $[\mathsf{A}]_{j,:} = ([\mathsf{A}]_{:,j})^T = \mathbf{e}_j$*. That is, any entry in the j-th row or j-th column of* A *is zero except that the (j,j)-entry* $[\mathsf{A}]_{j,j} = 1$*;*

(3) $\mathcal{S}\mathsf{A} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$ *for some nonnegative integers* $s_1', \dots, s_4'$ *such that* $s_1' + s_2' + s_3' + s_4' = s$*.*

Proof. Let $\mathsf{A} = (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}\mathsf{B}_{\mathsf{Q}_1}$ be constructed as in Algorithm 3.1, where $\mathsf{Q}_1 := \mathsf{Q}(\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$, $\mathsf{B}_{(-k,k)}$ is constructed in the inner `while` loop of Algorithm 3.1, and $\mathsf{B}_1, \dots, \mathsf{B}_r$ is constructed in the `for` loop of Algorithm 3.1. If $k_2 \neq -k_1$, then $\mathsf{B}_1 = \cdots = \mathsf{B}_r = \mathsf{B}_{(-k,k)} = I_s$ and A is simply $\mathsf{B}_{\mathsf{Q}_1}$, where $\mathsf{Q}_1 = \mathsf{Q}$ is of the form in (3.3.4) with either $\operatorname{coeff}(\mathsf{Q}_1, -k) = \mathbf{0}$ or $\operatorname{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$. In this case, by the construction of $\mathsf{B}_{\mathsf{Q}_1}$ as in Section 3.2, all items in Lemma 3.7 hold. We are already done. So, without loss of generality, we assume that $k_2 = -k_1 = k$.

We first show that the coefficient support of $\mathsf{B}_1 \cdots \mathsf{B}_r$ is contained inside $[-1,1]$. Let $\mathsf{p}_j := [\mathsf{Q}]_{j,:}$, $\mathsf{B}_0 := I_s$, and $\mathsf{q}_j := \mathsf{p}_j\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$ for $j = 1, \dots, r$. Suppose we already show that $\operatorname{coeffsupp}(\mathsf{B}_0 \cdots \mathsf{B}_{j-1}) \subseteq [-1,1]$ for $j \geqslant 1$. Then, according to Algorithm 3.1, $\mathsf{B}_j = \mathsf{B}_{\mathsf{q}_j}$ if $\operatorname{coeffsupp}(\mathsf{p}_j) = \operatorname{coeffsupp}(\mathsf{q}_j)$, $|\operatorname{coeffsupp}(\mathsf{q}_j)| \geqslant 2$, and one of $\operatorname{coeff}(\mathsf{q}_j, k)$ and $\operatorname{coeff}(\mathsf{q}_j, -k)$ is nonzero; otherwise $\mathsf{B}_j = I_s$. Note that $\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$ is paraunitary and satisfies $\mathcal{S}(\mathsf{B}_0 \cdots \mathsf{B}_{j-1}) = (\mathcal{S}\theta)^*\mathcal{S}\theta$ with $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$. By item (1) of Lemma 3.6, the coefficient support of $\mathsf{B}_0 \cdots \mathsf{B}_{j-1}\mathsf{B}_j$ is also contained inside $[-1,1]$. By induction, the coefficient support of $\mathsf{B}_1 \cdots \mathsf{B}_r$

is contained inside $[-1,1]$. Moreover, $\mathsf{B}_1 \cdots \mathsf{B}_r$ takes the form in (3.5.1). Next, since $\mathsf{B}_{(-k,k)}$ is constructed recursively from pairs $(\mathsf{q}_1, \mathsf{q}_2)$ of $\mathsf{Q}_0 := \mathsf{Q}(\mathsf{B}_1 \cdots \mathsf{B}_r)$, by applying induction again and using item (2) of Lemma 3.6, we conclude that the coefficient support of $\mathsf{B} := (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1,1]$.

Due to the properties (P1), (P2) of B_q and (P3), (P4) of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_1, \ldots, \mathsf{B}_r$ and $\mathsf{B}_{(-k,k)}$ reduce Q of the form in (3.3.3) to $\mathsf{Q}_1 = \mathsf{Q}(\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)} = \mathsf{Q}\mathsf{B}$ of the form in (3.3.4) with at least one of $\operatorname{coeff}(\mathsf{Q}_1, -k)$ and $\operatorname{coeff}(\mathsf{Q}_1, k)$ being $\mathbf{0}$. As constructed in Section 3.2, we have $\mathsf{B}_{\mathsf{Q}_1} = I_s$ for the case that $\operatorname{coeff}(\mathsf{Q}_1, -k) = \operatorname{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$, or $\mathsf{B}_{\mathsf{Q}_1} = \operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ for the case $\operatorname{coeff}(\mathsf{Q}_1, k) \neq \mathbf{0}$, or $\mathsf{B}_{\mathsf{Q}_1} := \operatorname{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ for the case that $\operatorname{coeff}(\mathsf{Q}_1, -k) \neq \mathbf{0}$. We next show that $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1,1]$.

Let Q, Q_1 take the form in (3.3.3), (3.3.4), respectively, with $\operatorname{coeff}(\mathsf{Q}_1, k) \neq \mathbf{0}$. Then $\mathsf{B}_{\mathsf{Q}_1} := \operatorname{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ with U_1, W_1, and E being constructed as in Section 3.2. B takes the form in (3.5.1). Define

$$[G_1, G_2, F_3, F_4, G_5, G_6, F_7, F_8] := \begin{bmatrix} G_{11} & G_{21} & F_{31} & F_{41} & G_{51} & G_{61} & F_{71} & F_{81} \\ G_{12} & G_{22} & F_{32} & F_{42} & G_{52} & G_{62} & F_{72} & F_{82} \end{bmatrix}.$$

By $\operatorname{coeff}(\mathsf{Q}_1, k) = \operatorname{coeff}(\mathsf{Q}, k-1)\operatorname{coeff}(\mathsf{B}, 1) + \operatorname{coeff}(\mathsf{Q}, k)\operatorname{coeff}(\mathsf{B}, 0)$, we have

$$\begin{aligned} \widetilde{G}_1 &= G_5A_1 + G_6A_2 + F_7B_1 - F_8B_2 + G_1D_1 + F_3B_1 + F_4B_2; \\ \widetilde{G}_2 &= G_5A_3 + G_6A_4 - F_7C_1 + F_8C_2 + G_2D_2 + F_3C_1 + F_4C_2; \\ \mathbf{0} &= F_7A_5 + F_8A_6 + G_1B_3 + G_2C_3 + F_3D_3 =: \widetilde{F}_3; \\ \mathbf{0} &= F_7A_7 + F_8A_8 + G_1B_4 + G_2C_4 + F_4D_4 =: \widetilde{F}_4, \end{aligned} \tag{3.5.3}$$

where $\widetilde{G}_1, \widetilde{G}_2$ are matrices defined in (3.3.5). Then $U_1 = \operatorname{diag}(U_{\widetilde{G}_1}, U_{\widetilde{G}_2})$ and W_1 is defined as in (3.3.6). By the coefficient supports of B and $\mathsf{B}_{\mathsf{Q}_1}$, we only need to check that $\text{coeff}(\mathsf{B}\text{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}), -2) = \mathbf{0}$. Let $V_{11}, V_{12}, V_{21}, V_{22}$ be diagonal matrices of size $s_1 \times s_1$, $s_1 \times s_2$, $s_2 \times s_1$, $s_2 \times s_2$, respectively, and satisfy $\text{diag}(V_{j\ell}) = [\mathbf{1}_{m_1}, \mathbf{0}]$ for $j, \ell = 1, 2$, where m_1 is the rank of $\widetilde{G}_1$. Then

$$\text{coeff}(\mathsf{B}\text{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}), -2) = \text{coeff}(\mathsf{B}, -1) \cdot \text{coeff}(\text{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}), -1)$$
$$= \begin{bmatrix} A_1 & -A_3 & B_3 & -B_4 \\ -A_2 & A_4 & -C_3 & C_4 \\ \mathbf{0} & \mathbf{0} & A_5 & -A_7 \\ \mathbf{0} & \mathbf{0} & -A_6 & A_8 \end{bmatrix} \begin{bmatrix} U_{\widetilde{G}_1}V_{11} & U_{\widetilde{G}_1}V_{12} & \mathbf{0} & \mathbf{0} \\ U_{\widetilde{G}_2}V_{21} & U_{\widetilde{G}_2}V_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Thus, we need to show that for each $j = 1, 2$, $A_1U_{\widetilde{G}_1}V_{1j} - A_3U_{\widetilde{G}_2}V_{2j} = \mathbf{0}$ and $A_2U_{\widetilde{G}_1}V_{1j} - A_4U_{\widetilde{G}_2}V_{2j} = \mathbf{0}$, which is equivalent to showing that for each $j = 1, 2$, $V_{j1}U^*_{\widetilde{G}_1}A^*_1 - V_{j2}U^*_{\widetilde{G}_2}A^*_3 = \mathbf{0}$ and $V_{j1}U^*_{\widetilde{G}_1}A^*_2 - V_{j2}U^*_{\widetilde{G}_2}A^*_4 = \mathbf{0}$. Since $\widetilde{G}_1U_{\widetilde{G}_1} = [R, \mathbf{0}]$ and $\widetilde{G}_2U_{\widetilde{G}_2} = [R, \mathbf{0}]$, for some lower triangular matrix R of full rank m_1, it is equivalent to proving that $\widetilde{G}_1A^*_1 - \widetilde{G}_2A^*_3 = \mathbf{0}$ and $\widetilde{G}_1A^*_2 - \widetilde{G}_2A^*_4 = \mathbf{0}$. By (3.5.3), we have,

$$\begin{aligned}
\widetilde{G}_1A^*_1 - \widetilde{G}_2A^*_3 &= \widetilde{G}_1A^*_1 - \widetilde{G}_2A^*_3 + \widetilde{F}_3B^*_3 - \widetilde{F}_4B^*_4 \\
&= (G_5A_1 + G_6A_2 + F_7B_1 - F_8B_2 + G_1D_1 + F_3B_1 + F_4B_2)A^*_1 \\
&\quad - (G_5A_3 + G_6A_4 - F_7C_1 + F_8C_2 + G_2D_2 + F_3C_1 + F_4C_2)A^*_3 \\
&\quad + (F_7A_5 + F_8A_6 + G_1B_3 + G_2C_3 + F_3D_3)B^*_3 \\
&\quad - (F_7A_7 + F_8A_8 + G_1B_4 + G_2C_4 + F_4D_4)B^*_4 \\
&= G_5(A_1A^*_1 - A_3A^*_3) + G_6(A_2A^*_1 - A_4A^*_3) \\
&\quad + F_7(B_1A^*_1 + C_1A^*_3 + A_5B^*_3 - A_7B^*_4) \\
&\quad + F_8(-B_2A^*_1 - C_2A^*_3 + A_6B^*_3 - A_8B^*_4) \\
&\quad + G_1(D_1A^*_1 + B_3B^*_3 - B_4B^*_4) + G_2(-D_2A^*_3 + C_3B^*_3 - C_4B^*_4) \\
&\quad + F_3(B_1A^*_1 - C_1A^*_3 + D_3B^*_3) + F_4(B_2A^*_1 - C_2A^*_3 - D_4B^*_4) = \mathbf{0},
\end{aligned}$$

where the last identity follows from $\mathsf{B}\mathsf{B}^* = I_s$ and $\mathrm{coeff}(\mathsf{Q}\mathsf{B}, k+1) = \mathbf{0}$. Similarly, $\widetilde{G}_1 A_2^* - \widetilde{G}_2 A_4^* = \mathbf{0}$. The computation for showing that $\mathrm{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1, 1]$ with $\mathsf{B}_{\mathsf{Q}_1} = \mathrm{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ is similar. Consequently, $\mathrm{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1, 1]$. Therefore, item (1) holds. Item (2) is due to the property (3.2.2) of $U_{\mathtt{f}}$ and U_G.

Note that $\mathcal{S}\mathsf{B} = (\mathcal{S}\theta)^*\mathcal{S}\theta$ with $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$. And by the construction of $\mathsf{B}_{\mathsf{Q}_1}$, $\mathcal{S}\mathsf{B}_{\mathsf{Q}_1} = (\mathcal{S}\theta)^*[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$ for some nonnegative integers $s_1', \dots, s_4'$ depending on the rank of $\widetilde{G}_1$ or $\widetilde{G}_3$ (see Section 3.2). Consequently, item (3) holds. This also completes the proof of Algorithm 3.1. $\square$

Now, we are ready to prove Theorems 3.1 and 3.2.

Proof of Theorems 3.1 and 3.2: The sufficiency part of Theorem 3.2 is obvious. We only need to show the necessary part. Suppose $\mathcal{S}\mathsf{P} = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$. Let $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^*\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ and $\mathrm{coeffsupp}(\mathsf{Q}) := [k_1, k_2]$. Then $\mathcal{S}\mathsf{Q}$ satisfies (3.3.1). By Lemma 3.7, the step of support reduction in Algorithm 3.1 produces a sequence of paraunitary matrices $\mathsf{A}_1, \dots, \mathsf{A}_J$ with coefficient support contained inside $[-1, 1]$ such that $\mathsf{Q}\mathsf{A}_1 \cdots \mathsf{A}_J = [I_r, \mathbf{0}]$. Due to item (1) of Lemma 3.7, $J \leqslant \lceil \frac{k_2-k_1}{2} \rceil$. Let $\mathsf{P}_j := \mathsf{A}_j^*$, $\mathsf{P}_0 := \mathsf{U}_{\mathcal{S}\theta_2}^*$ and $\mathsf{P}_{J+1} := \mathrm{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$. Then $\mathsf{P}_e := \mathsf{P}_{J+1}\mathsf{P}_J \cdots \mathsf{P}_1\mathsf{P}_0$ satisfies $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$. By item (3) of Lemma 3.7, $(\mathsf{P}_{j+1}, \mathsf{P}_j)$ has mutually compatible symmetry for all $0 \leqslant j \leqslant J$. The claim that $|\mathrm{coeffsupp}([\mathsf{P}_e]_{k,j})| \leqslant \max_{1 \leqslant n \leqslant r} |\mathrm{coeffsupp}([\mathsf{P}]_{n,j})|$ for $1 \leqslant j, k \leqslant s$ follows from item (2) of Lemma 3.7. Hence, all claims in Theorems 3.1 and 3.2 have been verified. $\square$

3.6 Conclusions and Remarks

In this chapter, we introduce the general problem of matrix extension with symmetry. We successfully solve this problem for any r, s such that $1 \leqslant r \leqslant s$. More importantly, we obtain a complete representation of any $r \times s$ paraunitary matrix P having compatible symmetry with $1 \leqslant r \leqslant s$. This representation leads to a step-by-step algorithm for deriving a desired matrix P_e from a given matrix P.

Moreover, we obtain an optimal result in the sense of (3.1.7) on controlling the coefficient support of the desired matrix P_e derived from a given matrix P by our algorithm. This is of importance in both theory and application, since short support of a filter is a highly desirable property, which usually means a fast algorithm and simple implementation in practice.

Furthermore, we introduce the notion of compatibility of symmetry, which plays a critical role in the study of the general matrix extension problem with symmetry for the multi-row case ($r \geqslant 1$). We provide a complete analysis and a systematic construction algorithm for d-band symmetric filter banks and symmetric orthonormal multiwavelets.

Finally, most of the literature on the matrix extension problem only consider Laurent polynomials with coefficients in the special field $\mathbb{C}$ ([54]) or $\mathbb{R}$ ([6, 59]). In this chapter, our setting is under a more general field $\mathbb{F}$, which can be any subfield of $\mathbb{C}$ satisfying (3.1.6).

In next chapter, we shall study the matrix extension problem with symmetry for the biorthogonal case, which can be applied to the construction of biorthogonal multiwavelets from a pair of dual d-refinable function vectors.

Chapter 4

Matrix Extension with Symmetry: Biorthogonal Generalization

4.1 Introduction and Main Results

In this chapter, we shall consider the matrix extension problem for the construction of biorthogonal multiwavelets.

Due to the flexibility of the biorthogonality relation, the restriction in (3.1.6) for a coefficient field $\mathbb{F}$ can be released and $\mathbb{F}$ can be relaxed as any subfield in $\mathbb{C}$. In this chapter, we shall refer $\mathbb{F}$ as a subfield of $\mathbb{C}$.

Now we generalize the matrix extension problem to the biorthogonal case as follows: Let $\mathsf{P}, \widetilde{\mathsf{P}}$ be two $r \times s$ matrices of Laurent polynomials with coefficients in $\mathbb{F}$ such that $\mathsf{P}(z)\widetilde{\mathsf{P}}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$, the symmetry of each P and $\widetilde{\mathsf{P}}$ is compatible, and $\mathcal{S}\mathsf{P} = \mathcal{S}\widetilde{\mathsf{P}}$. Find two $s \times s$ square matrices $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ of Laurent polynomials with coefficients in $\mathbb{F}$ and with symmetry such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}, [I_r, \mathbf{0}]\widetilde{\mathsf{P}}_e = \widetilde{\mathsf{P}}$, (that is, the submatrix of the first r rows of $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ is the given matrix $\mathsf{P}, \widetilde{\mathsf{P}}$, respectively), the symmetry of P_e and $\widetilde{\mathsf{P}}_e$ is compatible, and $\mathsf{P}_e(z)\widetilde{\mathsf{P}}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$. The coefficient support of $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ can be controlled by that of $\mathsf{P}, \widetilde{\mathsf{P}}$ in some way.

Due to the flexibility of biorthogonality, the above extension problem becomes far more complicated than the matrix extension problem we considered in Chapter 3. The difficulty here is not the symmetry patterns of the extension matrices, but the support control of the extension matrices. Without considering any issue on support control, almost all results of Theorems 3.1 and 3.2 can be transferred to the biorthogonal case without much difficulty. In Chapter 3, we showed that the length of the coefficient support of the extension matrix can never exceed the length of the coefficient support of the given matrix. Yet, for the extension matrices in the biorthogonal extension case, we can no longer expect such a nice result, that is, in this case, the length of the coefficient supports of the extension matrices might not be controlled by one of the given matrices. Let us present an example here to show why we might not have such a result.

Example 4.1. Consider two 1×3 vectors of Laurent polynomials $\mathsf{p}(z) = [1, 0, a(z)]$ and $\widetilde{\mathsf{p}}(z) = [1, \widetilde{a}(z), 0]$ with $|\mathrm{coeffsupp}(a)| > 0, |\mathrm{coeffsupp}(\widetilde{a})| > 0$. We have $\mathsf{p}\widetilde{\mathsf{p}}^* = 1$. Let P_e and $\widetilde{\mathsf{P}}_e$ be their extension matrices such that $\mathsf{P}_e\widetilde{\mathsf{P}}_e^* = I_3$. Then $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ must be of the form:

$$\mathsf{P}_e = \begin{bmatrix} 1 & 0 & a(z) \\ -b_1(z)\widetilde{a}^*(z) & b_1(z) & c_1(z) \\ -b_2(z)\widetilde{a}^*(z) & b_2(z) & c_2(z) \end{bmatrix}, \widetilde{\mathsf{P}}_e = \begin{bmatrix} 1 & \widetilde{a}(z) & 0 \\ -\widetilde{c}_1(z)a^*(z) & \widetilde{b}_1(z) & \widetilde{c}_1(z) \\ -\widetilde{c}_2(z)a^*(z) & \widetilde{b}_2(z) & \widetilde{c}_2(z) \end{bmatrix}.$$

It is easy to show that $\det(\mathsf{P}_e) = b_1(z)c_2(z) - b_2(z)c_1(z)$. Since P_e is invertible with $\mathsf{P}_e^{-1} = \widetilde{\mathsf{P}}_e^*$, we know that $\det(\mathsf{P}_e)$ must be a monomial. Without loss of generality, we can assume $b_1(z)c_2(z) - b_2(z)c_1(z) = 1$. Using the cofactors of P_e, it is easy to show that $\widetilde{\mathsf{P}}_e = (\mathsf{P}_e^{-1})^*$ must be of the form:

$$\widetilde{\mathsf{P}}_e = \begin{bmatrix} 1 & \widetilde{a}(z) & 0 \\ b_2^*(z)a^*(z) & c_2^*(z) + \widetilde{a}(z)a^*(z)b_2^*(z) & -b_2^*(z) \\ -b_1^*(z)a^*(z) & -c_1^*(z) - \widetilde{a}(z)a^*(z)b_1^*(z) & b_1^*(z) \end{bmatrix}.$$

One the one hand, if $|\text{coeffsupp}(b_1(z))| > 0$ or $|\text{coeffsupp}(b_2(z))| > 0$, then we see that one of the extension matrices will have support length exceeding the maximal length of the given columns. One the other hand, if both $|\text{coeffsupp}(b_1(z))| = 0$ and $|\text{coeffsupp}(b_2(z))| = 0$ (that is, both b_1 and b_2 are monomials), then the length of the coefficient support of $c_1(z)$ and $c_2(z)$ in $\widetilde{\mathsf{P}}_e$ must be comparable with $\widetilde{a}^*(z)a(z)$ so that the support length of $\widetilde{\mathsf{P}}_e$ can be controlled by that of p or $\widetilde{\mathsf{p}}$, which in turn will result in longer support length of P_e.

The above example shows that it is difficult to control the support length of the coefficient support of the extension matrices independently by only one given vector in the biorthogonal setting. Nevertheless, we have the following result:

Theorem 4.1. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Let $\mathsf{P}, \widetilde{\mathsf{P}}$ be two $r \times s$ matrices of Laurent polynomials with coefficients in $\mathbb{F}$ such that the symmetry of each $\mathsf{P}, \widetilde{\mathsf{P}}$ is compatible: $\mathcal{S}\mathsf{P} = \mathcal{S}\widetilde{\mathsf{P}} = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$ for some $1\times r$, $1\times s$ vectors θ_1, θ_2 of Laurent polynomials with symmetry. $\mathsf{P}(z)\widetilde{\mathsf{P}}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$. Then there exist two $s \times s$ square matrices $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(i) $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}, [I_r, \mathbf{0}]\widetilde{\mathsf{P}}_e = \widetilde{\mathsf{P}}$, *that is, the submatrices of the first r rows of $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ are $\mathsf{P}, \widetilde{\mathsf{P}}$, respectively;*

(ii) P_e *and* $\widetilde{\mathsf{P}}_e$ *are biorthogonal:* $\mathsf{P}_e(z)\widetilde{\mathsf{P}}_e^*(z) = I_s$ *for all* $z \in \mathbb{C}\backslash\{0\}$*;*

(iii) *The symmetry of each* $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ *is compatible:* $\mathcal{S}\mathsf{P}_e = \mathcal{S}\widetilde{\mathsf{P}}_e = (\mathcal{S}\theta)^*\mathcal{S}\theta_2$ *for some* $1 \times s$ *vector* θ *of Laurent polynomials with symmetry.*

(iv) $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ *can be represented as:*

$$\mathsf{P}_e(z) = \mathsf{P}_J(z) \cdots \mathsf{P}_1(z), \quad \widetilde{\mathsf{P}}_e(z) = \widetilde{\mathsf{P}}_J(z) \cdots \widetilde{\mathsf{P}}_1(z), \tag{4.1.1}$$

where $\mathsf{P}_j, \widetilde{\mathsf{P}}_j, 1 \leqslant j \leqslant J$ *are* $s \times s$ *matrices of Laurent polynomials with symmetry that satisfy* $\mathsf{P}_j(z)\widetilde{\mathsf{P}}_j^*(z) = I_s$*. Moreover, each pair of* $(\mathsf{P}_{j+1}, \mathsf{P}_j)$ *and* $(\widetilde{\mathsf{P}}_{j+1}, \widetilde{\mathsf{P}}_j)$ *has mutually compatible symmetry for all* $j = 1, \ldots, J-1$.

(v) *If* $r = 1$*, then the coefficient supports of* $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ *are controlled by that of* $\mathsf{P}, \widetilde{\mathsf{P}}$ *in the following sense:*

$$\begin{aligned}\max_{1\leqslant j,k\leqslant s}\{|\mathrm{coeffsupp}([\mathsf{P}_e]_{j,k})|\} &\leqslant \max_{1\leqslant \ell\leqslant s}|\mathrm{coeffsupp}([\mathsf{P}]_\ell)| + \max_{1\leqslant \ell\leqslant s}|\mathrm{coeffsupp}([\widetilde{\mathsf{P}}]_\ell)|\\ \max_{1\leqslant j,k\leqslant s}\{|\mathrm{coeffsupp}([\widetilde{\mathsf{P}}_e]_{j,k})|\} &\leqslant \max_{1\leqslant \ell\leqslant s}|\mathrm{coeffsupp}([\mathsf{P}]_\ell)| + \max_{1\leqslant \ell\leqslant s}|\mathrm{coeffsupp}([\widetilde{\mathsf{P}}]_\ell)|.\end{aligned} \tag{4.1.2}$$

For $r = 1$, Goh et al. in [17] considered this matrix extension problem without symmetry. They provided a step-by-step algorithm for deriving the extension matrices, yet they did not concern about the support control of the extension matrices nor the symmetry patterns of the extension matrices. For $r > 1$, there are only a few results in the literature [3, 9] and most of them concern only about some special cases. The difficulty still comes from the flexibility of the biorthogonality relation between the given two matrices. In this chapter, we shall mainly consider this matrix extension problem with symmetry for the biorthogonal case and shall provide an extension algorithm from which the extension matrices can have both symmetry and support control as stated in Theorem 4.1.

Here is the structure of this chapter. In Section 4.2, we shall introduce some auxiliary results, prove Theorem 4.1, and also provide a step-by-step algorithm for the construction of the extension matrices. In Section 4.3, we shall discuss the applications of our main result to the construction of symmetric biorthogonal multiwavelets in wavelet analysis. Examples will be provided to illustrate our algorithms. Conclusions and remarks shall be given in the last section.

4.2 Proof of Theorem 4.1 and an Algorithm

First, let us introduce some auxiliary results.

Lemma 4.2. *Let $\mathtt{f}, \widetilde{\mathtt{f}}$ be two nonzero $1 \times n$ vectors in $\mathbb{F}$. Then the following statements hold.*

(1) *If $\mathtt{f}\widetilde{\mathtt{f}}^* \neq 0$, then there exist two $n \times n$ matrices $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$, $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})}$ in $\mathbb{F}$ such that $U_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{\widetilde{\mathtt{f}}}{\overline{\widetilde{c}}})^*, F]$, $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{\mathtt{f}}{c})^*, \widetilde{F}]$, and $U_{(\mathtt{f},\widetilde{\mathtt{f}})}\widetilde{U}^*_{(\mathtt{f},\widetilde{\mathtt{f}})} = I_n$, where $F, \widetilde{F}$ are $n \times (n-1)$ constant matrices in $\mathbb{F}$ and $c, \widetilde{c}$ are two nonzero numbers in $\mathbb{F}$ such that $\mathtt{f}\widetilde{\mathtt{f}}^* = c\overline{\widetilde{c}}$. In this case, $\mathtt{f}U_{(\mathtt{f},\widetilde{\mathtt{f}})} = c\mathbf{e}_1$ and $\widetilde{\mathtt{f}}\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = \widetilde{c}\mathbf{e}_1$.*

(2) *If $\mathtt{f}\widetilde{\mathtt{f}}^* = 0$, then there exist two $n \times n$ matrices $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$, $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})}$ in $\mathbb{F}$ such that $U_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{\mathtt{f}}{\overline{\widetilde{c}_1}})^*, (\frac{\widetilde{\mathtt{f}}}{c_2})^*, F]$, $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{\mathtt{f}}{c_1})^*, (\frac{\widetilde{\mathtt{f}}}{\overline{\widetilde{c}_2}})^*, \widetilde{F}]$, and $U_{(\mathtt{f},\widetilde{\mathtt{f}})}\widetilde{U}^*_{(\mathtt{f},\widetilde{\mathtt{f}})} = I_n$, where $F, \widetilde{F}$ are $n \times (n-2)$ constant matrices in $\mathbb{F}$ and*

$c_1, c_2, \widetilde{c}_1, \widetilde{c}_2$ are nonzero numbers in $\mathbb{F}$ such that $\|\mathtt{f}\|^2 = c_1\overline{\widetilde{c}_1}, \|\widetilde{\mathtt{f}}\|^2 = c_2\overline{\widetilde{c}_2}$. In this case, $\mathtt{f}U_{(\mathtt{f},\widetilde{\mathtt{f}})} = c_1\mathbf{e}_1$ and $\widetilde{\mathtt{f}}\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = c_2\mathbf{e}_2$.

Proof. If $\mathtt{f}\widetilde{\mathtt{f}}^* \neq 0$, there exists $\{\mathtt{f}_2, \ldots, \mathtt{f}_n\}$ being a basis of the orthogonal compliment of the linear span of $\{\mathtt{f}\}$ in $\mathbb{F}^n$. Let $F := [\mathtt{f}_2^*, \ldots, \mathtt{f}_n^*]$ and $U_{(\mathtt{f},\widetilde{\mathtt{f}})} := [(\frac{\widetilde{\mathtt{f}}}{\widetilde{c}})^*, F]$. Then $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$ is invertible. Let $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} := (U^{-1}_{(\mathtt{f},\widetilde{\mathtt{f}})})^*$. It is easy to show that $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$ and $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})}$ are the desired matrices.

If $\mathtt{f}\widetilde{\mathtt{f}}^* = 0$, let $\{\mathtt{f}_3, \ldots, \mathtt{f}_n\}$ be a basis of the orthogonal compliment of the linear span of $\{\mathtt{f}, \widetilde{\mathtt{f}}\}$ in $\mathbb{F}^n$. Let $U_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{\mathtt{f}}{\widetilde{c}_1})^*, (\frac{\widetilde{\mathtt{f}}}{c_2})^*, F]$ with $F := [\mathtt{f}_3^*, \ldots, \mathtt{f}_n^*]$. Then $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$ and $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} := (U^{-1}_{(\mathtt{f},\widetilde{\mathtt{f}})})^*$ are the desired matrices. ☐

Lemma 4.3. *Let $\mathsf{p}, \widetilde{\mathsf{p}}$ be two $1 \times s$ vectors of Laurent polynomials with symmetry such that $\mathsf{p}\widetilde{\mathsf{p}}^* = 1$ and $\mathcal{S}\mathsf{p} = \mathcal{S}\widetilde{\mathsf{p}} = \varepsilon z^c[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}] =: \mathcal{S}\theta$ for some nonnegative integers $s_1, \ldots, s_4$ satisfying $s_1 + \cdots + s_4 = s$ and $\varepsilon \in \{1, -1\}, c \in \{0, 1\}$. Suppose* coeffsupp$(\mathsf{p}) > 0$. *Then there exist two $s \times s$ matrices $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ of Laurent polynomials with symmetry such that*

(1) *$\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ are biorthogonal: $\mathsf{B}(z)\widetilde{\mathsf{B}}(z)^* = I_n$;*

(2) *$\mathcal{S}\mathsf{B} = \mathcal{S}\widetilde{\mathsf{B}} = (\mathcal{S}\theta)^*\mathcal{S}\theta_1$ with $\mathcal{S}\theta_1 = \varepsilon z^c[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$ for some nonnegative integers $s_1', \ldots, s_4'$ such that $s_1' + \cdots + s_4' = s$;*

(3) *the length of the coefficient support of p is reduced by that of B. $\widetilde{\mathsf{B}}$ does not increase the length of the coefficient support of $\widetilde{\mathsf{p}}$. That is, $|\text{coeffsupp}(\mathsf{p}\mathsf{B})| \leqslant |\text{coeffsupp}(\mathsf{p})| - |\text{coeffsupp}(\mathsf{B})|$ and $|\text{coeffsupp}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}})| \leqslant |\text{coeffsupp}(\widetilde{\mathsf{p}})|$.*

Proof. We shall only prove the case that $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$. The proofs for other cases are similar. By their symmetry patterns, p and

$\widetilde{\mathsf{p}}$ must take the form as follows with $\ell > 0$ and $\operatorname{coeff}(\mathsf{p}, -\ell) \neq \mathbf{0}$:

$$\begin{aligned} \mathsf{p} = [\mathtt{f}_1, -\mathtt{f}_2, \mathtt{g}_1, -\mathtt{g}_2] z^{-\ell} + [\mathtt{f}_3, -\mathtt{f}_4, \mathtt{g}_3, -\mathtt{g}_4] z^{-\ell+1} + \cdots \\ + [\mathtt{f}_3, \mathtt{f}_4, \mathtt{g}_1, \mathtt{g}_2] z^{\ell-1} + [\mathtt{f}_1, \mathtt{f}_2, \mathbf{0}, \mathbf{0}] z^{\ell}; \end{aligned} \tag{4.2.1}$$

$$\begin{aligned} \widetilde{\mathsf{p}} = [\widetilde{\mathtt{f}}_1, -\widetilde{\mathtt{f}}_2, \widetilde{\mathtt{g}}_1, -\widetilde{\mathtt{g}}_2] z^{-\widetilde{\ell}} + [\widetilde{\mathtt{f}}_3, -\widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_3, -\widetilde{\mathtt{g}}_4] z^{-\widetilde{\ell}+1} + \cdots \\ + [\widetilde{\mathtt{f}}_3, \widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2] z^{\widetilde{\ell}-1} + [\widetilde{\mathtt{f}}_1, \widetilde{\mathtt{f}}_2, \mathbf{0}, \mathbf{0}] z^{\widetilde{\ell}}; \end{aligned} \tag{4.2.2}$$

Then, either $\|\mathtt{f}_1\| + \|\mathtt{f}_2\| \neq 0$ or $\|\mathtt{g}_1\| + \|\mathtt{g}_2\| \neq 0$. Considering $\|\mathtt{f}_1\| + \|\mathtt{f}_2\| \neq 0$, due to $\mathsf{p}\widetilde{\mathsf{p}}^* = 1$ and $\operatorname{coeffsupp}(\mathsf{p}) > 0$, we have $\mathtt{f}_1 \widetilde{\mathtt{f}}_1^* - \mathtt{f}_2 \widetilde{\mathtt{f}}_2^* = 0$. Let $c := \mathtt{f}_1 \widetilde{\mathtt{f}}_1^* = \mathtt{f}_2 \widetilde{\mathtt{f}}_2^*$. Then there are at most three cases: (a) $c \neq 0$; (b) $c = 0$ but both $\mathtt{f}_1, \mathtt{f}_2$ are nonzero vectors; (c) $c = 0$ and one of $\mathtt{f}_1, \mathtt{f}_2$ is $\mathbf{0}$.

Case (a): In this case, we have $\mathtt{f}_1 \widetilde{\mathtt{f}}_1^* \neq 0$ and $\mathtt{f}_2 \widetilde{\mathtt{f}}_2 \neq 0$. By Lemma 4.2, we can construct two pairs of biorthogonal matrices $(U_{(\mathtt{f}_1, \widetilde{\mathtt{f}}_1)}, \widetilde{U}_{(\mathtt{f}_1, \widetilde{\mathtt{f}}_1)})$ and $(U_{(\mathtt{f}_2, \widetilde{\mathtt{f}}_2)}, \widetilde{U}_{(\mathtt{f}_2, \widetilde{\mathtt{f}}_2)})$ with respect to the pairs $(\mathtt{f}_1, \widetilde{\mathtt{f}}_1)$ and $(\mathtt{f}_2, \widetilde{\mathtt{f}}_2)$ such that

$$\begin{aligned} U_{(\mathtt{f}_1, \widetilde{\mathtt{f}}_1)} = [(\tfrac{\widetilde{\mathtt{f}}_1}{\widetilde{c}_1})^*, F_1], \quad \widetilde{U}_{(\mathtt{f}_1, \widetilde{\mathtt{f}}_1)} = [(\tfrac{\mathtt{f}_1}{c_1})^*, \widetilde{F}_1], \quad \mathtt{f}_1 U_{(\mathtt{f}_1, \widetilde{\mathtt{f}}_1)} = c_1 \mathbf{e}_1, \quad \widetilde{\mathtt{f}}_1 \widetilde{U}_{(\mathtt{f}_1, \widetilde{\mathtt{f}}_1)} = \widetilde{c}_1 \mathbf{e}_1, \\ U_{(\mathtt{f}_2, \widetilde{\mathtt{f}}_2)} = [(\tfrac{\widetilde{\mathtt{f}}_2}{\widetilde{c}_1})^*, F_2], \quad \widetilde{U}_{(\mathtt{f}_2, \widetilde{\mathtt{f}}_2)} = [(\tfrac{\mathtt{f}_1}{c_1})^*, \widetilde{F}_2], \quad \mathtt{f}_2 U_{(\mathtt{f}_2, \widetilde{\mathtt{f}}_2)} = c_1 \mathbf{e}_1, \quad \widetilde{\mathtt{f}}_2 \widetilde{U}_{(\mathtt{f}_2, \widetilde{\mathtt{f}}_2)} = \widetilde{c}_1 \mathbf{e}_1, \end{aligned}$$

where $c_1, \widetilde{c}_1$ are constants in $\mathbb{F}$ such that $c = c_1 \overline{\widetilde{c}_1}$. Define $\mathsf{B}_0(z), \widetilde{\mathsf{B}}_0(z)$ as follows:

$$\begin{aligned} \mathsf{B}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_1}{\widetilde{c}_1})^* & F_1 & \frac{1-z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_1}{\widetilde{c}_1})^* & \mathbf{0} & \mathbf{0} \\ \frac{1-z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_2}{\widetilde{c}_1})^* & \mathbf{0} & \frac{1+z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_2}{\widetilde{c}_1})^* & F_2 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array}\right], \\ \widetilde{\mathsf{B}}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_1}{c_1})^* & \widetilde{F}_1 & \frac{1-z^{-1}}{2}(\frac{\mathtt{f}_1}{c_1})^* & \mathbf{0} & \mathbf{0} \\ \frac{1-z^{-1}}{2}(\frac{\mathtt{f}_2}{c_1})^* & \mathbf{0} & \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_2}{c_1})^* & \widetilde{F}_2 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array}\right]. \end{aligned} \tag{4.2.3}$$

Direct computation shows that $\mathsf{B}_0(z)\widetilde{\mathsf{B}}_0(z)^* = I_s$ and $\mathsf{B}_0(z)$, $\widetilde{\mathsf{B}}_0(z)$ reduce the lengths of the coefficient support of $\mathsf{p}, \widetilde{\mathsf{p}}$ by 1, respectively. Moreover,

$$\mathcal{S}(\mathsf{p}\mathsf{B}_0) = \mathcal{S}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0) = [z^{-1}, \mathbf{1}_{s_1-1}, -z^{-1}, -\mathbf{1}_{s_2-1}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}].$$

Let E be a permutation matrix such that

$$\mathcal{S}(\mathsf{p}\mathsf{B}_0)E = \mathcal{S}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0)E = [\mathbf{1}_{s_1-1}, -\mathbf{1}_{s_2-1}, z^{-1}\mathbf{1}_{s_3+1}, -z^{-1}\mathbf{1}_{s_4+1}] =: \mathcal{S}\theta_1.$$

Define $\mathsf{B}(z) = \mathsf{B}_0(z)E$ and $\widetilde{\mathsf{B}}(z) = \widetilde{\mathsf{B}}_0(z)E$. Then $\mathsf{B}(z)$ and $\widetilde{\mathsf{B}}(z)$ are the desired matrices.

Case (b): In this case, both $\mathtt{f}_1, \mathtt{f}_2$ are nonzero vectors. We have $\mathtt{f}_1\mathtt{f}_1^* \neq 0$ and $\mathtt{f}_2\mathtt{f}_2^* \neq 0$. Again, by Lemma 4.2, we can construct two pairs of biorthogonal matrices $(U_{(\mathtt{f}_1,\mathtt{f}_1)}, \widetilde{U}_{(\mathtt{f}_1,\mathtt{f}_1)})$ and $(U_{(\mathtt{f}_2,\mathtt{f}_2)}, \widetilde{U}_{(\mathtt{f}_2,\mathtt{f}_2)})$ with respect to the pairs $(\mathtt{f}_1, \mathtt{f}_1)$ and $(\mathtt{f}_2, \mathtt{f}_2)$ such that

$$\begin{aligned} U_{(\mathtt{f}_1,\mathtt{f}_1)} &= [(\frac{\mathtt{f}_1}{\widetilde{c}_1})^*, F_1], \quad \widetilde{U}_{(\mathtt{f}_1,\mathtt{f}_1)} = [(\frac{\mathtt{f}_1}{c_0})^*, F_1], \quad \mathtt{f}_1 U_{(\mathtt{f}_1,\mathtt{f}_1)} = c_0\mathbf{e}_1, \\ U_{(\mathtt{f}_2,\mathtt{f}_2)} &= [(\frac{\mathtt{f}_2}{\widetilde{c}_2})^*, F_2], \quad \widetilde{U}_{(\mathtt{f}_2,\mathtt{f}_2)} = [(\frac{\mathtt{f}_1}{c_0})^*, F_2], \quad \mathtt{f}_2 U_{(\mathtt{f}_2,\mathtt{f}_2)} = c_0\mathbf{e}_1, \end{aligned}$$

where $c_0, \widetilde{c}_1, \widetilde{c}_2$ are constants in $\mathbb{F}$ such that $\mathtt{f}_1\mathtt{f}_1^* = c_0\overline{\widetilde{c}_1}$ and $\mathtt{f}_2\mathtt{f}_2^* = c_0\overline{\widetilde{c}_2}$. Let $\mathsf{B}_0, \widetilde{\mathsf{B}}_0(z)$ be defined as follows:

$$\begin{aligned} \mathsf{B}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_1}{\widetilde{c}_1})^* & F_1 & \frac{1-z^{-1}}{2}(\frac{\mathtt{f}_1}{\widetilde{c}_1})^* & \mathbf{0} & \mathbf{0} \\ \frac{1-z^{-1}}{2}(\frac{\mathtt{f}_2}{\widetilde{c}_2})^* & \mathbf{0} & \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_2}{\widetilde{c}_2})^* & F_2 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array}\right], \\ \widetilde{\mathsf{B}}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_1}{c_0})^* & F_1 & \frac{1-z^{-1}}{2}(\frac{\mathtt{f}_1}{c_0})^* & \mathbf{0} & \mathbf{0} \\ \frac{1-z^{-1}}{2}(\frac{\mathtt{f}_2}{c_0})^* & \mathbf{0} & \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_2}{c_0})^* & F_2 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array}\right]. \end{aligned} \tag{4.2.4}$$

We can show that $\mathsf{B}_0(z)$ reduces the length of the coefficient support of p by 1, while $\widetilde{\mathsf{B}}_0(z)$ does not increase the support length of $\widetilde{\mathsf{p}}$. Moreover, similar to case (a), we can find a permutation matrix E such that

$$\mathcal{S}(\mathsf{p}\mathsf{B}_0)E = \mathcal{S}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0)E = [\mathbf{1}_{s_1-1}, -\mathbf{1}_{s_2-1}, z^{-1}\mathbf{1}_{s_3+1}, -z^{-1}\mathbf{1}_{s_4+1}] =: \mathcal{S}\theta_1.$$

Define $\mathsf{B}(z) = \mathsf{B}_0(z)E$ and $\widetilde{\mathsf{B}}(z) = \widetilde{\mathsf{B}}_0(z)E$. Then $\mathsf{B}(z)$ and $\widetilde{\mathsf{B}}(z)$ are the desired matrices.

Case (c): In this case, without loss of generality, we assume that $\mathtt{f}_1 \neq \mathbf{0}$ and $\mathtt{f}_2 = \mathbf{0}$. Construct a pair of matrices $(U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)}, \widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)})$ by Lemma 4.2 such that $\mathtt{f}_1 U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = c_1\mathbf{e}_1$ and $\widetilde{\mathtt{f}}_1 \widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = c_2\mathbf{e}_2$ (when $\widetilde{\mathtt{f}}_1 = \mathbf{0}$, the pair of matrices is given by $(U_{(\mathtt{f}_1,\mathtt{f}_1)}, \widetilde{U}_{(\mathtt{f}_1,\mathtt{f}_1)})$). Extend this pair to a pair of $s \times s$ matrices $(U, \widetilde{U})$ by $U := \mathrm{diag}(U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)}, I_{s_3+s_4})$ and $\widetilde{U} := \mathrm{diag}(\widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)}, I_{s_3+s_4})$. Then $\mathsf{p}U$ and $\widetilde{\mathsf{p}}\widetilde{U}$ must be of the form:

$$\begin{aligned} \mathsf{q} := \mathsf{p}U &= [c_1, 0, \dots, 0, -\mathtt{f}_2, \mathtt{g}_1, -\mathtt{g}_2]z^{-\ell} + [\mathtt{f}_3, -\mathtt{f}_4, \mathtt{g}_3, -\mathtt{g}_4]z^{-\ell+1} \\ &\quad + \cdots + [\mathtt{f}_3, \mathtt{f}_4, \mathtt{g}_1, \mathtt{g}_2]z^{\ell-1} + [c_1, 0, \dots, 0, \mathtt{f}_2, \mathbf{0}, \mathbf{0}]z^{\ell}; \end{aligned} \tag{4.2.5}$$

$$\begin{aligned} \widetilde{\mathsf{q}} := \widetilde{\mathsf{p}}\widetilde{U} &= [0, c_2, \dots, 0, -\widetilde{\mathtt{f}}_2, \widetilde{\mathtt{g}}_1, -\widetilde{\mathtt{g}}_2]z^{-\widetilde{\ell}} + [\widetilde{\mathtt{f}}_3, -\widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_3, -\widetilde{\mathtt{g}}_4]z^{-\widetilde{\ell}+1} \\ &\quad + \cdots + [\widetilde{\mathtt{f}}_3, \widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2]z^{\widetilde{\ell}-1} + [0, c_2, \dots, 0, , \widetilde{\mathtt{f}}_2, \mathbf{0}, \mathbf{0}]z^{\widetilde{\ell}}; \end{aligned} \tag{4.2.6}$$

If $[\widetilde{\mathsf{q}}]_1 \equiv 0$, we choose k such that $k = \arg\min_{\ell \neq 1}\{|\mathrm{coeffsupp}([\mathsf{q}]_1)| - |\mathrm{coeffsupp}([\mathsf{q}]_\ell)|\}$, i.e., k is an integer such that the length of coefficient support of $([\mathsf{q}]_1 - [\mathsf{q}]_k)$ is minimal among those of all $([\mathsf{q}]_1 - [\mathsf{q}]_\ell)$, $\ell = 2, \dots, s$; otherwise, due to $\mathsf{q}\widetilde{\mathsf{q}}^* = 0$, there must exist k such that

$$|\mathrm{coeffsupp}([\mathsf{q}]_1)| - |\mathrm{coeffsupp}([\mathsf{q}]_k)| \leqslant \max_{2 \leqslant j \leqslant s} |\mathrm{coeffsupp}([\widetilde{\mathsf{q}}]_j)| - |\mathrm{coeffsupp}([\widetilde{\mathsf{q}}]_1)|,$$

(k might not be unique, we can choose one of such k so that $|\mathrm{coeffsupp}([\mathsf{q}]_1)| -$

$|\text{coeffsupp}([\mathsf{q}]_k)|$ is minimal among all $|\text{coeffsupp}([\mathsf{q}]_1)|-|\text{coeffsupp}([\mathsf{q}]_\ell)|, \ell = 2,\dots,s)$. For such k (in the case of either $[\widetilde{\mathsf{q}}]_1 = 0$ or $[\widetilde{\mathsf{q}}]_1 \neq 0$), define two matrices $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ as follows:

$$\mathsf{B}(z) = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ -a(z) & 0 & \cdots & 1 & \\ \hline & & & & I_{s-k} \end{array}\right], \widetilde{\mathsf{B}}(z) = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & a(z)^* & \\ 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \\ \hline & & & & I_{s-k} \end{array}\right],$$

where $a(z)$ in $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ is a Laurent polynomial with symmetry such that $\mathcal{S}a(z) = \mathcal{S}([\mathsf{q}]_1)/\mathcal{S}([\mathsf{q}]_k)$, $|\text{coeffsupp}([\mathsf{q}_1]-a(z)[\mathsf{q}]_k)| < |\text{coeffsupp}([\mathsf{q}]_k)|$, and $|\text{coeffsupp}([\widetilde{\mathsf{q}}_k] - a(z)^*[\widetilde{\mathsf{q}}]_1)| \leqslant \max_{1\leqslant \ell\leqslant s}|\text{coeffsupp}([\widetilde{\mathsf{q}}]_\ell)|$. Such $a(z)$ can be easily obtained by long division. It is straightforward to show that $\mathsf{B}(z)\widetilde{\mathsf{B}}^*(z) = I_s$, $\mathsf{B}(z)$ reduces the length of the coefficient support of q by that of $a(z)$, and $\widetilde{\mathsf{B}}(z)$ does not increase the length of the coefficient support of $\widetilde{\mathsf{q}}$. Moreover, the symmetry patterns of both q and $\widetilde{\mathsf{q}}$ are preserved.

For $\|\mathtt{f}_1\| + \|\mathtt{f}_2\| = 0$, we must have $\|\mathsf{g}_1\| + \|\mathsf{g}_2\| \neq 0$. The discussion for this case is similar to above. We can find two matrices $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ such that all items in the lemma hold. In the case that $\mathsf{g}_1\widetilde{\mathsf{g}}_1^* = c_1\overline{\widetilde{c}_1} \neq 0$, the pair $(\mathsf{B}_0(z), \widetilde{\mathsf{B}}_0(z))$ similar to (4.2.4) is of the form:

$$\begin{aligned} \mathsf{B}_0(z) &= \left[\begin{array}{c|cc|cc} I_{s_1+s_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \frac{1+z}{2}(\frac{\widetilde{\mathsf{g}}_1}{\widetilde{c}_1})^* & G_1 & \frac{1-z}{2}(\frac{\widetilde{\mathsf{g}}_1}{\widetilde{c}_1})^* & \mathbf{0} \\ \mathbf{0} & \frac{1-z}{2}(\frac{\widetilde{\mathsf{g}}_2}{\widetilde{c}_1})^* & \mathbf{0} & \frac{1+z}{2}(\frac{\widetilde{\mathsf{g}}_2}{\widetilde{c}_1})^* & G_2 \end{array}\right], \\ \widetilde{\mathsf{B}}_0(z) &= \left[\begin{array}{c|cc|cc} I_{s_1+s_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \frac{1+z}{2}(\frac{\mathsf{g}_1}{c_1})^* & \widetilde{G}_1 & \frac{1-z}{2}(\frac{\mathsf{g}_1}{c_1})^* & \mathbf{0} \\ \mathbf{0} & \frac{1-z}{2}(\frac{\mathsf{g}_2}{c_1})^* & \mathbf{0} & \frac{1+z}{2}(\frac{\mathsf{g}_2}{c_1})^* & \widetilde{G}_2 \end{array}\right]. \end{aligned} \tag{4.2.7}$$

The pairs for other cases can be obtained similarly. We are done. □

Now, we can prove Theorem 4.1 using Lemma 4.3.

Proof of Theorem 4.1. Let $\mathsf{Q} := \mathsf{U}^*_{\mathcal{S}\theta_1}\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ and $\widetilde{\mathsf{Q}} := \mathsf{U}^*_{\mathcal{S}\theta_1}\widetilde{\mathsf{P}}\mathsf{U}_{\mathcal{S}\theta_2}$ (given θ, $\mathsf{U}_{\mathcal{S}\theta}$ is obtained by (3.3.2)). Then the symmetry of each row of Q or $\widetilde{\mathsf{Q}}$ is of the form $\varepsilon z^c[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some $\varepsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$.

Let $\mathsf{p} := [\mathsf{Q}]_{1,:}$ and $\widetilde{\mathsf{p}} := [\widetilde{\mathsf{Q}}]_{1,:}$ be the first row of $\mathsf{Q}, \widetilde{\mathsf{Q}}$, respectively. Applying Lemma 4.3 recursively, we can find $(\mathsf{B}_1, \widetilde{\mathsf{B}}_1)$, ..., $(\mathsf{B}_K, \widetilde{\mathsf{B}}_K)$ such that $\mathsf{p}\mathsf{B}_1 \cdots \mathsf{B}_K = [1, 0, \ldots, 0]$ and $\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_1 \cdots \widetilde{\mathsf{B}}_K = [1, \mathsf{q}(z)]$ for some $1 \times (s-1)$ vector of Laurent polynomials with symmetry. Now construct $\mathsf{B}_{K+1}(z), \widetilde{\mathsf{B}}_{K+1}(z)$ as follows:

$$\mathsf{B}_{K+1}(z) = \begin{bmatrix} 1 & 0 \\ \mathsf{q}^*(z) & I_{s-1} \end{bmatrix}, \ \widetilde{\mathsf{B}}_{K+1}(z) = \begin{bmatrix} 1 & -\mathsf{q}(z) \\ \mathbf{0} & I_{s-1} \end{bmatrix}.$$

B_{K+1} and $\widetilde{\mathsf{B}}_{K+1}$ are biorthogonal. Let $\mathsf{A} := \mathsf{B}_1 \cdots \mathsf{B}_K\mathsf{B}_{K+1}$ and $\widetilde{\mathsf{A}} := \widetilde{\mathsf{B}}_1 \cdots \widetilde{\mathsf{B}}_K\widetilde{\mathsf{B}}_{K+1}$. Then $\mathsf{p}\mathsf{A} = \widetilde{\mathsf{p}}\widetilde{\mathsf{A}} = \mathbf{e}_1$.

Note that $\mathsf{Q}\mathsf{A}$ and $\widetilde{\mathsf{Q}}\widetilde{\mathsf{A}}$ are of the form:

$$\mathsf{Q}\mathsf{A} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathsf{Q}_1(z) \end{bmatrix}, \ \widetilde{\mathsf{Q}}\widetilde{\mathsf{A}} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathsf{Q}}_1(z) \end{bmatrix}.$$

The rest of the proof is completed by employing the standard procedure of induction. □

Next, according to the proof of Theorem 4.1, we have an extension algorithm for Theorem 4.1 as follows:

Algorithm 4.1. *Input* $\mathsf{P}, \widetilde{\mathsf{P}}$ *as in Theorem 4.1 with* $\mathcal{S}\mathsf{P} = \mathcal{S}\widetilde{\mathsf{P}} = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$ *for two* $1\times r$, $1\times s$ *row vectors* θ_1, θ_2 *of Laurant polynomials with symmetry.*

1. Initialization: *Let* $\mathsf{Q} := \mathsf{U}^*_{\mathcal{S}\theta_1}\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ *and* $\widetilde{\mathsf{Q}} := \mathsf{U}^*_{\mathcal{S}\theta_1}\widetilde{\mathsf{P}}\mathsf{U}_{\mathcal{S}\theta_2}$. *Then both* Q *and* $\widetilde{\mathsf{Q}}$ *have the the same symmetry pattern as follows:*

$$\mathcal{S}\mathsf{Q} = \mathcal{S}\widetilde{\mathsf{Q}} = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}], \quad (4.2.8)$$

where all nonnegative integers $r_1, \ldots, r_4, s_1, \ldots, s_4$ *are uniquely determined by* $\mathcal{S}\mathsf{P}$. *Note that this step does not increase the lengths of the coefficient support of both* P *and* $\widetilde{\mathsf{P}}$.

2. Support Reduction: *Let* $\mathsf{U}_0 := \mathsf{U}^*_{\mathcal{S}\theta_2}$ *and* $\mathsf{A} = \widetilde{\mathsf{A}} := I_s$.

`for` k `from` 1 `to` r `do`

Let $\mathsf{p} := [\mathsf{Q}]_{k,k:s}$ *and* $\widetilde{\mathsf{p}} := [\widetilde{\mathsf{Q}}]_{k,k:s}$.

`while` $|\mathrm{coeffsupp}(\mathsf{p})| > 0$ `and` $|\mathrm{coeffsupp}(\widetilde{\mathsf{p}})| > 0$ `do`

Construct $\mathsf{B}(z)$, $\widetilde{\mathsf{B}}(z)$ *with respect to* p, $\widetilde{\mathsf{p}}$ *by Lemma 4.3 such that* $|\mathrm{coeffsupp}(\mathsf{p}\mathsf{B})| + |\mathrm{coeffsupp}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}})| < |\mathrm{coeffsupp}(\mathsf{p})| + |\mathrm{coeffsupp}(\widetilde{\mathsf{p}})|$.

Replace $\mathsf{p}, \widetilde{\mathsf{p}}$ *by* $\mathsf{p}\mathsf{B}$, $\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}$, *respectively.*

Set $\mathsf{A} := \mathsf{A}\mathrm{diag}(I_{k-1}, \mathsf{B})$ *and* $\widetilde{\mathsf{A}} := \widetilde{\mathsf{A}}\mathrm{diag}(I_{k-1}, \widetilde{\mathsf{B}})$.

`end while`

The pair $(\mathsf{p}, \widetilde{\mathsf{p}})$ *is of the form:* $([1, 0, \ldots, 0], [1, \mathsf{q}(z)])$ *for some* $1 \times (s-k)$ *vector of Laurent polynomials* $\mathsf{q}(z)$.

Construct $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ *as follows:*

$$\mathsf{B}(z) = \begin{bmatrix} 1 & 0 \\ \mathsf{q}^*(z) & I_{s-k} \end{bmatrix}, \widetilde{\mathsf{B}}(z) = \begin{bmatrix} 1 & -\mathsf{q}(z) \\ \mathbf{0} & I_{s-k} \end{bmatrix}.$$

Set $\mathsf{A} := \mathsf{A}\mathrm{diag}(I_{k-1}, \mathsf{B})$ *and* $\widetilde{\mathsf{A}} := \widetilde{\mathsf{A}}\mathrm{diag}(I_{k-1}, \widetilde{\mathsf{B}})$.

`end for`

3. Finalization: *Let* $\mathsf{U}_1 := \mathrm{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$. *Set* $\mathsf{P}_e := \mathsf{U}_1\mathsf{A}^*\mathsf{U}_0$ *and* $\widetilde{\mathsf{P}}_e := \mathsf{U}_1\widetilde{\mathsf{A}}^*\mathsf{U}_0$.

Output a pair of desired matrices $(\mathsf{P}_e, \widetilde{\mathsf{P}}_e)$ *satisfying all the properties in Theorem 4.1.*

4.3 Application to Biorthogonal Multiwavelets with Symmetry

In this section, we shall discuss the application of our results to biorthogonal multiwavelets with symmetry. Several examples are provided to demonstrate our results.

Let $a_0, \widetilde{a}_0 : \mathbb{Z} \to \mathbb{F}^{r\times r}$ with multiplicity r be finitely supported sequences of $r \times r$ matrices on $\mathbb{Z}$. Let d be a dilation factor and d_1, d_2 be two fixed number in $\mathbb{F}$ such that $\mathsf{d} = d_1 d_2$ (for instance $d_1 = 1, d_2 = 2$ for $\mathsf{d} = 2$ if $\mathbb{F} = \mathbb{Q}$). It is easily seen that $(a_0, \widetilde{a}_0)$ is a pair of dual masks (see (1.4.2)) with respect to a dilation factor d if

$$\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\widetilde{\mathsf{a}}^*_{0;\gamma}(z) = I_r, \qquad z \in \mathbb{C}\backslash\{0\}, \tag{4.3.1}$$

where $\mathsf{a}_{0;\gamma}$ and $\widetilde{\mathsf{a}}_{0;\gamma}$ are *subsymbols (polyphases)* of a_0 and $\widetilde{\mathsf{a}}_0$ defined to be

$$\begin{array}{l} \mathsf{a}_{0;\gamma}(z) := d_1 \sum_{k\in\mathbb{Z}} a_0(k+\mathsf{d}k)z^k, \\ \widetilde{\mathsf{a}}_{0;\gamma}(z) := d_2 \sum_{k\in\mathbb{Z}} \widetilde{a}_0(k+\mathsf{d}k)z^k, \end{array} \quad \gamma \in \mathbb{Z}. \tag{4.3.2}$$

We shall refer to such pair $(\mathsf{a}_0, \widetilde{\mathsf{a}}_0)$ (the symbol of $(a_0, \widetilde{a}_0)$) *a pair of* d*-band biorthogonal filters.*

To construct biorthogonal multiwavelets, we need to design high-pass filters $a_1, \dots, a_{\mathsf{d}-1} : \mathbb{Z} \to \mathbb{F}^{r\times r}$ and $\widetilde{a}_1, \dots, \widetilde{a}_{\mathsf{d}-1} : \mathbb{Z} \to \mathbb{F}^{r\times r}$ such that the polyphase matrices

$$\mathbf{P}(z) = \begin{bmatrix} \mathsf{a}_{0;0}(z) & \cdots & \mathsf{a}_{0;\mathsf{d}-1}(z) \\ \mathsf{a}_{1;0}(z) & \cdots & \mathsf{a}_{1;\mathsf{d}-1}(z) \\ \vdots & \vdots & \vdots \\ \mathsf{a}_{\mathsf{d}-1;0}(z) & \cdots & \mathsf{a}_{\mathsf{d}-1;\mathsf{d}-1}(z) \end{bmatrix}, \widetilde{\mathbf{P}}(z) = \begin{bmatrix} \widetilde{\mathsf{a}}_{0;0}(z) & \cdots & \widetilde{\mathsf{a}}_{0;\mathsf{d}-1}(z) \\ \widetilde{\mathsf{a}}_{1;0}(z) & \cdots & \widetilde{\mathsf{a}}_{1;\mathsf{d}-1}(z) \\ \vdots & \vdots & \vdots \\ \widetilde{\mathsf{a}}_{\mathsf{d}-1;0}(z) & \cdots & \widetilde{\mathsf{a}}_{\mathsf{d}-1;\mathsf{d}-1}(z) \end{bmatrix} \tag{4.3.3}$$

are biorthogonal, that is, $\mathbf{P}(z)\widetilde{\mathbf{P}}^*(z) = I_{\mathsf{d}r}$, where $\mathsf{a}_{m;\gamma}, \widetilde{\mathsf{a}}_{m;\gamma}$ are subsymbols of $\mathsf{a}_m, \widetilde{\mathsf{a}}_m$ defined similar to (4.3.2) for $m, \gamma = 0, \dots, \mathsf{d}-1$, respectively.

Let $(\phi, \widetilde{\phi})$ be a pair of dual d-refinable function vectors in $L_2(\mathbb{R})$ associated with a pair of d-band biorthogonal filters $(\mathsf{a}_0, \widetilde{\mathsf{a}}_0)$ and with $\phi = [\phi_1, \dots, \phi_r]^T$, $\widetilde{\phi} = [\widetilde{\phi}_1, \dots, \widetilde{\phi}_r]^T$. Define multiwavelet function vectors $\psi^m = [\psi_1^m, \dots, \psi_r^m]^T$, $\widetilde{\psi}^m = [\widetilde{\psi}_1^m, \dots, \widetilde{\psi}_r^m]^T$ associated with the high-pass filters $\mathsf{a}_m, \widetilde{\mathsf{a}}_m$, $m = 1, \dots, \mathsf{d}-1$, by

$$\widehat{\psi^m}(\mathsf{d}\xi) := \mathsf{a}_m(e^{-i\xi})\widehat{\phi}(\xi), \ \widehat{\widetilde{\psi}^m}(\mathsf{d}\xi) := \widetilde{\mathsf{a}}_m(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi), \ \xi \in \mathbb{R}. \tag{4.3.4}$$

It is well known that $\{\psi^1, \dots, \psi^{\mathsf{d}-1}; \widetilde{\psi}^1, \dots, \widetilde{\psi}^{\mathsf{d}-1}\}$ generates a biorthonormal multiwavelet basis in $L_2(\mathbb{R})$.

Now, for a pair of d-band biorthogonal low-pass filters $(\mathsf{a}_0, \widetilde{\mathsf{a}}_0)$ with multiplicity r satisfying (3.4.3), we have the following algorithm to construct high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}$ and $\widetilde{\mathsf{a}}_1, \dots, \widetilde{\mathsf{a}}_{\mathsf{d}-1}$ such that the polyphase matrices $\mathsf{P}(z)$ and $\widetilde{\mathsf{P}}(z)$ defined as in (4.3.3) satisfy $\mathsf{P}(z)\widetilde{\mathsf{P}}^*(z) = I_{\mathsf{d}r}$. In what follows, $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$ and $\widetilde{\mathsf{P}}_{\widetilde{\mathsf{a}}_0} := [\widetilde{\mathsf{a}}_{0;0}, \dots, \widetilde{\mathsf{a}}_{0;\mathsf{d}-1}]$ are the polyphase vectors of $\mathsf{a}_0, \widetilde{\mathsf{a}}_0$ obtained by (4.3.2), respectively.

Algorithm 4.2. *Input* $(\mathsf{a}_0, \widetilde{\mathsf{a}}_0)$ *a pair of biorthogonal* d*-band filters with multiplicity* r *and with the same symmetry as in* (3.4.3)*.*

(1) *Construct a pair of biorthogonal matrices* $(\mathsf{U}, \widetilde{\mathsf{U}})$ *in* $\mathbb{F}$ *similar to* (3.4.8) *such that both* $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ *and* $\widetilde{\mathsf{P}} = \widetilde{\mathsf{P}}_{\widetilde{\mathsf{a}}_0}\widetilde{\mathsf{U}}$ *are matrices of Laurent polynomials with coefficient in* $\mathbb{F}$ *having compatible symmetry:* $\mathcal{S}\mathsf{P} = \mathcal{S}\widetilde{\mathsf{P}} = [\varepsilon_1 z^{k_1}, \ldots, \varepsilon_r z^{k_r}]^T \mathcal{S}\theta$ *for some* $k_1, \ldots, k_r \in \mathbb{Z}$ *and some* $1 \times \mathsf{d}r$ *row vector* θ *of Laurent polynomials with symmetry.*

(2) *Derive* $\mathsf{P}_e, \widetilde{\mathsf{P}}_e$ *with all the properties as in Theorem* 4.1 *from* $\mathsf{P}, \widetilde{\mathsf{P}}$ *by Algorithm* 4.1*.*

(3) *Let* $\mathbf{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant \mathsf{d}-1}$*,* $\widetilde{\mathbf{P}} := \widetilde{\mathsf{P}}_e\widetilde{\mathsf{U}}^* =: (\widetilde{\mathsf{a}}_{m;\gamma})_{0\leqslant m,\gamma\leqslant \mathsf{d}-1}$ *as in* (4.3.3)*. For* $m = 1, \ldots, \mathsf{d}-1$*, define high-pass filters*

$$\mathsf{a}_m(z) := \frac{1}{d_1}\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{m;\gamma}(z^{\mathsf{d}})z^{\gamma}, \quad \widetilde{\mathsf{a}}_m(z) := \frac{1}{d_2}\sum_{\gamma=0}^{\mathsf{d}-1} \widetilde{\mathsf{a}}_{m;\gamma}(z^{\mathsf{d}})z^{\gamma}. \qquad (4.3.5)$$

Output symmetric filter banks $\{\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{d}-1}\}$ *and* $\{\widetilde{\mathsf{a}}_0, \widetilde{\mathsf{a}}_1, \ldots, \widetilde{\mathsf{a}}_{\mathsf{d}-1}\}$ *with the perfect reconstruction property, i.e.* $\mathbf{P}, \widetilde{\mathbf{P}}$ *in* (4.3.3) *are biorthogonal and all filters* $\mathsf{a}_m, \widetilde{\mathsf{a}}_m$*,* $m = 1, \ldots, \mathsf{d}-1$*, have symmetry:*

$$\begin{aligned}\mathsf{a}_m(z) &= \mathrm{diag}(\varepsilon_1^m z^{\mathsf{d}c_1^m}, \ldots, \varepsilon_r^m z^{\mathsf{d}c_r^m})\mathsf{a}_m(1/z)\mathrm{diag}(\varepsilon_1 z^{-c_1}, \ldots, \varepsilon_r z^{-c_r}),\\ \widetilde{\mathsf{a}}_m(z) &= \mathrm{diag}(\varepsilon_1^m z^{\mathsf{d}c_1^m}, \ldots, \varepsilon_r^m z^{\mathsf{d}c_r^m})\widetilde{\mathsf{a}}_m(1/z)\mathrm{diag}(\varepsilon_1 z^{-c_1}, \ldots, \varepsilon_r z^{-c_r}),\end{aligned} \qquad (4.3.6)$$

where $c_\ell^m := (k_\ell^m - k_\ell) + c_\ell \in \mathbb{R}$ *and all* $\varepsilon_\ell^m \in \{-1, 1\}$*,* $k_\ell^m \in \mathbb{Z}$*, for* $\ell = 1, \ldots, r$ *and* $m = 1, \ldots, \mathsf{d}-1$*, are determined by the symmetry pattern of* P_e *as follows:*

$$[\varepsilon_1 z^{k_1}, \ldots, \varepsilon_r z^{k_r}, \varepsilon_1^1 z^{k_1^1}, \ldots, \varepsilon_r^1 z^{k_r^1}, \ldots, \varepsilon_1^{\mathsf{d}-1} z^{k_1^{\mathsf{d}-1}}, \ldots, \varepsilon_r^{\mathsf{d}-1} z^{k_r^{\mathsf{d}-1}}]^T \mathcal{S}\theta := \mathcal{S}\mathsf{P}_e. \qquad (4.3.7)$$

Since the high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}, \widetilde{\mathsf{a}}_1, \dots, \widetilde{\mathsf{a}}_{\mathsf{d}-1}$ satisfy (4.3.6), it is easy to verify that each $\psi^m = [\psi^m_1, \dots, \psi^m_r]^T$, $\widetilde{\psi}^m = [\widetilde{\psi}^m_1, \dots, \widetilde{\psi}^m_r]^T$ defined in (3.4.4) also has the following symmetry:

$$\begin{aligned}
&\psi^m_1(c^m_1 - \cdot) = \varepsilon^m_1 \psi^m_1, \quad \psi^m_2(c^m_2 - \cdot) = \varepsilon^m_2 \psi^m_2, \quad \dots, \quad \psi^m_r(c^m_r - \cdot) = \varepsilon^m_r \psi^m_r,\\
&\widetilde{\psi}^m_1(c^m_1 - \cdot) = \varepsilon^m_1 \widetilde{\psi}^m_1, \quad \widetilde{\psi}^m_2(c^m_2 - \cdot) = \varepsilon^m_2 \widetilde{\psi}^m_2, \quad \dots, \quad \widetilde{\psi}^m_r(c^m_r - \cdot) = \varepsilon^m_r \widetilde{\psi}^m_r.
\end{aligned} \tag{4.3.8}$$

In the following, let us present several examples to demonstrate our results and illustrate our algorithms.

Example 4.2. Let $\mathsf{d} = r = 2$ and $a_0, \widetilde{a}_0$ be a pair of dual masks obtained in Examples 1.1 and 1.6 of Chapter 1. That is, $a_0, \widetilde{a}_0$ with symbols $\mathsf{a}_0(z), \widetilde{\mathsf{a}}_0(z)$ are given by

$$\mathsf{a}_0(z) = \frac{1}{16}\begin{bmatrix} 8 & 6\,z^{-1} + 6 \\ 8\,z & -z^{-1} + 3 + 3\,z - z^2 \end{bmatrix},$$

$$\widetilde{\mathsf{a}}_0(z) = \frac{1}{384}\begin{bmatrix} -28\,z^{-1} + 216 - 28\,z & 112\,z^{-1} + 112 \\ 21\,z^{-1} - 18 + 330\,z - 18\,z^2 + 21\,z^3 & -36\,z^{-1} + 60 + 60\,z - 36\,z^2 \end{bmatrix}.$$

Both a_0 and $\widetilde{\mathsf{a}}_0$ have the same symmetry pattern and satisfy (3.4.3). Let $\mathsf{d} = d_1 d_2$ with $d_1 = 1$ and $d_2 = 2$. Then, $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}]$ and $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}]$ are as follows:

$$\mathsf{P}_{\mathsf{a}_0} = \frac{1}{16}\begin{bmatrix} 8 & 6 & 0 & 6\,z^{-1} \\ 0 & 3 - z & 8 & -z^{-1} + 3 \end{bmatrix},$$

$$\widetilde{\mathsf{P}}_{\widetilde{\mathsf{a}}_0} = \frac{1}{192}\begin{bmatrix} 216 & 112 & -28(z^{-1} + 1) & 112z^{-1} \\ -18(1 + z) & 12(5 - 3z) & 3(7z^{-1} + 110 + 7z) & 12(5 - 3z^{-1}) \end{bmatrix}.$$

Let U and $\widetilde{\mathsf{U}}$ be defined by

$$\mathsf{U} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & -z \end{bmatrix}, \widetilde{\mathsf{U}} := \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & z & 0 & -z \end{bmatrix}.$$

Then we have $\mathsf{U}\widetilde{\mathsf{U}}^* = I_4$. $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ and $\widetilde{\mathsf{P}} := \widetilde{\mathsf{P}}_{\widetilde{\mathsf{a}}_0}\mathsf{U}$ satisfy $\mathcal{S}\mathsf{P} = \mathcal{S}\widetilde{\mathsf{P}} = [1, z]^T[1, 1, z^{-1}, -1]$ and are given as follows:

$$\mathsf{P} = \frac{1}{8}\begin{bmatrix} 4 & 6 & 0 & 0 \\ 0 & 1(1+z) & 4 & 2(1-z) \end{bmatrix},$$
$$\widetilde{\mathsf{P}} = \frac{1}{192}\begin{bmatrix} 216 & 112 & -28(1+z^{-1}) & 0 \\ -18(1+z) & 12(1+z) & 3(7z^{-1}+110+7z) & 48(1-z) \end{bmatrix}.$$

Now applying Algorithm 4.1, we obtain two extension matrices P_e and $\widetilde{\mathsf{P}}_e$ as follows:

$$\mathsf{P}_e = \frac{1}{192}\begin{bmatrix} 96 & 144 & 0 & 0 \\ 0 & 24(1+z) & 96 & 48(1-z) \\ -112 & -3(z^{-1}-70+z) & -12(1+z^{-1}) & -6(z^{-1}-z) \\ 0 & -6(z-z^{-1}) & -24(1-z^{-1}) & 12(z+14+z^{-1}) \end{bmatrix},$$
$$\widetilde{\mathsf{P}}_e = \frac{1}{192}\begin{bmatrix} 216 & 112 & -28(1+z^{-1}) & 0 \\ -18(1+z) & 12(1+z) & 3(7z^{-1}+110+7z) & 48(1-z) \\ -144 & 96 & -24(1+z^{-1}) & 0 \\ 0 & 0 & -96(1-z^{-1}) & 192 \end{bmatrix}.$$

Note that $\mathcal{S}\mathsf{P}_e = \mathcal{S}\widetilde{\mathsf{P}}_e = [1, z, 1, -1]^T[1, 1, z^{-1}, -1]$. Now from the polyphase matrices $\mathbf{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant 1}$ and $\widetilde{\mathbf{P}} := \widetilde{\mathsf{P}}_e\widetilde{\mathsf{U}}^* =: (\widetilde{\mathsf{a}}_{m;\gamma})_{0\leqslant m,\gamma\leqslant 1}$, we derive two high-pass filters $\mathsf{a}_1, \widetilde{\mathsf{a}}_1$ as follows:

$$\mathsf{a}_1(z) = \frac{1}{384}\begin{bmatrix} -8(3z+28+3z^{-1}) & 3(z^2-3z+70+70z^{-1}-3z^{-2}+z^{-3}) \\ -48(z-z^{-1}) & 6(z^2-3z+28-28z^{-1}+3z^{-2}-z^{-3}) \end{bmatrix},$$

$$\widetilde{\mathsf{a}}_1(z) = \frac{1}{16}\begin{bmatrix} -(z+6+z^{-1}) & 4(1+z^{-1}) \\ -4(z-z^{-1}) & 8(1-z^{-1}) \end{bmatrix}.$$

See Figure 4.1 for the graphs of $\phi = [\phi_1, \phi_2]^T$, $\psi = [\psi_1, \psi_2]^T$, $\widetilde{\phi} = [\widetilde{\phi}_1, \widetilde{\phi}_2]^T$, and $\widetilde{\psi} = [\widetilde{\psi}_1, \widetilde{\psi}_2]^T$.

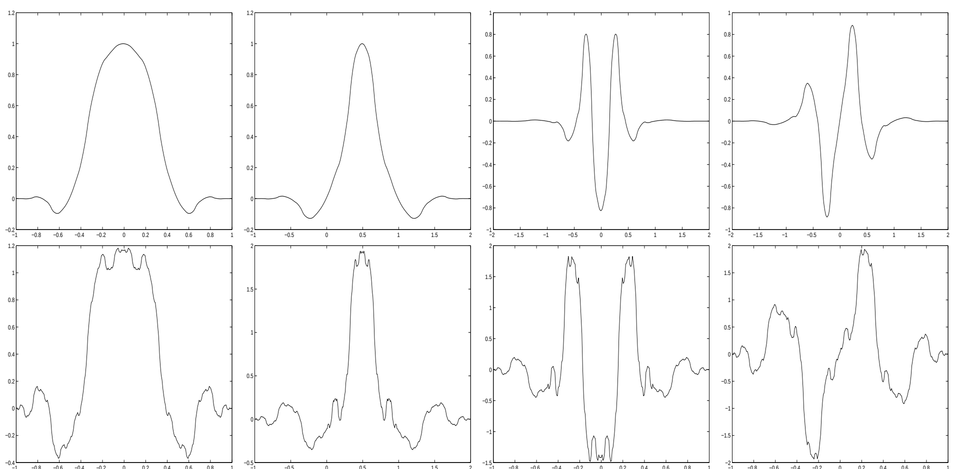

FIGURE 4.1: The graphs of $\phi = [\phi_1, \phi_2]^T, \psi = [\psi_1, \psi_2]^T$ (top, left to right), and $\widetilde{\phi} = [\widetilde{\phi}_1, \widetilde{\phi}_2]^T, \widetilde{\psi} = [\widetilde{\psi}_1, \widetilde{\psi}_2]^T$ (bottom, left to right) in Example 4.2.

Example 4.3. Let $\mathsf{d} = 3, r = 2$, and $a_0, \widetilde{a}_0$ be a pair of dual masks obtained in Examples 1.2 and 1.7 of Chapter 1 (see (1.3.1) and (1.4.8)). The low-pass filters a_0 and $\widetilde{a}_0$ do not satisfy (3.4.3). However, we can employ a very simple orthogonal transform $E := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ to $\mathsf{a}_0, \widetilde{\mathsf{a}}_0$ so that the symmetry in (3.4.3) holds. That is, for $\mathsf{b}_0(z) := E\mathsf{a}_0(z)E^{-1}$ and $\widetilde{\mathsf{b}}_0(z) := E^{-1}\widetilde{\mathsf{a}}_0(z)E$, it is easy to verify that b_0 and $\widetilde{\mathsf{b}}_0$ satisfy (3.4.3) with $c_1 = c_2 = 1/2$ and $\varepsilon_1 = 1, \varepsilon_2 = -1$. Let $\mathsf{d} = d_1 d_2$ with $d_1 = 1$ and $d_2 = 3$. Construct $\mathsf{P}_{\mathsf{b}_0} := [\mathsf{b}_{0;0}, \mathsf{b}_{0;1}, \mathsf{b}_{0;2}]$ and $\widetilde{\mathsf{P}}_{\widetilde{\mathsf{b}}_0} := [\widetilde{\mathsf{b}}_{0;0}, \widetilde{\mathsf{b}}_{0;1}, \widetilde{\mathsf{b}}_{0;2}]$ from b_0 and $\widetilde{\mathsf{b}}_0$. Let U be given by:

$$\mathsf{U} = \begin{bmatrix} z^{-1} & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & z^{-1} & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\widetilde{\mathsf{U}} := (\mathsf{U}^*)^{-1}$. Then $\mathsf{P} := \mathsf{P}_{\mathsf{b}_0}\mathsf{U}$ and $\widetilde{\mathsf{P}} := \widetilde{\mathsf{P}}_{\widetilde{\mathsf{b}}_0}\widetilde{\mathsf{U}}$ satisfy $\mathcal{S}\mathsf{P} = \mathcal{S}\widetilde{\mathsf{P}} = [z^{-1}, -z^{-1}]^T[1,-1,-1,1,1,-1]$, and are given by

$$\mathsf{P} = c \begin{bmatrix} t_{11}(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & t_{14} & t_{15}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \end{bmatrix},$$

$$\widetilde{\mathsf{P}} = \widetilde{c} \begin{bmatrix} \widetilde{t}_{11}(1+\frac{1}{z}) & \widetilde{t}_{12}(1-\frac{1}{z}) & \widetilde{t}_{13}(1-\frac{1}{z}) & \widetilde{t}_{14} & \widetilde{t}_{15}(1+\frac{1}{z}) & \widetilde{t}_{16}(1-\frac{1}{z}) \\ \widetilde{t}_{21}(1-\frac{1}{z}) & \widetilde{t}_{22}(1+\frac{1}{z}) & \widetilde{t}_{23}(1+\frac{1}{z}) & \widetilde{t}_{24}(1-\frac{1}{z}) & \widetilde{t}_{25}(1-\frac{1}{z}) & \widetilde{t}_{26}(1+\frac{1}{z}) \end{bmatrix},$$

where $c = \frac{1}{486}, \widetilde{c} = \frac{3}{34884}$ and t_{jk}'s, $\widetilde{t}_{jk}$'s are constants defined as follows:

$$\begin{array}{llllll} t_{11} = 162; & t_{12} = 34; & t_{13} = -196; & t_{14} = 0; & t_{15} = 81; & t_{16} = 29; \\ t_{21} = -126; & t_{22} = -14; & t_{13} = 176; & t_{24} = -36; & t_{15} = -99; & t_{16} = -31; \\ \widetilde{t}_{11} = 5814; & \widetilde{t}_{12} = -1615; & \widetilde{t}_{13} = -7160; & \widetilde{t}_{14} = 0; & \widetilde{t}_{15} = 5814; & \widetilde{t}_{16} = 2584; \\ \widetilde{t}_{21} = -5551; & \widetilde{t}_{22} = 5808; & \widetilde{t}_{13} = 7740; & \widetilde{t}_{24} = -1358; & \widetilde{t}_{15} = -6712; & \widetilde{t}_{16} = -4254. \end{array}$$

Applying Algorithm 4.1, we obtain P_e and $\widetilde{\mathsf{P}}_e$ as follows:

$$\mathsf{P}_e = c \left[\begin{array}{cccccc} t_{11}(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & t_{14} & t_{15}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \\ \hline t_{31}(1+\frac{1}{z}) & t_{32}(1-\frac{1}{z}) & t_{33}(1-\frac{1}{z}) & t_{34}(1+\frac{1}{z}) & t_{35}(1+\frac{1}{z}) & t_{36}(1-\frac{1}{z}) \\ t_{41} & 0 & 0 & t_{44} & t_{45} & 0 \\ \hline 0 & t_{52} & t_{53} & 0 & 0 & t_{56} \\ t_{61}(1-\frac{1}{z}) & t_{62}(1+\frac{1}{z}) & t_{63}(1+\frac{1}{z}) & t_{64}(1-\frac{1}{z}) & t_{65}(1-\frac{1}{z}) & t_{66}(1+\frac{1}{z}) \end{array}\right],$$

where all t_{jk}'s are constants given by:

$$
\begin{array}{llll}
t_{31} = 24; & t_{32} = \dfrac{472}{27}; & t_{33} = -\dfrac{148}{27}; & \\
t_{34} = -36; & t_{35} = -24; & t_{36} = -\dfrac{112}{27}; & \\
t_{41} = \dfrac{109998}{533}; & t_{44} = \dfrac{94041}{533}; & t_{45} = -\dfrac{109989}{533}; & \\
t_{52} = 406c_0; & t_{53} = 323c_0; & t_{56} = 1142c_0; & c_0 = \dfrac{1609537}{13122}; \\
t_{61} = 24210c_1; & t_{62} = 14318c_1; & t_{63} = -11807c_1; & t_{64} = -26721c_1; \\
t_{65} = -14616c_1; & t_{66} = -1934c_1; & c_1 = 200/26163. &
\end{array}
$$

And

$$
\widetilde{\mathsf{P}}_e = \widetilde{c} \left[\begin{array}{cccccc}
\widetilde{t}_{11}(1+\frac{1}{z}) & \widetilde{t}_{12}(1-\frac{1}{z}) & \widetilde{t}_{13}(1-\frac{1}{z}) & \widetilde{t}_{14} & \widetilde{t}_{15}(1+\frac{1}{z}) & \widetilde{t}_{16}(1-\frac{1}{z}) \\
\widetilde{t}_{21}(1-\frac{1}{z}) & \widetilde{t}_{22}(1+\frac{1}{z}) & \widetilde{t}_{23}(1+\frac{1}{z}) & \widetilde{t}_{24}(1-\frac{1}{z}) & \widetilde{t}_{25}(1-\frac{1}{z}) & \widetilde{t}_{26}(1+\frac{1}{z}) \\
\hline
\widetilde{t}_{31}(1+\frac{1}{z}) & \widetilde{t}_{32}(1-\frac{1}{z}) & \widetilde{t}_{33}(1-\frac{1}{z}) & \widetilde{t}_{34}(1+\frac{1}{z}) & \widetilde{t}_{35}(1+\frac{1}{z}) & \widetilde{t}_{36}(1-\frac{1}{z}) \\
\widetilde{t}_{41} & 0 & 0 & \widetilde{t}_{44} & \widetilde{t}_{45} & 0 \\
\hline
0 & \widetilde{t}_{52} & \widetilde{t}_{53} & 0 & 0 & \widetilde{t}_{56} \\
\widetilde{t}_{61}(1-\frac{1}{z}) & \widetilde{t}_{62}(1+\frac{1}{z}) & \widetilde{t}_{63}(1+\frac{1}{z}) & \widetilde{t}_{64}(1-\frac{1}{z}) & \widetilde{t}_{65}(1-\frac{1}{z}) & \widetilde{t}_{66}(1+\frac{1}{z})
\end{array} \right],
$$

where all $\widetilde{t}_{jk}$'s are constants given by:

$$
\begin{array}{llll}
\widetilde{t}_{31} = 3483\widetilde{c}_0; & \widetilde{t}_{32} = 37427\widetilde{c}_0; & \widetilde{t}_{33} = 4342\widetilde{c}_0; & \widetilde{t}_{34} = -12222\widetilde{c}_0; \\
\widetilde{t}_{35} = -3483\widetilde{c}_0; & \widetilde{t}_{36} = -7267; & \widetilde{c}_0 = \dfrac{8721}{4264}; & \\
\widetilde{t}_{41} = 5814; & \widetilde{t}_{44} = 11628; & \widetilde{t}_{45} = -11628; & \\
\widetilde{t}_{52} = 3\widetilde{c}_1; & \widetilde{t}_{53} = 2\widetilde{c}_1; & \widetilde{t}_{56} = 10\widetilde{c}_1; & \widetilde{c}_1 = \dfrac{12680011}{243}; \\
\widetilde{t}_{61} = 18203\widetilde{c}_2; & \widetilde{t}_{62} = 101595\widetilde{c}_2; & \widetilde{t}_{63} = 1638\widetilde{c}_2; & \widetilde{t}_{64} = -33950\widetilde{c}_2; \\
\widetilde{t}_{65} = -10822\widetilde{c}_2; & \widetilde{t}_{66} = -36582\widetilde{c}_2; & \widetilde{c}_2 = \dfrac{26163}{213200}. &
\end{array}
$$

Note that P_e and $\widetilde{\mathsf{P}}_e$ satisfy

$$
\mathcal{S}\mathsf{P}_e = \mathcal{S}\mathsf{P}_e = [z^{-1}, -z^{-1}, z^{-1}, 1, -1, -z^{-1}]^T [1, -1, -1, 1, 1, -1].
$$

From the polyphase matrices $\mathbf{P} := \mathsf{P}_e\mathsf{U}^*$ and $\widetilde{\mathbf{P}} := \widetilde{\mathsf{P}}_e\mathsf{U}^*$, we derive high-pass filters $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$ as follows:

$$\mathsf{b}_1(z) = \begin{bmatrix} b_{11}^1(z) & b_{12}^1(z) \\ b_{21}^1(z) & b_{22}^1(z) \end{bmatrix}, \mathsf{b}_2(z) = \begin{bmatrix} b_{11}^2(z) & b_{12}^2(z) \\ b_{21}^2(z) & b_{22}^2(z) \end{bmatrix},$$

where

$$\begin{aligned}
b_{11}^1(z) &= \frac{199}{6561} + \frac{125}{6561}z^3 - \frac{4}{81}z^2 + \frac{199}{6561}z - \frac{4}{81}z^{-1} + \frac{125}{6561}z^{-2};\\
b_{12}^1(z) &= -\frac{361}{6561} - \frac{125}{6561}z^3 - \frac{56}{6561}z^2 + \frac{361}{6561}z + \frac{56}{6561}z^{-1} + \frac{125}{6561}z^{-2};\\
b_{21}^1(z) &= \frac{679}{3198}z^3 + \frac{679}{3198}z - \frac{679}{1599}z^2;\\
b_{22}^1(z) &= \frac{387}{2132}z^3 - \frac{387}{2132}z;\\
b_{11}^2(z) &= c_3(323z^3 - 323z);\\
b_{12}^2(z) &= c_3(406z^3 + 2284z^2 + 406z);\\
b_{21}^2(z) &= c_4(-36017 + 12403\,z^3 - 29232\,z^2 + 36017\,z + 29232\,z^{-1} - 12403\,z^{-2});\\
b_{22}^2(z) &= c_4(41039 - 12403\,z^3 - 3868\,z^2 + 41039\,z - 3868\,z^{-1} - 12403\,z^{-2});\\
c_3 &= \frac{27}{3219074};\ c_4 = \frac{50}{6357609}.
\end{aligned}$$

And

$$\widetilde{\mathsf{b}}_1(z) = \begin{bmatrix} \widetilde{b}_{11}^1(z) & \widetilde{b}_{12}^1(z) \\ \widetilde{b}_{21}^1(z) & \widetilde{b}_{22}^1(z) \end{bmatrix}, \widetilde{\mathsf{b}}_2(z) = \begin{bmatrix} \widetilde{b}_{11}^2(z) & \widetilde{b}_{12}^2(z) \\ \widetilde{b}_{21}^2(z) & \widetilde{b}_{22}^2(z) \end{bmatrix},$$

where

$$\begin{aligned}
\widetilde{b}_{11}^1(z) &= -\frac{859}{17056} + \frac{7825}{17056}\,z^3 - \frac{3483}{8528}\,z^2 - \frac{859}{17056}\,z - \frac{3483}{8528}\,z^{-1} + \frac{7825}{17056}\,z^{-2};\\
\widetilde{b}_{12}^1(z) &= -\frac{49649}{17056} + \frac{25205}{17056}\,z^3 - \frac{559}{656}\,z^2 + \frac{49649}{17056}\,z + \frac{559}{656}\,z^{-1} - \frac{25205}{17056}\,z^{-2};\\
\widetilde{b}_{21}^1(z) &= \frac{1}{6}(z^3 + z - 2z^2);\\
\widetilde{b}_{22}^1(z) &= \frac{1}{3}(z^3 - z);
\end{aligned}$$

$\widetilde{b}^2_{11}(z) = 2\widetilde{c}_3(z^3 - z);$

$\widetilde{b}^2_{12}(z) = \widetilde{c}_3(3z^3 + 10z^2 + 3z);\ \widetilde{c}_3 = \dfrac{39257}{26244};$

$\widetilde{b}^2_{21}(z) = -\dfrac{9939}{170560} + \dfrac{59523}{852800}z^3 - \dfrac{16233}{426400}z^2 + \dfrac{9939}{170560}z + \dfrac{16233}{426400}z^{-1} - \dfrac{59523}{852800}z^{-2};$

$\widetilde{b}^2_{22}(z) = \dfrac{81327}{170560} + \dfrac{40587}{170560}z^3 - \dfrac{4221}{32800}z^2 + \dfrac{81327}{170560}z - \dfrac{4221}{32800}z^{-1} + \dfrac{40587}{170560}z^{-2}.$

Then the high-pass filters $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$ satisfy (4.3.6) with $c^1_1 = c^1_2 = 1/2$, $\varepsilon^1_1 = 1, \varepsilon^1_2 = 1$ and $c^2_1 = c^2_2 = 3/2$, $\varepsilon^1_1 = -1, \varepsilon^1_2 = -1$, respectively.

Let $\mathsf{a}_1, \mathsf{a}_2$ and $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ be high-pass filters constructed from $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$ by $\mathsf{a}_1(z) := E^{-1}\mathsf{b}_1(z)E, \mathsf{a}_2 := E^{-1}\mathsf{b}_2E$ and $\widetilde{\mathsf{a}}_1(z) := E\widetilde{\mathsf{b}}_1(z)E^{-1}, \widetilde{\mathsf{a}}_2 := E\widetilde{\mathsf{b}}_2E^{-1}$.

See Figure 4.4 for the graphs of the 3-refinable function vectors $\phi, \widetilde{\phi}$ associated with the low-pass filters $\mathsf{a}_0, \widetilde{\mathsf{a}}_0$, respectively, and the biorthogonal multiwavelet function vectors ψ^1, ψ^2 and $\widetilde{\psi}^1, \widetilde{\psi}^2$ associated with the high-pass filters $\mathsf{a}_1, \mathsf{a}_2$ and $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$, respectively. Also, see Figure 4.5 for the graphs of the 3-refinable function vectors $\eta, \widetilde{\eta}$ associated with the low-pass filters $\mathsf{b}_0, \widetilde{\mathsf{b}}_0$, respectively, and the biorthogonal multiwavelet function vectors ζ^1, ζ^2 and $\widetilde{\zeta}^1, \widetilde{\zeta}^2$ associated with the high-pass filters $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$, respectively.

4.4 Conclusions and Remarks

In this chapter, we study the general matrix extension problem with symmetry for the biothogonal case. We obtain a result on representing a pair of $r \times s$ biorthogonal matrices $(\mathsf{P}, \widetilde{\mathsf{P}})$ having the same compatible symmetry and provide a step-by-step algorithm for deriving a pair of $s \times s$ biorthogonal matrices from a given pair of biorthogonal matrices $(\mathsf{P}, \widetilde{\mathsf{P}})$. Our results

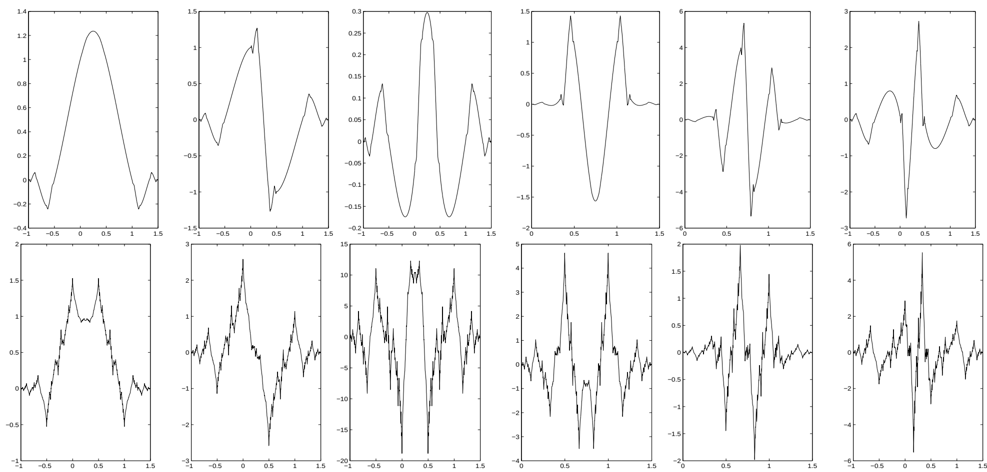

FIGURE 4.2: The graphs of $\eta = [\eta_1, \eta_2]^T$, $\zeta^1 = [\zeta_1^1, \zeta_2^1]^T$, and $\zeta^2 = [\zeta_1^2, \zeta_2^2]^T$ (top, left to right), and $\widetilde{\eta} = [\widetilde{\eta}_1, \widetilde{\eta}_2]^T$, $\widetilde{\zeta}^1 = [\widetilde{\zeta}_1^1, \widetilde{\zeta}_2^1]^T$, and $\widetilde{\zeta}^2 = [\widetilde{\zeta}_1^2, \widetilde{\zeta}_2^2]^T$ (bottom, left to right).

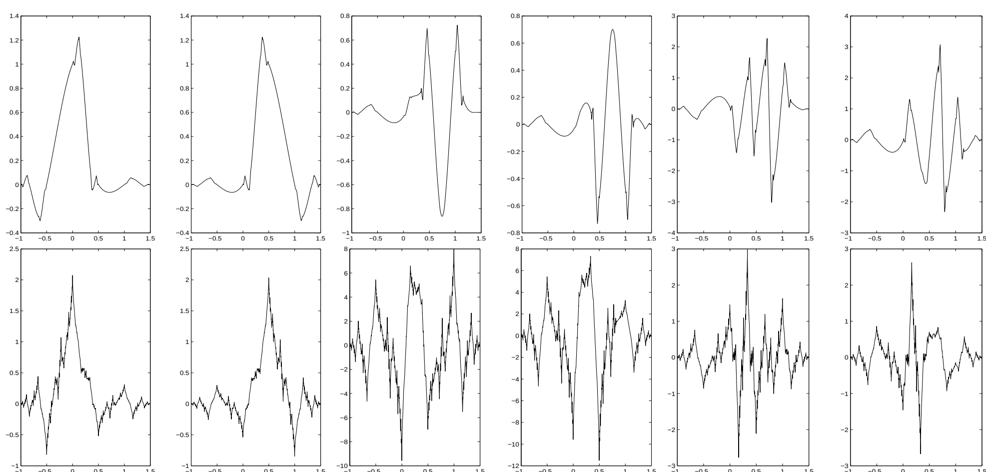

FIGURE 4.3: The graphs of $\phi = [\phi_1, \phi_2]^T$, $\psi^1 = [\psi_1^1, \psi_2^1]^T$, and $\psi^2 = [\psi_1^2, \psi_2^2]^T$ (top, left to right), and $\widetilde{\phi} = [\widetilde{\phi}_1, \widetilde{\phi}_2]^T$, $\widetilde{\psi}^1 = [\widetilde{\psi}_1^1, \widetilde{\psi}_2^1]^T$, and $\widetilde{\psi}^2 = [\widetilde{\psi}_1^2, \widetilde{\psi}_2^2]^T$ (bottom, left to right).

show that for the one row case ($r = 1$), the support lengths of the extension matrices can be controlled by the given pair of columns. We apply our results in this chapter to the derivation of biorthogonal multiwavelets from a pair of dual d-refinable functions constructed in Section 1.4.

Chapter 5

Future Research

In Chapters 1 and 2, we investigated refinable function vectors with many desirable properties, such as interpolation, symmetry, orthogonality, and so on. In Chapters 3 and 4, we mainly studied the matrix extension problem with symmetry, which plays a fundamental role in the construction of orthogonal and biorthogonal multiwavelets with symmetry. Except in Chapter 2 for some symmetric interpolating refinable function vectors in high dimensions, in previous chapters, we mainly focus on the construction of one-dimensional refinable function vectors and their corresponding multiwavelets. We did not address any results related to the construction of high dimensional wavelets. In this chapter, we shall discuss several possible future research topics related to matrix extension and high dimensional wavelets.

5.1 Matrix Extension in Dimension One

For the matrix extension technique, it can be applied not only to the construction of orthogonal or biorthogonal multiwavelets, but also to the construction of tight wavelet frames (multiframes). For example, for a d-refinable function ϕ in $L_2(\mathbb{R})$, if its associated mask (low-pass filter) a_0 satisfies

$$0 \leqslant \sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\mathsf{a}_{0;\gamma}^*(z) \leqslant 1, \tag{5.1.1}$$

then by Riesz lemma, one can derive an extra element $\mathsf{a}_{0;\mathsf{d}}(z)$ such that the polyphase vector $\mathsf{p}(z) := [\mathsf{a}_{0;0}(z), \ldots, \mathsf{a}_{0,\mathsf{d}-1}(z), \mathsf{a}_{0;\mathsf{d}}(z)]$ satisfies the matrix extension condition, i.e., $\mathsf{p}(z)\mathsf{p}^*(z) = 1$. Consequently, using our matrix extension algorithm, one can construct tight d-wavelet frames having symmetry with only d wavelet generators for any integer $\mathsf{d} \geqslant 2$ provided p satisfying certain symmetry condition. This is the case for scalar refinable functions.

For d-refinable function vectors ϕ associated with a matrix mask a_0 with multiplicity r, condition (5.1.1) becomes

$$I_r - \sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\mathsf{a}_{0;\gamma}^*(z) \geqslant 0, \tag{5.1.2}$$

that is, the matrix of Laurent polynomials $I_r - \sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\mathsf{a}_{0;\gamma}^*(z)$ is *positive semidefinite*. In order to obtain an extra element $\mathsf{a}_{0;\mathsf{d}}(z)$ so that $\mathsf{P}(z) := [\mathsf{a}_{0;0}(z), \ldots, \mathsf{a}_{0;\mathsf{d}-1}(z), \mathsf{a}_{0;\mathsf{d}}(z)]$ satisfies the matrix extension condition $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$, one needs to employ the *matrix-valued Riesz lemma* ([39]). In this case, there are two main issues need to be clearified. On the one hand, what types of d-refinable function vectors can satisfy (5.1.2)? In the scalar case, condition (5.1.1) can be easily satisfied if a mask $\widehat{a}(\xi)$ is obtained from $\widehat{b}(\xi) := \cos^{2n}(\xi/2)\sum_{j=0}^{\ell}\binom{n-1+j}{j}\sin^{2j}(\xi/2)$ via the Riesz lemma ($\ell \leqslant n-1$, see Section 1.1 of Chapter 1). Analogously, can one construct families of d-refinable function vectors satisfying (5.1.2)? On the other hand, in order to obtain tight wavelet multiframes with symmetry, the extra element $\mathsf{a}_{0;\mathsf{d}}(z)$ obtained via the matrix-valued Riesz lemma needs to satisfy certain symmetry condition. Though there are factorization algorithms ([39]) for the matrix-valued Riesz lemma, to our best knowledge, there is no factorization algorithm with symmetry for the matrix-valued Riesz lemma. Thus, to apply our matrix extension with symmetry to the construction of tight wavelet multiframes with symmetry, we need to develop factorization algorithms with symmetry for matrix-valued Riesz lemma.

In Chapter 4, we proposed an algorithm for deriving a pair of biorthgonal $s \times s$ matrices $(\mathsf{P}_e, \widetilde{\mathsf{P}}_e)$ from a pair of $r \times s$ matrices $(\mathsf{P}, \widetilde{\mathsf{P}})$ under a coefficient field $\mathbb{F}$. Though the condition, that the coefficient field $\mathbb{F}$ is only required to be a subfield of $\mathbb{C}$, is less restricted than that of $\mathbb{F}$ we considered in Chapter 3 (see (3.1.6)), in electronic engineering or computer science, filters with dyadic coefficients $\frac{m}{2^n}, m, n \in \mathbb{Z}$, are more preferred than that with rational numbers. Note that $\mathcal{R}_{\mathsf{d}} := \{\frac{m}{\mathsf{d}^n} : m, n \in \mathbb{Z}\}$ is a ring. A natural question is: "Can we construct pairs of d-band biorthogonal filters with coefficients in a subring $\mathcal{R}$ of $\mathbb{C}$, or more specifically, in $\mathcal{R}_{\mathsf{d}}$?" If the answer is *yes*, we need to develop extension algorithms for these types of filters so that the high-pass filters from such pairs forming filter banks with perfect reconstruction property also have coefficients in $\mathcal{R}$ or $\mathcal{R}_{\mathsf{d}}$.

Also, for the scalar case, our result (Theorem 4.1) says that the lengths of the coefficient supports of the extension matrices we obtained from a pair of vectors $(\mathsf{p}, \widetilde{\mathsf{p}})$ are controlled by the summation of both the lengths of p and $\widetilde{\mathsf{p}}$. This is not as nice as that in Theorem 3.1. For the vector case, our extension algorithm proceeds recursively row-by-row on a given pairs of $r \times s$ biorthogonal matrices $(\mathsf{P}, \widetilde{\mathsf{P}})$, which might result in even longer coefficient support of the pair $(\mathsf{P}_e, \widetilde{\mathsf{P}}_e)$ of extension matrices. To have better support control of the extension matrices, we need to develop algorithms that take into account the coefficients of the pair $(\mathsf{P}, \widetilde{\mathsf{P}})$ as coefficients of matrix types.

This line of research is under development and we point out that one can also apply our biorthogonal matrix extension technique to the construction of wavelet bi-frames.

5.2 Matrix Extension in High Dimensions

In high dimensions, a simple and common way to construct wavelets is via *tensor product* of one-dimensional wavelets. In this way, many properties, say symmetry, can be easily carried onto high dimensions. However, tensor product wavelets favor only two main directions: horizontal and vertical. To construct wavelets that favor directions other than horizontal and vertical directions, one may consider *non-tensor* product wavelets, which inevitably lead us to the study of the matrix extension problem in high dimensions. Also, tensor product wavelets can yield only a few symmetry patterns. Employing matrix extension with symmetry in high dimensions shall enrich the symmetry patterns of wavelets in high dimensions and might produce better results in applications.

Let M be a dilation matrix, $\mathsf{m} := |\det \mathsf{M}|$, and $\Omega_{\mathsf{M}} := \{\omega_0, \ldots, \omega_{\mathsf{m}-1}\}$ be an ordered of complete representatives of the cosets $\mathbb{Z}^d/\mathsf{M}\mathbb{Z}^d$. Suppose ϕ is an orthogonal M-refinable function vector in $L_2(\mathbb{R}^d)$ associated with a matrix mask a_0 with multiplicity r. Define $\mathsf{a}_{0;\omega}(z) := \sqrt{\mathsf{m}}\sum_{k\in\mathbb{Z}^d} a(\mathsf{M}k+\omega)z^k$, $\omega \in \Omega_{\mathsf{M}}$ to be the polyphases of a_0, where $z = (z_1, \ldots, z_d)$, $k = (k_1, \ldots, k_d)$, and $z^k = z_1^{k_1}\cdots z_d^{k_d}$. Then the orthogonality of ϕ implies that

$$\sum_{\omega\in\Omega_{\mathsf{M}}} \mathsf{a}_{0;\omega}(z)\mathsf{a}_{0;\omega}^*(z) = I_r. \tag{5.2.1}$$

Here for a matrix of Laurent polynomials $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}^d} P_k z^k$, $\mathsf{P}^*(z) := \sum_{k\in\mathbb{Z}^d} \overline{P_k}^T z^{-k}$. To construct high dimensional orthonormal multiwavelets from ϕ, we need to derive masks $a_1, \ldots, a_{\mathsf{m}-1}$ such that the polyphase matrix

$$\mathbf{P}(z) := \begin{bmatrix} \mathsf{a}_{0;\omega_0}(z) & \cdots & \mathsf{a}_{0;\omega_{\mathsf{m}-1}}(z) \\ \vdots & \ddots & \vdots \\ \mathsf{a}_{\mathsf{m}-1;\omega_0}(z) & \cdots & \mathsf{a}_{\mathsf{m}-1;\omega_{\mathsf{m}-1}}(z) \end{bmatrix} \tag{5.2.2}$$

is paraunity, i.e., $\mathbf{P}(z)\mathbf{P}^*(z) = I_{\mathsf{m}r}$.

Consequently, the matrix extension problem in high dimensions can be formulated as follows: Let $\mathsf{P}(z)$ be an $r \times s$ $(1 \leqslant r \leqslant s)$ matrix of Laurent polynomials in $\mathbb{R}^d$ such that $\mathsf{P}\mathsf{P}^* = I_r$. Find an $s \times s$ matrix P_e of Laurent polynomials in $\mathbb{R}^d$ such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$ and $\mathsf{P}\mathsf{P}^* = I_s$. Analogously, we can formulate the matrix extension problem for biorthogonal multiwavelets in high dimensions as: Given a pair of $r \times s$ matrices $(\mathsf{P}, \widetilde{\mathsf{P}})$ of Laurent polynomials in $\mathbb{R}^d$ such that $\mathsf{P}\widetilde{\mathsf{P}}^* = I_r$. Find a pair of matrices $(\mathsf{P}_e, \widetilde{\mathsf{P}}_e)$ such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$, $[I_r, \mathbf{0}]\widetilde{\mathsf{P}}_e = \widetilde{\mathsf{P}}$ and $\mathsf{P}\widetilde{\mathsf{P}}^* = I_s$.

In high dimensions, two main issues for the matrix extension are symmetry and support control. Without considering any symmetry issue, for the biorthogonal case, the matrix extension problem is guaranteed by Quillen-Suslin theorem ([69]) and there are constructive algorithms to derive the extension matrices. For high dimensional orthogonal multiwavelets, to our best knowledge, there is still no constructive algorithm to derive the corresponding extension matrix. When integrating symmetry to the matrix extension problem, it becomes even more complicated. Firstly, the Quillen-Suslin theorem does not guarantee symmetry of the extension system. Secondly, as we mentioned and studied in Chapter 2, symmetry in high dimensions is related to some symmetry groups, which is highly nontrivial compared to symmetry in dimension one. Lastly but not least, we do not know whether the support lengths of the extension matrices can be controlled by the given matrices. These issues surely complicate the design of the extension algorithm.

Nevertheless, to study the high dimensional matrix extension problem, we may consider some specific cases first in order to gain a rough idea of matrix extension in the high dimensional situation. For example, we may study and develop possible algorithms for the matrix extension problem without symmetry and then try to integrate symmetry to the algorithms. For simplicity, we can consider the case for *bivariate* orthogonal multiwavelets from a bivariate M-refinable function vectors in $L_2(\mathbb{R}^2)$ which is symmetric with respect to a symmetry group G. To further simplify our analysis, we can consider $\mathsf{M} = 2I_2$ and choose the symmetry group G to be D_4 introduced in Section 2.2 (see (2.3.7)). This line of research is under development.

5.3 Tight Wavelet Frames with Directionality in High Dimensions

Deriving high dimensional wavelets via matrix extension has its own rights in many aspects. Though we can have directions other than horizonal and vertical, we must point out that orthogonal (multi)wavelets derived via matrix extension can only possess directions no more than $|\det \mathsf{M}| - 1$. Yet in applications, for instance, image processing, wavelets (not necessarily orthogonal) that can achieve as many directions as possible are desired, especially in image denoising and edge detection. Moreover, redundant systems, say tight wavelet frames, are preferred and generally produce better results in image denoising/inpainting than that of non-redundant systems. For orthogonal wavelet bases, they are of course highly non-redundant systems compared with tight wavelet frames.

Also, from the point of view of approximation, orthogonal wavelets are not good at representing smooth functions with discontinuity along some piecewise smooth edges in dimension two. It is known ([13]) that the optimal approximation rate of the best n-term wavelet approximation for any orthogonal system is n^{-2}, i.e.

$$\|f - f_n\|^2_{L_2(\mathbb{R}^2)} \asymp n^{-2}, \quad n \to \infty, \tag{5.3.1}$$

where f_n is the approximation obtained by using the largest n coefficients in the orthogonal expansion. When considering a function in $\mathbb{R}^2$ that is C^2 away from a discontinuity along a curve of finite length, the best n-term approximation can only achieve approximation rate n^{-1}, i.e., $\|f - f_n\|^2_{L_2(\mathbb{R}^2)} \asymp n^{-1}$ ([13, 56]), which is way far from the optimal approximation rate n^{-2}. One of the reasons is again the lack of directionality of orthogonal wavelets, which makes it impossible for representing the edge curve with only few coefficients.

To overcome the above shortcomings of orthogonal wavelet systems, Candès and Donoho in [1] introduced the so-called *Curvelets* and showed that these systems can achieve nearly optimal approximation rate on representing objects with C^2 singularity, i.e., $\|f - f_n\|^2_{L_2(\mathbb{R}^2)} \asymp n^{-2}(\log n)^3$. Curvelets are tight frames, which of course have redundancy and can achieve better results in image denoising. However, we must point out that curvelets are not *wavelet* frames and involve with parabolic dilation matrix and rotation opertators, which makes the design of the algorithms for curvelet transform in applications very complicated, while for wavelet frames, the algorithms for frame transforms are fast and can be easily designed.

The above discussion raises a question:"Can we achieve directionality and optimal representation under the framework of *tight wavelet fames*?".

In the following, we shall show that *directionality* can be easily achieved under the framework of tight wavelet frames.

Let us first construct two univariate functions $\boldsymbol{\eta}, \boldsymbol{\zeta}$ in frequency domain such that $\boldsymbol{\eta}(\xi)^2 + \boldsymbol{\zeta}(\xi)^2 = \boldsymbol{\eta}(\xi/2)^2$. Define $f(x) := e^{-\frac{1}{x^2}}, x > 0$ and $f(x) := 0, x \leqslant 0$. Let $g(x) := \int_{-1}^{x} f(1+t)f(1-t)dx$. Define

$$\gamma(x) := \frac{g(x)}{\sqrt{g(-x)^2 + g(x)^2}}. \tag{5.3.2}$$

Then one can show that $\gamma \in C^\infty(\mathbb{R})$ and satisfies

$$\gamma(x) = 0, x < -1;\ \gamma(-x)^2 + \gamma^2(x) = 1;\ \gamma(x) = 1, x \geqslant 1. \tag{5.3.3}$$

Let $c, \varepsilon > 0$ be such that $c - \varepsilon > 0$. Define

$$\beta_{c,\varepsilon}(\xi) := \begin{cases} \gamma(\frac{\xi+c}{\varepsilon}), & -c-\varepsilon \leqslant \xi < -c+\varepsilon; \\ 1, & -c+\varepsilon \leqslant \xi \leqslant c-\varepsilon; \\ \gamma(\frac{-\xi+c}{\varepsilon}), & c-\varepsilon < \xi \leqslant c+\varepsilon; \\ 0, & \text{otherwise.} \end{cases} \tag{5.3.4}$$

Let $c_{\boldsymbol{\eta}}, \varepsilon_{\boldsymbol{\eta}}$ be such that $c_{\boldsymbol{\eta}} + \varepsilon_{\boldsymbol{\eta}} \leqslant \pi/2$. Define $\boldsymbol{\eta}(\xi) := \beta_{c_{\boldsymbol{\eta}},\varepsilon_{\boldsymbol{\eta}}}(\xi)$ and $\boldsymbol{\zeta}(\xi) := (\boldsymbol{\eta}(\xi/2)^2 - \boldsymbol{\eta}(\xi)^2)^{1/2}$. In this way, both $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are supported inside $[-\pi, \pi]$, $\boldsymbol{\eta}(\xi) \equiv 1$ in a neighborhood of the origin, and $\boldsymbol{\eta}(\xi)^2 + \boldsymbol{\zeta}(\xi)^2 = \boldsymbol{\eta}(\xi/2)^2$.

Using tensor product in the polar coordinates and splitting technique in [19], one can construct tight wavelet frames with directionality in dimension two using $\boldsymbol{\eta}, \boldsymbol{\zeta}$. Let us recall such an example from [29].

For a function $f : \mathbb{R}^d \to \mathbb{C}$ and a $d \times d$ real-valued invertible matrix U, in what follows, we shall adopt the notation:

$$\begin{aligned} &f_{U;\mathsf{k},\mathsf{n}}(x) := |\det U|^{1/2} e^{-i\mathsf{n}\cdot Ux} f(Ux - \mathsf{k}) \\ &\text{and} \quad f_{U;\mathsf{k}} := f_{U;\mathsf{k},0}, \quad x, \mathsf{k}, \mathsf{n} \in \mathbb{R}^d. \end{aligned} \tag{5.3.5}$$

Example 5.1. Consider $\mathsf{M} = 2I_2$ and $\mathsf{N} = (\mathsf{M}^T)^{-1}$. Let m be a positive integer and ρ be a parameter such that $0 \leqslant \rho < 1$. We are going to construct a nonstationary tight M-wavelet frame generated by $\boldsymbol{\Phi} = \{\boldsymbol{\varphi}\}$ and $\boldsymbol{\Psi} = \{\boldsymbol{\Psi}_j\}_{j=0}^{\infty}$, where $\boldsymbol{\Psi}_j = \{\boldsymbol{\psi}^{j,1}, \ldots, \boldsymbol{\psi}^{j,s_j}\}$ ad $s_j = m2^{\lfloor \rho j \rfloor}$, $j = 0, \ldots, \infty$, i.e., for all $f \in L_2(\mathbb{R}^2)$,

$$(2\pi)^d \|f\|^2_{L_2(\mathbb{R}^d)} = \sum_{\ell=1}^{r} \sum_{\mathsf{k}\in\mathbb{Z}^d} |\langle f, \boldsymbol{\varphi}^{\ell}_{\mathsf{N}^J;0,\mathsf{k}} \rangle|^2 + \sum_{j=J}^{\infty} \sum_{\ell=1}^{s_j} \sum_{\mathsf{k}\in\mathbb{Z}^d} |\langle f, \boldsymbol{\psi}^{j,\ell}_{\mathsf{N}^j;0,\mathsf{k}} \rangle|^2.$$

Let ε be such that $0 < \varepsilon \leqslant 2\pi/m$ and define a 2π-periodic function $\alpha_{m,\varepsilon} \in C^\infty(\mathbb{T})$ such that $\alpha_{m,\varepsilon}(\xi) := \beta_{\frac{\pi}{2m},\varepsilon}(\xi)$, $\xi \in [-\pi,\pi]$,where $\beta_{c,\varepsilon}$ is given in (5.3.4). Then one can verify that

$$\sum_{\ell=0}^{2m-1} |\alpha_{m,\varepsilon}(\xi + \frac{\pi\ell}{m})|^2 = 1, \quad \xi \in \mathbb{R}. \tag{5.3.6}$$

In the frequency domain, let $\mathsf{r}e^{i\theta}$ denote the point $(\mathsf{r}\cos\theta, \mathsf{r}\sin\theta)$. Define

$$\boldsymbol{\varphi}(\mathsf{r}e^{i\theta}) = \boldsymbol{\eta}(\mathsf{r}), \quad \boldsymbol{\psi}(\mathsf{r}e^{i\theta}) = \boldsymbol{\zeta}(\mathsf{r}), \quad \mathsf{r} \geqslant 0,\, \theta \in [-\pi,\pi].$$

Now, we can define $\boldsymbol{\psi}^{j,\ell}$ by splitting $\boldsymbol{\psi}$ using $\alpha_{m,\varepsilon}$ and its property (5.3.6) as follows:

$$\boldsymbol{\psi}^{j,\ell}(\mathsf{r}e^{i\theta}) := \boldsymbol{\psi}(\mathsf{r}e^{i\theta}) \left(\alpha_{m,\varepsilon}(2^{\lfloor \rho j \rfloor}\theta + \frac{(\ell-1)\pi}{m}) + \alpha_{m,\varepsilon}(-2^{\lfloor \rho j \rfloor}\theta - \frac{(\ell-1)\pi}{m}) \right). \tag{5.3.7}$$

Then due to (5.3.6), we have $\sum_{\ell=1}^{s_j} |\boldsymbol{\psi}^{j,\ell}(\mathsf{r}e^{i\theta})|^2 = |\boldsymbol{\psi}(\mathsf{r}e^{i\theta})|^2 = \boldsymbol{\zeta}(\mathsf{r})^2$. Consequently, $|\boldsymbol{\varphi}(\xi)|^2 + \sum_{\ell=1}^{s_j} |\boldsymbol{\psi}^{j,\ell}(\xi)|^2 = |\boldsymbol{\varphi}(\xi/2)|^2$ for all $\xi \in \mathbb{R}^2$. By [29, Corollary 17], we conclude that $\boldsymbol{\Phi}, \boldsymbol{\Psi}_j, j = 0, \ldots, \infty$ generate a tight M-wavelet frame.

According to the construction, $\boldsymbol{\varphi}, \boldsymbol{\psi}^{j,\ell}$ have the following properties:

(1) All functions in $\boldsymbol{\Phi}, \boldsymbol{\Psi}$ are compactly supported $C^\infty(\mathbb{R}^2)$ functions. $\boldsymbol{\varphi} \equiv 1$ in a neighborhood of the origin and all $\boldsymbol{\psi}^{j,\ell}$ vanish in a neighborhood of the origin;

(2) Refinability: there exist $2\pi\mathbb{Z}^2$-periodic functions $\mathbf{a}$ (low-pass filter) and $\mathbf{b}_{j,\ell}$ (high-pass filters) such that $\boldsymbol{\varphi}(2\xi) = \mathbf{a}(\xi)\boldsymbol{\varphi}(\xi)$ and $\boldsymbol{\psi}^{j,\ell}(2\xi) = \mathbf{b}_{j,\ell}(\xi)\boldsymbol{\varphi}(\xi)$ (see [29]);

(3) Fixed $j \geqslant 0$, $\boldsymbol{\psi}^{j,\ell}$ are obtained via rotating $\boldsymbol{\psi}^{j,1}$: $\boldsymbol{\psi}^{j,\ell}(\xi) = \boldsymbol{\psi}^{j,1}(R_{\theta_\ell}\xi)$ with R_{θ_ℓ} being the standard rotation operator about angle $\theta_\ell = \frac{(\ell-1)\pi}{m2^{\lfloor \rho j \rfloor}}$;

(4) $\boldsymbol{\psi}^{j,1}_{2^{-j}I_2;0,\mathsf{k}}$ is symmetric about the origin and its support has two parts with each part obeys width≈length$^{1-\rho}$. More precisely, for all $\mathsf{k} \in \mathbb{Z}^2$ and $j \geqslant 0$,

$$\operatorname{supp}\boldsymbol{\psi}^{j,1}_{2^{-j}I_2;0,\mathsf{k}} = \{\mathsf{r}e^{i\theta}, \mathsf{r}e^{-i\theta} : 2^j\mathsf{r}_1 \leqslant \mathsf{r} \leqslant 2^j\mathsf{r}_2, -2^{-\lfloor \rho j \rfloor}\theta_0 \leqslant \theta \leqslant 2^{-\lfloor \rho j \rfloor}\theta_0\}.$$

We next illustrate some graphs to show the directionality of the tight wavelet frame generated by $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}^{j,\ell}$'s. See Figure 5.1 for the graphs of $\boldsymbol{\varphi}, \boldsymbol{\psi}$ and their graphs in time domain. See Figure 5.2 for the splitting effect of $\alpha_{m,\varepsilon}$ on $\boldsymbol{\psi}$ and Figure 5.3 for the rotation effect.

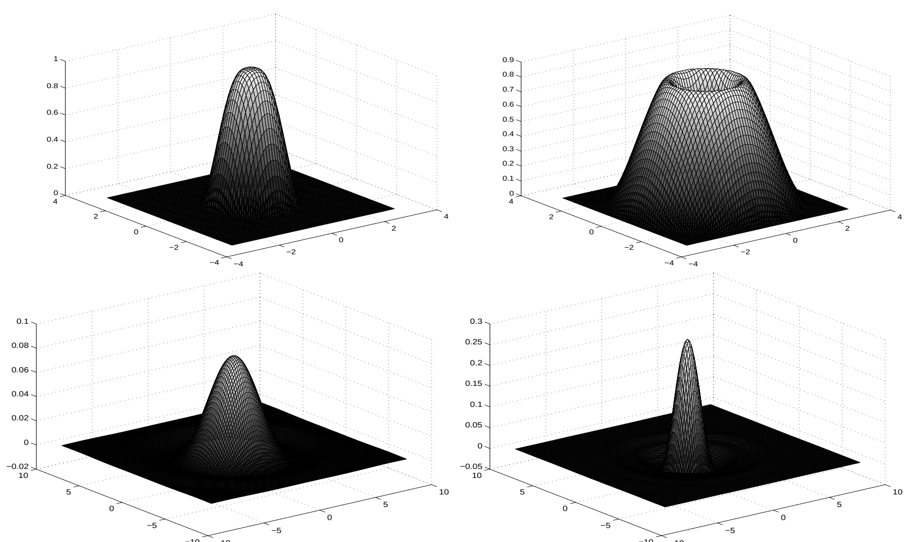

FIGURE 5.1: The graphs of $\boldsymbol{\varphi}, \boldsymbol{\psi}$ (top 2) in frequency domain and their corresponding graphs (bottom 2) in time domain.

For the above example, in the frequency domain, the wavelet generators $\boldsymbol{\psi}^{j,\ell}$'s are compactly supported and C^∞. In the time domain, they are essentially supported in rectangles obeying width≈length$^{1-\rho}$ (see Figures 5.2

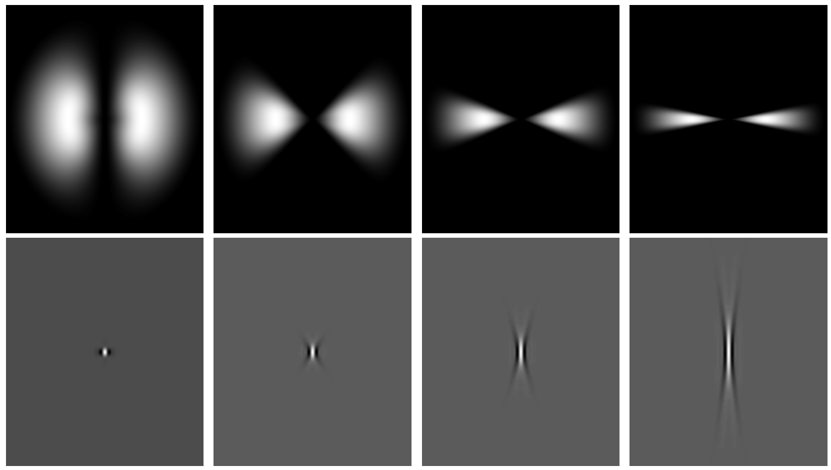

FIGURE 5.2: The splitting effect of $\alpha_{m,\varepsilon}$ on $\boldsymbol{\psi}$. The top row is the graphs of the supports of $\boldsymbol{\psi}^{0,1}$ with respect to $m = 2, 4, 8, 16$ (left to right). The bottom row is the graphs of corresponding effect on the supports of $\boldsymbol{\psi}^{0,1}$ in the time domain.

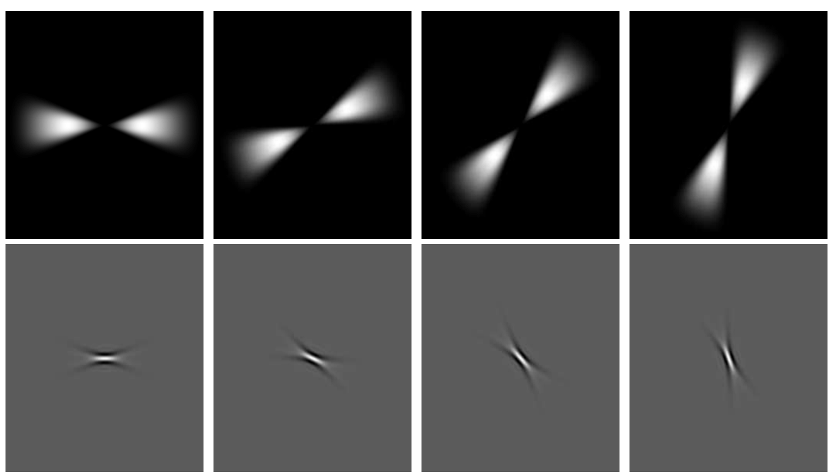

FIGURE 5.3: The rotation effect on $\boldsymbol{\psi}^{0,1}$ with $m = 8$. The top row is the graphs of the supports of $\boldsymbol{\psi}^{0,\ell}$ with respect to $\ell = 1, 2, 3, 4$ (left to right). The bottom row is the graphs of corresponding effect on the supports of $\boldsymbol{\psi}^{0,\ell}$ in the time domain.

and 5.3). For implementation, the C^∞ smoothness does not necessarily be needed. Indeed, we can modify function $\gamma(x)$ in (5.3.2) to obtain wavelet generators with regularity of C^k (γ can actually be a polynomial). Alternatively, one may consider the construction of compactly supported bivariate tight wavelet frames with directionality in time domain. That is, the wavelet generators $\boldsymbol{\varphi}$, $\boldsymbol{\psi}^{j,\ell}$'s are compactly supported in time domain with symmetry and certain regularity (vanishing moments).

Comparing with curvelets (or shearlets) using anisotropic dilation $\mathsf{M} = \mathrm{diag}(4,2)$, here we use isotropic dilation matrix $\mathsf{M} = 2I_2$, which significantly simplify the algorithm design. Also curvelets are essentially supported in rectangles with width≈length$^{1/2}$. In our case, the choice of the ratio ρ in width≈length$^{1-\rho}$ is flexible, which might provide us more freedom in the investigation of the optimal approximation rate of the representation of objects with C^2 singularities. Along this direction, we would like to establish results similar to [1] on representing objects with C^2 singularities using the tight M-wavelet frame with directionality we constructed above.

Finally, let us end this chapter by commenting on possible applications of the tight wavelet frames constructed in this chapter. Bearing the properties of redundancy and directional sensitivity, the tight wavelet frames constructed in this chapter have the potential applications in signal/image processing, or even in 3-D objects modeling and reconstructing. Our future work shall continue along this line of research.

Appendix A

Proofs of Proposition 1.5 and Theorem 2.9

A.1 Proof of Proposition 1.5

Proof. By $\phi(x) = \mathsf{d}\sum_{k\in\mathbb{Z}} a(k)\phi(dx-k)$, we can deduce that $\phi_\ell(x) = \mathsf{d}\sum_{k\in\mathbb{Z}} a_\ell(k)\phi_{R_k+1}(dx-Q_k)$, for $\ell = 1,\dots,r$. Hence, for each $j\in\mathbb{Z}$, substituting $x_0 := \frac{R_j}{\mathsf{d}r} + \frac{Q_j}{\mathsf{d}}$ to the above and using the interpolation property of ϕ, we have $\phi_\ell(x_0) = \mathsf{d}\sum_{k\in\mathbb{Z}} a_\ell(k)\phi_{R_k+1}(\frac{R_j}{r} + Q_j - Q_k) = \mathsf{d}a_\ell(j)$.

On the other hand, $\phi_{r-\ell+1}(\frac{r-1}{r} - x_0) = \mathsf{d}\sum_{k\in\mathbb{Z}} a_{r-\ell+1}(k)\phi_{R_k+1}(\frac{\mathsf{d}(r-1)}{r} - \frac{R_j}{r} - Q_j - Q_k) = \mathsf{d}a_{r-\ell+1}(r(Q_t-Q_j)+R_t)$, where $\mathsf{d}(r-1) - R_j = rQ_t + R_t$ with $R_t, Q_t \in \mathbb{Z}$ and $0 \leqslant R_t \leqslant r-1$. Consequently, by $\phi_\ell(x) = \phi_{r-\ell+1}(\frac{r-1}{r} - x)$, we conclude that

$$\begin{aligned} a_\ell(j) &= a_{r-\ell+1}(r(Q_t-Q_j)+R_t) = a_{r-\ell+1}(rQ_t+R_t-rQ_j) \\ &= a_{r-\ell+1}(\mathsf{d}(r-1)-(rQ_j+R_j)) = a_{r-\ell+1}(-j+(r-1)\mathsf{d}). \end{aligned}$$

Therefore, (1.5.1) holds. The computations for showing (1.5.2) is similar.

Conversely, suppose (1.5.1) holds and $\mathsf{d}-1=k_0r$ for some integer $k_0\geqslant 1$. Since ϕ is interpolating, we automatically obtain

$$\phi_\ell(\frac{m}{r}+j)=\phi_{r-\ell+1}(\frac{r-1}{r}-(\frac{m}{r}+j))\quad j\in\mathbb{Z};0\leqslant m\leqslant r-1;\ell=1,\ldots,r.$$

Suppose we have proved that for $n\geqslant 1$, $0\leqslant m\leqslant r-1$ and $1\leqslant\ell\leqslant r$,

$$\phi_\ell(\frac{1}{\mathsf{d}^{n-1}}(\frac{m}{r}+j))=\phi_{r-\ell+1}(\frac{r-1}{r}-\frac{1}{\mathsf{d}^{n-1}}(\frac{k}{r}+j)),\quad j\in\mathbb{Z}.$$

Then

$$\begin{aligned}
&\phi_\ell(\frac{1}{\mathsf{d}^n}(\frac{m}{r}+j))=\mathsf{d}\sum_{k\in\mathbb{Z}}a_\ell(k)\phi_{R_k+1}(\frac{1}{\mathsf{d}^{n-1}}(\frac{m}{r}+j)-Q_k)\\
&=\mathsf{d}\sum_{k\in\mathbb{Z}}a_{r-\ell+1}(-k+(r-1)\mathsf{d})\phi_{r-R_k}(\frac{r-1}{r}-\frac{1}{\mathsf{d}^{n-1}}(\frac{m}{r}+j)+Q_k)\\
&=\mathsf{d}\sum_{k\in\mathbb{Z}}a_{r-\ell+1}(-k+(r-1)\mathsf{d})\times\\
&\qquad\qquad\phi_{r-R_k}\left(\mathsf{d}(\frac{r-1}{r}-\frac{1}{\mathsf{d}^n}(\frac{m}{r}+j))-\frac{(\mathsf{d}-1)(r-1)}{r}+Q_k\right).
\end{aligned}$$

Now, let $k'=-k+(r-1)\mathsf{d}$. We have $k'=-rQ_k-R_k+(r-1)(k_0r+1)=-r(Q_k-(r-1)k_0)+r-1-R_k$. Hence, $R_{k'}=r-1-R_k$ and $Q_{k'}=Q_k-(r-1)k_0=Q_k-\frac{(\mathsf{d}-1)(r-1)}{r}$. Consequently,

$$\begin{aligned}
\phi_\ell(\frac{1}{\mathsf{d}^n}(\frac{m}{r}+j))&=\mathsf{d}\sum_{k'\in\mathbb{Z}}a_{r-\ell+1}(k')\phi_{R_{k'}+1}(\mathsf{d}(\frac{r-1}{r}-\frac{1}{\mathsf{d}^n}(\frac{m}{r}+j))-Q_{k'})\\
&=\phi_{r-\ell+1}(\frac{r-1}{r}-\frac{1}{\mathsf{d}^n}(\frac{m}{r}+j)).
\end{aligned}$$

By induction, we have $\phi_\ell(\frac{1}{\mathsf{d}^n}(\frac{m}{r}+j))=\phi_{r-\ell+1}(\frac{r-1}{r}-\frac{1}{\mathsf{d}^n}(\frac{k}{r}+j))$, for $0\leqslant m\leqslant r-1$, $1\leqslant\ell\leqslant r$, $j\in\mathbb{Z}$, and all $n\in\mathbb{N}$. Therefore, by the continuity of ϕ and density of $\{\frac{1}{\mathsf{d}^n}(\frac{m}{r}+j) : j \in \mathbb{Z}; n \in \mathbb{N}; m = 0, \ldots, r-1\}$ in $\mathbb{R}$, we conclude that $\phi_\ell = \phi_{r-\ell+1}(\frac{r-1}{r} - \cdot)$, $\ell = 1, \ldots, r$. The proof for the other case is similar. □

A.2 Proof of Theorem 2.9

Proof. Existence. Let $G := \{(\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d : \beta_1 + \cdots + \beta_d \text{ is odd}\}$. It is easy to verify that $G \cup \Gamma = \mathbb{Z}^d$ and $G = \gamma + \Gamma$ for any $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d$ with $\gamma_1 + \cdots + \gamma_d$ being an odd number. That is, $\mathbb{Z}^d/\Gamma = \{0, \gamma\}$ for any γ with $|\gamma|$ being an odd number. Together with item (1), item (3) becomes

$$\sum_{\beta \in \Gamma} a(\gamma+\beta)p(\gamma+\beta) = \frac{1}{2}\delta(\beta)p(\beta), \quad |\gamma| \text{ is odd}; p \in \Pi_{|m|},$$

which is equivalent to $\sum_{\beta \in G} a(\beta)\beta^\mu = \frac{1}{2}\delta(\mu)$, $|\mu| \leqslant |m|$.

Claim 1: Let $G_S := G \cap S$. Then, $(\beta^\mu)_{\beta \in G_S, |\mu| \leqslant |m|}$ is of full row rank.

In fact, define $|m|$ hyperplanes:

$$H_{2j} := \{(x_1, \cdots, x_s) \in \mathbb{R}^d : x_1 + \cdots + x_d = |m| - 2j\};$$
$$H_{2j+1} := \{(x_1, \cdots, x_s) \in \mathbb{R}^d : x_1 + \cdots + x_d = -(|m| - 2j)\},$$

for $j = 0, \ldots, \frac{|m|-1}{2}$. Note that $\#(G_S \cap H_{2j}) = \#(G_S \cap H_{2j+1}) \leqslant \#\{\mu : |\mu| \leqslant 2j\}$. Due to the special structure of the points in G_S, it is easy to extend G_S to G_{S_e} such that $G_S \subseteq G_{S_e}$ and $\#(G_{S_e} \cap H_j) = \#\{\mu : |\mu| \leqslant j\}$ for $j = 0, \ldots, |m|$. Moreover, G_{S_e} satisfies the "Node Configuration A in $\mathbb{R}^d$" as in [6]. Consequently, by [6, Theorem 4], we conclude that

$(\beta^\mu)_{\beta\in G_{S_e},|\mu|\leqslant|m|}$ is of full rank. Since $G_S\subseteq G_{S_e}$, $(\beta^\mu)_{\beta\in G_S,|\mu|\leqslant|m|}$ is of full row rank.

Claim 2: Let $\Gamma_0:=\{\mu\in\mathbb{N}_0^d : |\mu|\leqslant|m| \text{ and } \mu_\ell\leqslant m_\ell \text{ for } \ell=1,\dots,d\}$. Then $(\beta^\mu)_{\beta\in G_S,|\mu|\leqslant|m|}$ and $(\beta^\mu)_{\beta\in G_S,\mu\in\Gamma_0}$ have the same column rank.

In fact, for each $\mu\notin\Gamma_0$ and $|\mu|\leqslant|m|$, using long division of polynomials, x^μ with $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ can be represented as

$$x^\mu=q_1^\mu(x)\prod_{j=-m_1}^{m_1}(x_1-j)+\cdots+q_d^\mu(x)\prod_{j=-m_d}^{m_d}(x_d-j)+P_\mu(x),$$

where $q_1^\mu(x),\dots,q_d^\mu(x)$ are polynomials of d variables and $P_\mu(x)$ is a linear combination of x^ν with $\nu\in\Gamma_0$. Hence, we have $P_\mu(0)=0$. Now, the conclusion follows from $\beta^\mu=P_\mu(\beta)$ for any $\beta\in G_S$.

Claim 3: Let $G_+:=G_S\cap\mathbb{N}_0^d$ and $\Gamma_1:=\{2\mu : 2\mu\in\Gamma_0\}$. Then $\#G_+=\#\Gamma_1$ and $(\beta^\mu)_{\beta\in G_+,\mu\in\Gamma_1}$ is nonsingular.

In fact, $\Gamma_1=\bigcup_{j=0}^{\frac{|m|-1}{2}}\{2\mu\in\mathbb{N}_0^d : |\mu|=j,\mu_\ell\leqslant m_\ell;\ell=1,\dots,d\}$ and $G_+=\bigcup_{j=0}^{\frac{|m|-1}{2}}\{\beta\in\mathbb{N}_0^d : |\beta|=2j+1,\beta_\ell\leqslant m_\ell;\ell=1,\dots,d\}$. For $j=2k+1$ with $0\leqslant 2k+1\leqslant\frac{|m|-1}{2}$, we have $\#\{2\mu\in\mathbb{N}_0^d : |\mu|=2k+1,\mu_\ell\leqslant m_\ell;\ell=1,\dots,d\}=\#\{\beta\in\mathbb{N}_0^d : |\beta|=2k+1,\beta_\ell\leqslant m_\ell;\ell=1,\dots,d\}$. And for $j=2k$ with $0\leqslant 2k\leqslant\frac{|m|-1}{2}$, we have $\#\{2\mu\in\mathbb{N}_0^d : |\mu|=2k,\mu_\ell\leqslant m_\ell;\ell=1,\dots,d\}=\#\{\beta\in\mathbb{N}_0^d : |\beta|=|m|-2k,\beta_\ell\leqslant m_\ell;\ell=1,\dots,d\}$. Therefore $\#G_+=\#\Gamma_1$.

Next, we show that $(\beta^\mu)_{\beta\in G_+,\mu\in\Gamma_1}$ is nonsingular. Suppose not, there exists $\{c_\beta\}_{\beta\in G_+}$ with $c_\beta\neq 0$ for some $\beta\in G_+$ such that $\sum_{\beta\in G_+}c_\beta\beta^\mu=0,\forall\mu\in\Gamma_1$.

Let $\mathcal{E}:=\{E_\varepsilon : E_\varepsilon=\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_d),\varepsilon=(\varepsilon_1,\dots,\varepsilon_d)$ with each $\varepsilon_\ell=\pm1$ for $\ell=1,\dots,d\}$. By the symmetry of G_S, we have $G_S=\{E_\varepsilon\beta :$

$\beta \in G_+; E_\varepsilon \in \mathcal{E}\}$. By the evenness of Γ_1, $\sum_{\beta \in G_S} c_\beta \beta^\mu = 0$, $\forall \mu \in \Gamma_1$, where for each $\beta = (\beta_1, \ldots, \beta_d) \in G_S$, $c_\beta := c_{\beta^+}$ with $\beta^+ = (|\beta_1|, \ldots, |\beta_d|) \in G_+$.

For any $\mu \in \mathbb{N}_0^d$ such that $|\mu|$ is odd, by the symmetry of G_S and oddness of μ, we have

$$\begin{aligned}\sum_{\beta \in G_S} c_\beta \beta^\mu &= \sum_{E_\varepsilon \in \mathcal{E}} \sum_{\beta \in G_+} c_{E_\varepsilon \beta} (E_\varepsilon \beta)^\mu \\ &= \sum_{\varepsilon_1 = \pm 1} \cdots \sum_{\varepsilon_d \in \pm 1} \sum_{\beta \in G_+} c_{E_{(\varepsilon_1, \ldots, \varepsilon_d)} \beta} (E_{(\varepsilon_1, \ldots, \varepsilon_d)} \beta)^\mu = 0.\end{aligned}$$

Moreover, by Claim 2, β^μ is the linear combination of $\beta^\nu, \nu \in \Gamma_0$ for any $|\mu| \leqslant |m$. Consequently, $\sum_{\beta \in G_S} c_\beta \beta^\mu = 0$, $\forall |\mu| \leqslant |m|$. This contradicts to the Claim 1. Therefore, $(\beta^\mu)_{\beta \in G_+, \mu \in \Gamma_1}$ must be nonsingular.

By Claim 3, we can choose a subset Γ_2 with $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_0$ and $\#\Gamma_2 = \#G_S$ such that $(\beta^\mu)_{\beta \in G_+, \mu \in \Gamma_2}$ is a nonsingular matrix. Solve the linear system $\sum_{\beta \in G_S} c_\beta \beta^\mu = \frac{1}{2}\delta(\mu)$, $\mu \in \Gamma_2$ for $\{c_\beta : \beta \in G_S\}$. Then we also have $\sum_{\beta \in G_S} c_\beta \beta^\mu = \frac{1}{2}\delta(\mu)$ for all $|\mu| \leqslant |m|$. Construct the mask a_m to be $a_m(\beta) = c_\beta$ for $\beta \in G_S$, $a_m(0) = \frac{1}{2}$, and $a_m(\beta) = 0$ otherwise. Then a_m satisfies all conditions in item (1)–(3).

Uniqueness: Suppose there is another mask b satisfies conditions in item (1)–(3). Then

$$\sum_{\beta \in G_S} (a_m(\beta) - b(\beta))\beta^\mu = 0, \forall |\mu| \leqslant |m|. \Longrightarrow \sum_{\beta \in G_S} (a_m(\beta) - b(\beta))\beta^\mu = 0,\ \mu \in \Gamma_2.$$

By the nonsingularity of $(\beta^\mu)_{\beta \in G_+, \mu \in \Gamma_2}$, we have $a_m(\beta) = b(\beta)$ for all $\beta \in G_S$. Consequently, the mask a_m must be unique. ☐

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