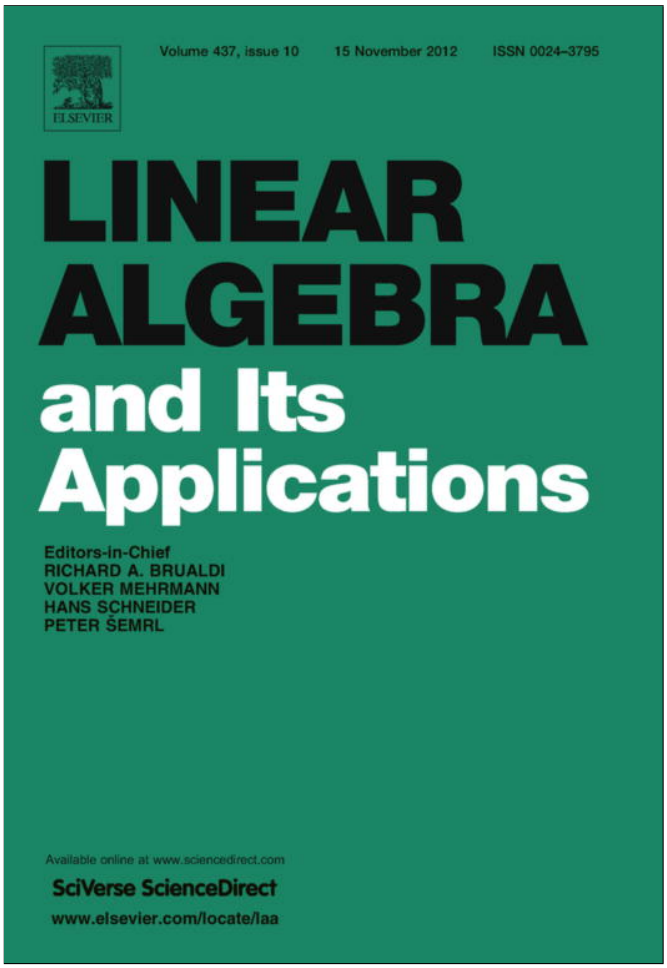

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Matrix splitting with symmetry and dyadic framelet filter banks over algebraic number fields

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ABSTRACT

Algebraic number fields are of particular interest and play an important role in both mathematics and engineering since an algebraic number field can be viewed as a finite dimensional linear space over the rational number field $\mathbb{Q}$. Algorithms using algebraic number fields can be efficiently implemented involving only integer arithmetics. In this paper, we properly formulate the matrix splitting problem over any general subfield of $\mathbb{C}$, including an algebraic number field as a special case, and provide a simple necessary and sufficient condition for a 2×2 matrix of Laurent polynomials with symmetry to be able to be factorized by a 2×2 matrix of Laurent polynomials with certain symmetry structure. We propose an effective algorithm on how to obtain the factorization matrix step by step. As an application, we obtain a satisfactory algorithm for constructing dyadic framelet filter banks with the perfect reconstruction property and with symmetry over algebraic number fields. Several examples are provided to illustrate the algorithms proposed in this paper.

1. Introduction and motivation

Nowadays, raw data from musical albums, cameras, films, scientific research observations, etc., are typically digitized so that they can be further analyzed by digital devices. In the point of view of engineering or wavelet analysis, digital data are usually analyzed by representation systems (filter banks or wavelet systems) using convolutions. Hence, in signal/image processing, one of the primary goals is to obtain filter systems in engineering or wavelet systems in wavelet analysis with desirable

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properties that are suitable for processing, storage, transmission, and recovery of those data of interest. In the digital world, due to machine precision, data are often stored using integers, rational numbers, or more precisely, dyadic numbers, e.g., an 8-bit gray-scale image or the signal of a song in a CD album. The rational number field $\mathbb{Q}$ therefore has many advantages in scientific computing. Using integer arithmetic and simple hardware implementation, convolution operations using rational number field are much more efficient than those using floating point arithmetic. Moreover, there is no roundup error using integer arithmetic.

Among many types of wavelet systems such as orthonormal bases, biorthogonal bases, Riesz bases, frames, etc., redundant system is one of the most popular types of representation systems because of its robustness property under noise, quantization, and data loss. One of the main objectives in wavelet analysis is to construct wavelet systems with many nice properties, e.g., smoothness, high order of vanishing moments, symmetry, etc. Wavelet systems such as symmetric tight wavelet frames (or framelet filter banks in engineering) have been successfully applied in various image processing problems including debluring, denoising, inpainting, and separation (see [26] and many references therein). Motivated by [15,19], in this paper, we are interested in the construction of dyadic tight wavelet frames in wavelet analysis or dyadic framelet filter banks in engineering with symmetry over an algebraic number field.

As pointed out in [15], almost all existing tight framelet filter banks have their coefficients coming from some algebraic number fields, and even though a filter bank with the perfect reconstruction property can have its low-pass filter coefficients coming from the rational number field $\mathbb{Q}$, it is not necessary that its high-pass filters have coefficients coming from the rational number field as well. Taking the famous Ron-Shen dyadic framelet filter bank for example (see [22]), its low-pass filter is given by $a = \{a(k)\}_{k=-1,0,1} = \frac{1}{4}\{1, 2, 1\}_{k=-1,0,1}$, or in terms of its symbol $\mathsf{a}(z) = \frac{1}{4}(z^{-1} + 2 + z)$, which is a Laurent polynomial with coefficients in the rational number field $\mathbb{Q}$. On the one hand, by [19, Theorem 5.2], a framelet filter bank $\{a, b_1, b_2\}$ constructed from a has all its filters with coefficients coming from the rational number field $\mathbb{Q}$ only if $1 - \mathsf{a}(z)\mathsf{a}(z^{-1}) - \mathsf{a}(-z)\mathsf{a}(-z^{-1}) = \mathsf{d}(z^2)\mathsf{d}(z^{-2})$ for all $z \in \mathbb{C}\backslash\{0\}$ with d being a Laurent polynomial with rational coefficients. However, it is easy to show that $1 - \mathsf{a}(z)\mathsf{a}(z^{-1}) - \mathsf{a}(-z)\mathsf{a}(-z^{-1}) = \mathsf{d}(z^2)\mathsf{d}(z^{-2})$ with $\mathsf{d}(z) = \frac{1}{2\sqrt{2}}(z^2 - 1)$. Consequently, it is impossible to have rational-coefficient framelet filter bank $\{a, b_1, b_2\}$ from a. On the other hand, it is well-known that the Ron-Shen example consisting of the filter system $\{a, b_1, b_2\}$ with $b_1 = \frac{1}{4}\{1, -2, 1\}_{k=-1,0,1}$ and $b_2 = \frac{\sqrt{2}}{4}\{-1, 0, 1\}_{k=-1,0,1}$ is a framelet filter bank with the perfect reconstruction property. Obviously, though the high-pass filters have coefficients not necessarily coming from the rational number field, each of its high-pass filters has its coefficients in an algebraic number field $\mathbb{Q}(\sqrt{t})$ (more precisely, $\sqrt{t}\mathbb{Q}$) with t being some fixed positive integer. Recall that an algebraic number field $\mathbb{A}$ is a finite field extension of the rational number field $\mathbb{Q}$, i.e., $\mathbb{A} = \mathbb{Q}(\sqrt{t_1}, \ldots, \sqrt{t_n})$ for $t_1, \ldots, t_n$ being roots of some polynomial with integer coefficients. An algebraic number field $\mathbb{A}$ can be viewed as a finite dimensional vector space over $\mathbb{Q}$. Consequently, the arithmetic over $\mathbb{A}$ can be implemented by combining integer arithmetic and matrix/vector operations from linear algebra (for example, see [1,18]). In other words, algorithms over an algebraic number field $\mathbb{A}$ have the same computational complexity as those over the rational number field $\mathbb{Q}$. Moreover, when filters are with coefficients in $\sqrt{t}\mathbb{Q}$, the implementation of the framelet transform can be *exact* since the square root in the analysis side or the synthesis side can be realized in only one side; see [15, Section 3] for more details on implementation of the fast framelet transform for a filter bank over an algebraic number field. Therefore, it is natural and necessary to consider framelet filter banks over algebraic number fields. The main objective of this paper is to provide a proper characterization along with simple algorithms for the construction of dyadic framelet filter banks with the perfect reconstruction property and with the symmetry property. To this end, we shall discuss the matrix splitting problem next.

1.1. The matrix splitting problem

The matrix splitting problem plays a fundamental role in the construction of framelet filter banks. It is well-known that the construction of framelet filter banks can be reduced to a matrix splitting

problem, which we will explain with more details in Section 3. In this paper, we are going to investigate the matrix splitting problem with symmetry and its application to the construction of dyadic framelet filter banks with symmetry over a general subfield $\mathbb{F}$ of the complex number field $\mathbb{C}$. Before introducing the matrix splitting problem, let us introduce some necessary notation and definitions.

We denote $\mathbb{F}$ to be a subfield of $\mathbb{C}$ satisfying

$$\bar{x} \in \mathbb{F} \quad \text{if} \quad x \in \mathbb{F}. \tag{1.1}$$

Let $\mathsf{p}(z) = \sum_{k\in\mathbb{Z}} p_k z^k$, $z \in \mathbb{C}\backslash\{0\}$ be a Laurent polynomial with complex coefficients $p_k \in \mathbb{C}$. We say that p has *symmetry* if its coefficient sequence $\{p_k\}_{k\in\mathbb{Z}}$ has symmetry; more precisely, there exist $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ such that

$$p_{c-k} = \epsilon p_k \qquad \forall\, k \in \mathbb{Z}. \tag{1.2}$$

If $\epsilon = 1$, then p is symmetric about the point $c/2$; if $\epsilon = -1$, then p is antisymmetric about the point $c/2$. Symmetry of a Laurent polynomial can be conveniently expressed using a symmetry operator S defined by

$$\mathsf{S}\mathsf{p}(z) := \frac{\mathsf{p}(z)}{\mathsf{p}(z^{-1})}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{1.3}$$

When p is not identically zero, it is evident that (1.2) holds if and only if $\mathsf{S}\mathsf{p}(z) = \epsilon z^c$. For the zero polynomial, it is very natural that $\mathsf{S}0$ can be assigned any symmetry pattern; that is, for every occurrence of $\mathsf{S}0$ appearing in an identity in this paper, $\mathsf{S}0$ is understood to take an appropriate choice of ϵz^c for some $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ so that the identity holds.

An $r \times s$ matrix P of Laurent polynomials can be written as $\mathsf{P}(z) := \sum_{k=m}^{n} P_k z^{-k}$ with $m, n, k \in \mathbb{Z}$, $P_k \in \mathbb{C}^{r\times s}$ being $r \times s$ matrices of complex numbers, and $P_m \neq 0$, $P_n \neq 0$. We can define its filter support and length to be $\mathrm{fsupp}(\mathsf{P}) := [m, n]$ and $\mathrm{len}(\mathsf{P}) := n - m$. Moreover, we shall use $\mathsf{P}^\star(z) = \mathsf{P}(z)^\star := \overline{\mathsf{P}(\bar{z}^{-1})}^T = \sum_{k\in\mathbb{Z}} P_k^\star z^{-k}$ with $P_k^\star := \overline{P_k}^{\mathsf{T}}$ being the transpose of the complex conjugate of the matrix P_k. With the above $\star$ notation, we often work on $\mathsf{P}(z)$ with $z \in \mathbb{T} := \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$. If P is an $r \times s$ matrix of Laurent polynomials with symmetry, then we can apply the operator S to each entry of P; that is, $\mathsf{S}\mathsf{P}$ is an $r \times s$ matrix such that $[\mathsf{S}\mathsf{P}]_{j,k} := \mathsf{S}([\mathsf{P}]_{j,k})$ for $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant s$, where $[\mathsf{P}]_{j,k}$ denotes the (j, k)-entry of the matrix P.

For two matrices P and Q of Laurent polynomials with symmetry, even though all the entries in P and Q have symmetry, their sum $\mathsf{P} + \mathsf{Q}$, difference $\mathsf{P} - \mathsf{Q}$, or product $\mathsf{P}\mathsf{Q}$, if well defined, generally may not have symmetry anymore. This is one of the difficulties for matrix splitting or extension with symmetry. In order for $\mathsf{P} \pm \mathsf{Q}$ or $\mathsf{P}\mathsf{Q}$ to possess some symmetry, the symmetry patterns of P and Q should be compatible. For example, if $\mathsf{S}\mathsf{P} = \mathsf{S}\mathsf{Q}$; that is, both P and Q have the same symmetry pattern, then indeed $\mathsf{P} \pm \mathsf{Q}$ has symmetry and $\mathsf{S}(\mathsf{P} \pm \mathsf{Q}) = \mathsf{S}\mathsf{P} = \mathsf{S}\mathsf{Q}$. In the following, we discuss the compatibility of symmetry patterns of matrices of Laurent polynomials. For an $r \times s$ matrix P of Laurent polynomials with symmetry, we say that *the symmetry of* P *is compatible* or P *has compatible symmetry*, if

$$\mathsf{S}\mathsf{P}(z) = (\mathsf{S}\theta_1)^\star(z)\mathsf{S}\theta_2(z), \tag{1.4}$$

for some $1 \times r$ vector θ_1 and $1 \times s$ vector θ_2 of Laurent polynomials with symmetry. For an $r \times s$ matrix P and an $s \times t$ matrix Q of Laurent polynomials with symmetry, we say that (P, Q) *has mutually compatible symmetry* if

$$\mathsf{S}\mathsf{P}(z) = (\mathsf{S}\theta_1)^\star(z)\mathsf{S}\theta(z) \quad \text{and} \quad \mathsf{S}\mathsf{Q}(z) = (\mathsf{S}\theta)^\star(z)\mathsf{S}\theta_2(z) \tag{1.5}$$

for some $1 \times r$, $1 \times s$, $1 \times t$ row vectors $\theta_1, \theta, \theta_2$ of Laurent polynomials with symmetry, respectively. If (P, Q) has mutually compatible symmetry as in (1.5), then it is easy to verify that their product $\mathsf{P}\mathsf{Q}$ has compatible symmetry and in fact $\mathsf{S}(\mathsf{P}\mathsf{Q}) = (\mathsf{S}\theta_1)^\star\mathsf{S}\theta_2$.

Now we are ready to introduce the matrix splitting problem with symmetry. Let M be an $r \times r$ matrix of Laurent polynomials with symmetry in $\mathbb{F}[z, z^{-1}]$ such that $\mathsf{M} = \mathsf{M}^\star$ and $\mathsf{SM} = (\mathsf{S}\theta)^\star(\mathsf{S}\theta)$ for some $1 \times r$ vector θ of Laurent polynomials with symmetry. The matrix splitting problem with symmetry is to find necessary and sufficient conditions on M along with a constructive algorithm to derive an $r \times s$ matrix U of Laurent polynomials such that

(1) $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$; i.e., M is factorized by an $r \times s$ matrix U of Laurent polynomials.
(2) Each column of U is a vector of Laurent polynomials with symmetry in $\mathbb{F}[z, z^{-1}]$ up to a multiplicative constant, and $\mathsf{SU} = (\mathsf{S}\theta)^\star\mathsf{S}\theta_1$ for some $1 \times s$ vector θ_1 of Laurent polynomials with symmetry; i.e., U has certain symmetry pattern as well.
(3) The support of U is controlled by that of M in some sense.

Let us make some remarks. Item (1) is to guarantee the perfect reconstruction property of a framelet filter bank. Filters, either low-pass or high-pass filters, are constructed from the factorization matrix U and item (2) is related to the symmetry property and coefficients of the filters in the framelet filter bank. We shall see the precise meaning of "up to a multiplicative constant" in the next section. This simple relaxation allows one to construct a large family of framelet filter banks over algebraic number fields. Moreover, it is desirable to have filters in a framelet filter bank with as short support as possible. Item (3) means that the lengths of the supports of high-pass filters from the factorization matrix U should not be too long comparing to that of the input low-pass filter.

For an important case $r = s = 2$, our result – see Theorem 1 in Section 2 – provides a simple characterization on M to be able to be factorized by a 2×2 matrix U with many desirable properties as indicated above. We show that as long as M is positive semi-definite on the torus $\mathbb{T}$ and the determinant of M satisfies $\det(\mathsf{M}) = c_0\mathsf{d}(z)\mathsf{d}^\star(z)$ for some Laurent polynomial d in $\mathbb{F}[z, z^{-1}]$ and some positive constant $c_0 \in \mathbb{F}$, then a 2×2 matrix U of Laurent polynomials with symmetry and with many nice properties can be derived by a simple step-by-step algorithm (see Algorithm 1 in Section 2). This result plays a crucial role in the construction of dyadic framelet filter banks $\{a, b_1, b_2\}$ with symmetry over algebraic number fields, which we shall discuss in Section 3. In fact, the necessary and sufficient condition allows one to design a framelet filter bank with many desirable properties, for example, the vanishing moments, the sum rules, symmetry constrain, etc., by imposing constrains only on the low-pass filter a. Once the low-pass filter with the desirable properties is obtained, one can construct the matrix M associated with a and the derivation of the high-pass filters b_1, b_2 is straightforward as shown by our algorithm in Section 3.

1.2. Related work

Without considering any symmetry issue, it is well-known by the matrix-valued Fejér-Riesz lemma that an $r \times r$ matrix M of Laurent polynomials in $\mathbb{C}[z, z^{-1}]$ can be factorized as $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$ for some $r \times r$ matrix U of Laurant polynomials in $\mathbb{C}[z, z^{-1}]$ if and only if M is positive semi-definite on the torus $\mathbb{T}$, see [16]. To our best knowledge, there is no general result, except for a very few special cases, concerning the matrix splitting problem for any $2 \leqslant r \leqslant s$. The most appearing case is $r = 2$. Considering the symmetry constrain, Chui and He [2] (also see [12,20]) showed that for $r = 2, s = 3$, and $\mathbb{F} = \mathbb{R}$, there exists an $r \times s$ matrix U of Lauent polynomials with symmetry in $\mathbb{R}[z, z^{-1}]$ such that $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$ provided M is positive semi-definite on the torus $\mathbb{T}$. However, their techniques for deriving the existence of such a splitting matrix U are based on those for the matrix extension problem, not for the matrix splitting problem. A more interesting and desirable case is for $r = s = 2$ since only two framelet generators are needed in this case for a tight wavelet frame, which is important in terms of computational efficiency and storage. Petukhov [21] and Jiang [17] studied such a construction of tight wavelet frames with only two framelet generators, again under the setting of matrix extension; see [10,14,15,27] for more results on matrix extension with symmetry. In the setting of matrix splitting, the matrix splitting problem with symmetry and with $r = s = 2$ has been investigated by Han and Mo in [13] for $\mathbb{F} = \mathbb{R}$, and Mo and Li in [19] for $\mathbb{F} = \mathbb{Q}$. But as pointed out in [15], examples obtained in [13,19] are either with irrational coefficients or with poor vanishing moments for the corresponding

framelet generators in a framelet filter bank. For more about the construction of dyadic tight wavelet frames or dyadic framelet filter banks, see [3–6,11,23,24] and many references therein.

1.3. Our contributions

Our major contributions of this paper lie in many aspects. First, we give a proper formulation of the 2×2 matrix splitting problem with symmetry over a general subfield $\mathbb{F}$ of $\mathbb{C}$. Second, we provide a simple necessary and sufficient condition for a 2×2 matrix M of Laurent polynomials with symmetry in $\mathbb{F}[z, z^{-1}]$ to be able to be factorized as $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$ such that items (1) – (3) are satisfied in the above matrix splitting problem for $r = s = 2$. Third, we provide a step-by-step algorithm for deriving the desirable factorization matrix U by employing the extended Euclidean algorithm and solving a simple system of linear equations if necessary. Finally, as an application to the construction of tight wavelet frames or framelet filter banks with symmetry over algebraic number fields, we show that high-pass filters with coefficients in an algebraic number field can be derived from the low-pass filters using our matrix splitting algorithm. We would like to further point out that our necessary and sufficient condition not only recovers many existing examples (e.g., the Ron-Shen tight framelets), but also yields many new examples of framelet generators with higher order of vanishing moments and higher order of regularity, and at the same time having many nice properties and advantages as those framelet systems over the rational number fields; see examples in Section 4.

1.4. Contents

The structure of this paper is as follows. In Section 2, we shall introduce the main result for the matrix splitting problem with symmetry over a general subfield of the complex number field and provide a step-by-step algorithm for deriving the factorization matrix. In Section 3, we make the connection between the construction of tight wavelet frames or framelet filter banks with symmetry to the matrix splitting problem with symmetry. We also provide a step-by-step algorithm detailing the construction of the low-pass filter satisfying the necessary and sufficient condition and the derivation of the high-pass filters from the low-pass filter having the desirable properties. Illustrative examples shall be given in Section 4 for showing the simplicity and efficiency of our algorithms. Proofs of some lemmas are postponed in Section 5. Final remarks are given in the last section.

2. Main results

In this section, we shall investigate the matrix splitting problem with symmetry over a general subfield $\mathbb{F}$. We shall first introduce our main theorem on the matrix splitting problem with symmetry and then provide a step-by-step algorithm based on the constructive proof of our main theorem. Our results in this section play a key part in the construction of tight wavelet frames or framelet filter banks in the next section.

2.1. Main theorem

Throughout the paper, $\mathbb{F}$ always denotes a subfield of $\mathbb{C}$ satisfying (1.1). Then, $\mathbb{F}[z, z^{-1}]$ is a unique factorization domain (UFD). We shall use $\gcd(\mathsf{p}_1, \ldots, \mathsf{p}_k)$ to denote the great common divisor of Laurent polynomials $\mathsf{p}_1, \ldots, \mathsf{p}_k$ in $\mathbb{F}[z, z^{-1}]$. Also, $\det \mathsf{P}$ is the determinant of a square matrix P of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$. Our main result in this section is as follows.

Theorem 1. *Let* p, q, *and* r *be three Laurent polynomials with symmetry in* $\mathbb{F}[z, z^{-1}]$ *such that* $\gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}) = \gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}^\star) = 1$. *Let* M *be a* 2×2 *matrix of Laurent polynomials defined to be*

$$\mathsf{M}(z) := \begin{bmatrix} \mathsf{p}(z) & \mathsf{r}(z) \\ \mathsf{r}^\star(z) & \mathsf{q}(z) \end{bmatrix}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{2.1}$$

Then M *is positive semi-definite on* $\mathbb{T}$, *i.e.,* $\mathsf{M}(z) \geqslant 0$ *for all* $z \in \mathbb{T}$, *and* $\det \mathsf{M} = \mathsf{d}\mathsf{d}^\star$ *with* $\mathsf{d} = c_0\mathsf{d}_0$ *for some Laurent polynomial* d_0 *in* $\mathbb{F}[z, z^{-1}]$ *with symmetry and some constant* $c_0 \in \mathbb{C}$ *satisfying* $|c_0|^2 \in \mathbb{F}$, *if and only if, the following statements hold*

(1) M *can be factorized as*

$$\mathsf{M} = \mathsf{U}\mathsf{U}^\star \quad \textit{with} \quad \mathsf{U}(z) = \begin{bmatrix} \mathsf{u}_1(z) & \mathsf{v}_1(z) \\ \mathsf{u}_2(z) & \mathsf{v}_2(z) \end{bmatrix} \quad \forall z \in \mathbb{C} \setminus \{0\} \tag{2.2}$$

for some Laurent polynomials $\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2$ *with symmetry.*

(2) $\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2$ *satisfy*

$$\mathsf{u}_1 = c_1\tilde{\mathsf{u}}_1, \ \mathsf{u}_2 = c_1\tilde{\mathsf{u}}_2, \ \mathsf{v}_1 = c_2\tilde{\mathsf{v}}_1, \ \mathsf{v}_2 = c_2\tilde{\mathsf{v}}_2 \tag{2.3}$$

with $\tilde{\mathsf{u}}_1, \tilde{\mathsf{u}}_2, \tilde{\mathsf{v}}_1, \tilde{\mathsf{v}}_2$ *being Laurent polynomials in* $\mathbb{F}[z, z^{-1}]$ *and* $c_1, c_2 \in \mathbb{C}$ *being constants such that* $|c_1|^2, |c_2|^2 \in \mathbb{F}$. *Moreover, the symmetry of* $\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2$ *satisfies*

$$\frac{\mathsf{S}\mathsf{u}_1}{\mathsf{S}\mathsf{u}_2} = \frac{\mathsf{S}\mathsf{v}_1}{\mathsf{S}\mathsf{v}_2} \tag{2.4}$$

(3) *The supports of* $\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2$ *satisfy*

$$\begin{aligned} &\max(\mathrm{len}(\mathsf{u}_1\mathsf{u}_1^\star), \mathrm{len}(\mathsf{v}_1\mathsf{v}_1^\star)) \leqslant \mathrm{len}(\mathsf{p}), \\ &\max(\mathrm{len}(\mathsf{u}_2\mathsf{u}_2^\star), \mathrm{len}(\mathsf{v}_2\mathsf{v}_2^\star)) \leqslant \mathrm{len}(\mathsf{q}). \end{aligned} \tag{2.5}$$

Let us make some remarks about the necessary conditions of the main theorem. When applied to the construction of framelet filter banks, M is usually constructed from a low-pass filter and the conditions $\gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}) = \gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}^\star) = 1$ and $\mathsf{M} \geqslant 0$ are automatically satisfied (see Section 3 and examples in Section 4). Since $\mathsf{M} \geqslant 0$, $\det \mathsf{M}$ can be always factorized as $\det \mathsf{M} = \mathsf{d}\mathsf{d}^\star$ in view of the Fejér-Riesz lemma. Our only requirement is the symmetry constrains on the polynomial entries $\mathsf{p}, \mathsf{q}, \mathsf{r}$ of M and the determinant factor d of $\det \mathsf{M}$. Again, the symmetry of $\mathsf{p}, \mathsf{q}, \mathsf{r}$ will be automatically satisfied when the input low-pass filter has symmetry. And the condition on the determinant factor d of $\det \mathsf{M}$ can be pre-designed when constructing the low-pass filter (see Algorithm 2 in Section 3). We should point out that the key difference comparing to [19] is the relaxation of the condition $\det \mathsf{M} = \mathsf{d}\mathsf{d}^\star$, where d in [19] is required to be a Laurent polynomial in $\mathbb{Q}[z, z^{-1}]$ while here in our paper d is relaxed to be a Laurent polynomial in $\mathbb{F}[z, z^{-1}]$ up to a multiplicative constant. Such a relaxation condition yields a great many more examples with nice properties. More importantly, such a relaxation condition does not affect the advantages of using integer arithmetic over the rational number field $\mathbb{Q}$ as discussed in [14,19].

For the sufficient part, item (1) guarantees the perfect reconstruction property of a framelet filter bank derived via matrix splitting technique. Item (2) shows that each column of the factorization U is a vector of Laurent polynomials with symmetry in $\mathbb{F}[z, z^{-1}]$ up to a multiplicative constant. Item (3) provides a subtle support control of the entries in the factorization matrix U, which implies that the support length of U is about half of that of M.

Now, let us turn to the proof of the main theorem in this section. To this end, we need to introduce some auxiliary results. Before that, let us lay out the main idea of the proof of Theorem 1. The proof is constructive, which consists of mainly two parts. The first part is to take out the common factors of p and r, or q and r. More precisely, the first part is aiming at constructing a diagonal matrix $\mathsf{P} = \mathrm{diag}(\theta_1^{-1}, \theta_2^{-1})$ of Laurent polynomials such that $\mathsf{M}_1 := \mathsf{P}\mathsf{M}\mathsf{P}^\star$ has no common factors between pairs of entries. Then, the second part of the proof shows that M_1 must be of the form $\mathsf{M}_1 = \mathsf{U}_1\mathsf{U}_1^\star$ and U_1 can be easily obtained by simply solving a linear system of equations related to entries of M_1 (see (2.7)). Moreover, such a matrix U_1 satisfies properties as in items (1) – (3) in Theorem 1. In the following, Lemma 1 is about how to obtain the common factors of the entries of M and how to construct the

diagonal matrix P. Lemmas 2 and 3 are about the construction of the factorization matrix U_1 from M_1. We next introduce these lemmas for the proof of the main theorem.

Lemma 1. *Let* p, q, *and* r *be three Laurent polynomials in* $\mathbb{F}[z, z^{-1}]$ *with symmetry such that* $\gcd(p, q, r) = \gcd(p, q, r^\star) = 1$. *Let* M *be a* 2×2 *matrix of Laurent polynomials defined to be*

$$M(z) := \begin{bmatrix} p(z) & r(z) \\ r^\star(z) & q(z) \end{bmatrix}, \qquad z \in \mathbb{C}\backslash\{0\}.$$

Suppose that M *is positive semi-definite on* $\mathbb{T}$, *i.e.,* $M(z) \geqslant 0$ *for all* $z \in \mathbb{T}$, *and* $\det M = dd^\star$ *with* $d = c_0 d_0$ *for some Laurent polynomial* d_0 *in* $\mathbb{F}[z, z^{-1}]$ *with symmetry and some constant* $c_0 \in \mathbb{C}$ *satisfying* $|c_0|^2 \in \mathbb{F}$. *Define*

$$\eta_1(z) := \gcd(p(z), r(z)r^\star(z)), \quad \eta_2(z) := \gcd(q(z), r(z)r^\star(z)), \quad z \in \mathbb{C}\backslash\{0\}.$$

Then there exist Laurent polynomials θ_1 *and* θ_2 *in* $\mathbb{F}[z, z^{-1}]$ *with symmetry such that*

$$\eta_1(z) = c_1 z^{k_1}\theta_1(z)\theta_1^\star(z), \quad \eta_2(z) = c_2 z^{k_2}\theta_2(z)\theta_2^\star(z), \quad z \in \mathbb{C}\backslash\{0\} \tag{2.6}$$

with some trivial monomials $c_1 z^{k_1}$ *and* $c_2 z^{k_2}$ *in* $\mathbb{F}[z, z^{-1}]$. *Moreover,* $\gcd(\theta_1, \theta_2) = 1$.

Proof. We prove the result for η_1. The proof for the result of η_2 is similar.

(1) In view of $M \geqslant 0$ on $\mathbb{T}$, we have $p \geqslant 0$ and $q \geqslant 0$ on $\mathbb{T}$. Hence, by Fejér-Riesz lemma, $p = \tilde{p}\tilde{p}^\star$ and $q = \tilde{q}\tilde{q}^\star$ for some Laurent polynomials $\tilde{p}$ and $\tilde{q}$ in $\mathbb{C}[z, z^{-1}]$. Therefore, we have $M = M^\star$. Consequently, $\alpha \mid q$ implies $\alpha^\star \mid q$.

(2) We next show that $\eta_1 = c_1 z^{k_1}\theta_1\theta_1^\star$ for some Laurent polynomial θ_1 in $\mathbb{F}[z, z^{-1}]$. Let α be irreducible in $\mathbb{F}[z, z^{-1}]$ such that α is not a unit and $\alpha \mid \eta_1$. Since $\mathbb{F}[z, z^{-1}]$ is a UFD, every irreducible element is prime. Consequently, $\alpha \mid rr^\star$ implies $\alpha \mid r$ or $\alpha \mid r^\star$. Then, either $\alpha^\star \mid r^\star$ or $\alpha^\star \mid r$. In either case, we have $\alpha\alpha^\star \mid rr^\star$. Similarly, by that $dd^\star = pq - rr^\star$, we have $\alpha\alpha^\star \mid dd^\star$. Noting that $p = \frac{dd^\star + rr^\star}{q}$ and in addition with $\alpha \nmid q$ and $\alpha^\star \nmid q$ due to $\gcd(p, q, r) = \gcd(p, q, r^\star) = 1$, we conclude that $\alpha\alpha^\star \mid p$ and hence $\alpha\alpha^\star \mid \eta_1$. Consequently, η_1 must be of the form as in (2.6).

(3) We finally show the symmetry property of θ_1. If $\alpha \mid \eta_1$ is irreducible and without symmetry, i.e., $\alpha(z^{-1}) \nmid \alpha$ or equivalently $S\alpha$ is not a monomial. By $\alpha \mid rr^\star$, we have $\alpha \mid r$ or $\alpha \mid r^\star$. Then, by the symmetry of r (Sr is a monomial), we must have $\alpha(z^{-1}) \mid r$ or $\alpha(z^{-1}) \mid r^\star$. Let $\beta(z) := \alpha(z)\alpha(z^{-1})$. Then, β has symmetry and we have $\beta\beta^\star \mid rr^\star$. Similar to the proof in Step (2), we can also show that $\beta\beta^\star \mid p$. That is, $\beta\beta^\star \mid \eta_1$. Consequently, θ_1 must be a Laurent polynomial in $\mathbb{F}[z, z^{-1}]$ with symmetry.

That $\gcd(\theta_1, \theta_2) = 1$ follows from $\gcd(p, q, r) = \gcd(p, q, r^\star) = 1$. We are done. □

From the proof of Lemma 1, η_1 is of the form $\eta_1(z) = c_1 z^{k_1}\alpha_1(z)\alpha_1^\star(z)\cdots\alpha_n(z)\alpha_n^\star(z)$ for some irreducible Laurent polynomials $\alpha_1, \ldots, \alpha_n$ in $\mathbb{F}[z, z^{-1}]$. Each α_j satisfies $\alpha_j\alpha_j^\star \mid rr^\star$. Without loss of generality, we assume $\alpha_j \mid r$ for $j = 1, \ldots, n$. Choose $\theta_1 := \alpha_1 \cdots \alpha_n$. Then, $\eta_1(z) = c_1 z^{k_1}\theta_1(z)\theta_1^\star(z)$, and θ_1 satisfies $\theta_1 \mid r$ and $\theta_1^\star \mid r$. Similarly for η_2, we have $\eta_2(z) = c_2 z^{k_1}\theta_1(z)\theta_2^\star(z)$ with $\theta_2 \mid r^\star$ and $\theta_2^\star \mid r$. Consequently, we can define two operators as follows:

$$\eta_1(M) := \theta_1, \quad \eta_2(M) = \theta_2,$$

where $\theta_1 \mid r$ and $\theta_2 \mid r^\star$. We have the following corollary about taking all common row factors out of the 2×2 matrix M.

Corollary 1. *Let* p, q, r, d, *and* M *be the same as in Lemma 1. Let* $\theta_1 = \eta_1(\mathsf{M})$ *and* $\theta_2 = \eta_2(\mathsf{M})$ *be the two Laurent polynomials with symmetry obtained as in Lemma 1. Define*

$$\mathsf{P}(z) := diag\left(\frac{1}{\theta_1(z)}, \frac{1}{\theta_2(z)}\right) \text{ and } \tilde{\mathsf{M}}(z) := \mathsf{P}(z)\mathsf{M}(z)\mathsf{P}^{*}(z) =: \begin{bmatrix} \tilde{\mathsf{p}}(z) & \tilde{\mathsf{r}}(z) \\ \tilde{\mathsf{r}}^{*}(z) & \tilde{\mathsf{q}}(z) \end{bmatrix}, \ z \in \mathbb{C}\backslash\{0\}.$$

Then, $\tilde{\mathsf{M}}$ *is a* 2×2 *matrix of Laurent polynomials in* $\mathbb{F}[z, z^{-1}]$ *with symmetry. Moreover,* $\tilde{\mathsf{M}} \geqslant 0$ *on* $\mathbb{T}$, $\gcd(\tilde{\mathsf{p}}, \tilde{\mathsf{r}}\tilde{\mathsf{r}}^{*}) = \gcd(\tilde{\mathsf{q}}, \tilde{\mathsf{r}}\tilde{\mathsf{r}}^{*}) = 1$, *and* $\det \tilde{\mathsf{M}} = \tilde{\mathsf{d}}\tilde{\mathsf{d}}^{*}$ *with* $\tilde{\mathsf{d}} := \frac{\mathsf{d}}{\theta_1\theta_2}$.

Proof. This is a direct consequence of Lemma 1. □

We next introduce Lemmas 2 and 3. Since their proofs are a little bit longer, for simplicity of presentation, we postpone their proofs to Section 5 and proceed directly to the proof of Theorem 1. As stated in Corollary 1, once we take out the common factor of entries of M, we obtained a matrix $\tilde{\mathsf{M}}$ with certain property. We know by the matrix-valued Fejér-Riesz lemma (see [5]) that $\tilde{\mathsf{M}}$ can be factorized as $\tilde{\mathsf{M}} = \tilde{\mathsf{U}}\tilde{\mathsf{U}}^{*}$. The next lemma provide a way of obtaining such a $\tilde{\mathsf{U}}$ by solving a system of linear equations related to the entries of $\tilde{\mathsf{M}}$.

Lemma 2. *Let* p, q, *and* r *be nontrivial Laurent polynomials in* $\mathbb{C}[z, z^{-1}]$ *with symmetry such that* $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^{*}) = 1$. *Let* M *be defined as in* (2.1) *such that* M *is positive semi-definite on* $\mathbb{T}$ *and* $\det \mathsf{M} = \mathsf{d}\mathsf{d}^{*}$ *for some nontrivial Laurent polynomial* d *in* $\mathbb{C}[z, z^{-1}]$ *with symmetry. Suppose* $\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2$ *are Laurent polynomials with symmetry in* $\mathbb{C}[z, z^{-1}]$ *having the symmetry property as in* (2.4) *and the support control property as in* (2.5). *Then* (2.2) *holds, if and only if,* $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ *is a solution to the following linear system of equations*

$$\begin{cases} \mathsf{r}^{*}(z)\mathsf{u}_1(z) - \mathsf{d}(z)\mathsf{v}_1^{*}(z) - \mathsf{p}(z)\mathsf{u}_2(z) = 0 \\ \mathsf{r}^{*}(z)\mathsf{v}_1(z) + \mathsf{d}(z)\mathsf{u}_1^{*}(z) - \mathsf{p}(z)\mathsf{v}_2(z) = 0 \end{cases} \tag{2.7}$$

with the following normalization condition

$$|\mathsf{u}_1(1)|^2 + |\mathsf{v}_1(1)|^2 = \mathsf{p}(1). \tag{2.8}$$

Lemma 2 shows that if the factorization matrix U for $\mathsf{M} = \mathsf{U}\mathsf{U}^{*}$ exists, then it necessarily comes from a system of linear equations. Although we know that the existence of U is guaranteed by the matrix-valued Fejér-Riesz lemma, we do not know the symmetry structure of U. The next lemma shows the existence and symmetry structure of a nontrivial solution $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ to the system of linear equations as in (2.7).

Lemma 3. *Let* p, q, r *be nontrivial Laurent polynomials in* $\mathbb{C}[z, z^{-1}]$ *with symmetry such that* $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^{*}) = 1$. *Let* M *be defined as in* (2.1) *such that* M *is positive semi-definite on* $\mathbb{T}$ *and* $\det \mathsf{M} = \mathsf{d}\mathsf{d}^{*}$ *for some nontrivial Laurent polynomial* d *in* $\mathbb{C}[z, z^{-1}]$ *with symmetry. Then there exists* $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ *of Laurent polynomials in* $\mathbb{C}[z, z^{-1}]$ *with symmetry such that* (2.4), (2.5), (2.7), *and* (2.8) *hold.*

With the above lemmas in hand, we are ready to prove Theorem 1. Again, the main idea is to take out the common factor of M first and then factor the resulted matrix by solving a system of linear equations. In addition, we need to show that the solution to the system of equations is indeed of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ up to multiplicative constants.

Proof of Theorem 1. By Corollary 1, we can assume that $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^{*}) = 1$; otherwise, replace M by $\mathsf{P}\mathsf{M}\mathsf{P}^{*}$ with $\mathsf{P} := \operatorname{diag}\left(\frac{1}{\eta_1(\mathsf{M})}, \frac{1}{\eta_2(\mathsf{M})}\right)$ as in Corollary 1.

If $\mathsf{r}(z) \equiv 0$, by $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^{*}) = 1$, then p must be a positive constant and all the claims can be easily verified by taking $\mathsf{u}_1 = \sqrt{\mathsf{p}}$, $\mathsf{u}_2 = 0$, $\mathsf{v}_1 = 0$, and $\mathsf{v}_2 = \mathsf{d}/\sqrt{\mathsf{p}}$. If $\mathsf{d} \equiv 0$, then $\mathsf{p}\mathsf{q} = \mathsf{r}\mathsf{r}^{*}$. Since

$\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^\star) = 1$, p must be a positive constant. All the claims hold by taking $\mathsf{u}_1 = \sqrt{\mathsf{p}}$, $\mathsf{u}_2 = \mathsf{r}^\star/\sqrt{\mathsf{p}}$, $\mathsf{v}_1 = \mathsf{v}_2 = 0$. So, we can assume that both r and d are not identically zero.

First, by Lemma 2, we conclude that if $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ satisfies (2.4) and (2.5), then (2.2) holds if and only if $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ is the solution to the system of linear equation as in (2.7) with the normalization condition (2.8).

Second, by Lemma 3, there indeed exists $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ of Laurent polynomials in $\mathbb{C}[z, z^{-1}]$ with symmetry such that (2.4), (2.5), (2.7), and (2.8) hold.

Finally, we need to show that the solution $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ to the system of linear equations in (2.7) indeed satisfies the specific form as in (2.3). In fact, the linear system of equations (2.7) is equivalent to

$$\begin{cases} \mathsf{r}^\star(z)\frac{\mathsf{u}_1(z)}{c_0} - \mathsf{d}_0(z)\mathsf{v}_1^\star(z) - \mathsf{p}(z)\frac{\mathsf{u}_2(z)}{c_0} = 0 \\ \mathsf{r}^\star(z)\mathsf{v}_1(z) + |c_0|^2\mathsf{d}_0(z)\frac{\mathsf{u}_1^\star(z)}{\bar{c}_0} - \mathsf{p}(z)\mathsf{v}_2(z) = 0 \end{cases},$$

which is equivalent to

$$\begin{cases} \mathsf{r}^\star(z)\tilde{\mathsf{u}}_1(z) - \mathsf{d}_0(z)\mathsf{v}_1^\star(z) - \mathsf{p}(z)\tilde{\mathsf{u}}_2(z) = 0 \\ \mathsf{r}^\star(z)\mathsf{v}_1(z) + |c_0|^2\mathsf{d}_0(z)\tilde{\mathsf{u}}_1^\star(z) - \mathsf{p}(z)\mathsf{v}_2(z) = 0 \end{cases}. \tag{2.9}$$

That is, $\{\tilde{\mathsf{u}}_1, \tilde{\mathsf{u}}_2, \tilde{\mathsf{v}}_1, \tilde{\mathsf{v}}_2\}$ is a solution to (2.9) if and only if

$$\{\mathsf{u}_1 := c_0\tilde{\mathsf{u}}_1, \mathsf{u}_2 := c_0\tilde{\mathsf{u}}_2, \mathsf{v}_1 := \tilde{\mathsf{v}}_1, \mathsf{v}_2 := \tilde{\mathsf{v}}_2\}$$

is a solution to (2.7). Note that all coefficients of (2.9) are in $\mathbb{F}$. Hence, the solution $\{\tilde{\mathsf{u}}_1, \tilde{\mathsf{u}}_2, \tilde{\mathsf{v}}_1, \tilde{\mathsf{v}}_2\}$ must be of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$. In view of the normalization condition (2.8), the solution $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ to (2.7) must be of the form as in (2.3). We are done. □

2.2. Algorithm for matrix splitting with symmetry

In this subsection, we shall provide a step-by-step algorithm for deriving the factorization matrix U as in Theorem 1. From the constructive proof of Theorem 1, we know that the main steps of obtaining the factorization matrix U from M are taking out common factors and solving a system of linear equations. In the algorithm, we shall also employ *the extended Euclidean algorithm*, which reduces the complexity of the matrix splitting problem by lowering the support of the matrix M. More importantly, the extended Euclidean algorithm significantly reduces the complexity of the system of linear equations or sometimes it can even avoid solving such a system of linear equations. We next discuss about the extended Euclidean algorithm. See examples in Section 4 for its application.

We say that an $r \times r$ matrix M of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ is *Hermitian* if $\mathsf{M}^\star = \mathsf{M}$. We say that two $r \times r$ matrices A and B of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ are *complex congruent* if there exists an $r \times r$ *invertible matrix* P in $\mathbb{F}[z, z^{-1}]$; that is, $\det \mathsf{P}$ is a monomial in $\mathbb{F}[z, z^{-1}]$, such that $\mathsf{P}\mathsf{A}\mathsf{P}^\star = \mathsf{B}$. It is well-known that complex congruent is an equivalent relation.

Let M be a 2×2 matrix of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ with symmetry. Then, there exists a Laurent polynomial w with symmetry such that $\mathsf{S}\mathsf{M} = [1, \mathsf{S}w]^\star[1, \mathsf{S}w]$. Define

$$\mathcal{H}_w := \{\mathsf{M} : \mathsf{M} \text{ is a } 2 \times 2 \text{ Hermitian matrix satisfying } \mathsf{S}\mathsf{M} = [1, \mathsf{S}w]^\star[1, \mathsf{S}w]\}.$$

For each Laurent polynomial w with symmetry, there exist an $n \in \mathbb{Z}$ and a $w_0 \in \{1, 1+z, 1-z, z-z^{-1}\}$ such that $[\mathsf{S}w] = z^{2n}[\mathsf{S}w_0]$. Hence for each $\mathsf{M} \in \mathcal{H}_w$, there exists an $\mathsf{M}_0 \in \mathcal{H}_{w_0}$ such that

$$\mathsf{M}_0(z) = \begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} \mathsf{M}(z) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix},$$

i.e., M and M_0 are complex congruent. Thus, up to a complex congruent relation, we can assume $w \in \{1, 1+z, 1-z, z-z^{-1}\}$ and we take this assumption from here to the end of this subsection. A direct conclusion of this assumption is that if $\mathsf{Sp} = \mathsf{S}w$ for a Laurent polynomial p with symmetry, then $\mathsf{p} = w\mathsf{q}$ with q satisfying $\mathsf{Sq} = 1$.

Now for each $\mathsf{M} \in \mathcal{H}_w$, we want to use the extended Euclidean algorithm to find its canonical form under complex congruent relation. More precisely, we wish to construct an invertible matrix P such that $\mathsf{PMP}^\star$ has as small support length as possible. We first introduce the *one-step extended Euclidean algorithm*, which lower the support length of the non-diagonal entries of M.

Lemma 4 (One-step extended Euclidean algorithm). *Let $\mathsf{A} \in \mathcal{H}_w$ for some $w \in \{1, 1+z, 1-z, z-z^{-1}\}$. Then, there exists an invertible matrix P such that $\mathsf{B} := \mathsf{PAP}^\star \in \mathcal{H}_w$ and B satisfies*

$$\max\{\mathrm{len}([\mathsf{B}]_{1,1}), \mathrm{len}([\mathsf{B}]_{2,2})\} \leqslant \min\{\mathrm{len}([\mathsf{A}]_{1,1}), \mathrm{len}([\mathsf{A}]_{2,2})\}$$

and

$$\mathrm{len}([\mathsf{B}]_{1,2}) - \mathrm{len}(w) \leqslant \min\{\mathrm{len}([\mathsf{A}]_{1,1}), \mathrm{len}([\mathsf{A}]_{2,2})\} - 2.$$

Proof. Let $\mathsf{A} =: \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^\star & \mathsf{q} \end{bmatrix}$.

Without loss of generality, we can assume that $\mathrm{len}(\mathsf{p}) \leqslant \mathrm{len}(\mathsf{q})$; otherwise, consider $\Theta\mathsf{A}\Theta^\star \in \mathcal{H}_{w^\star}$ with $\Theta := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By the Euclidean algorithm and the fact that $\mathsf{SA} = [1, \mathsf{S}w]^\star[1, \mathsf{S}w]$, there exist Laurent polynomials s, t with symmetry such that

$$\mathsf{r}(z) = \mathsf{p}(z)s(z)w(z) + t(z). \tag{2.10}$$

Moreover, $\mathsf{S}s = 1$, $\mathsf{S}t = \mathsf{S}w$, and $\mathrm{len}(t) - \mathrm{len}(w) < \mathrm{len}(\mathsf{p})$. Since $\mathsf{S}t = \mathsf{S}w$, $\mathrm{len}(t) - \mathrm{len}(w)$ is even. Moreover, $\mathrm{len}(\mathsf{p})$ is also even since $\mathsf{Sp} = 1$. Thus,

$$\mathrm{len}(t) - \mathrm{len}(w) \leqslant \mathrm{len}(\mathsf{p}) - 2.$$

Now define

$$\mathsf{P} := \begin{bmatrix} 1 & \\ -s^\star w^\star & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{B} := \mathsf{PAP}^\star = \begin{bmatrix} \mathsf{p} & t \\ t^\star & \tilde{\mathsf{q}} \end{bmatrix}, \tag{2.11}$$

where $\tilde{\mathsf{q}} = (sw)(sw)^\star\mathsf{p} - sw\mathsf{r}^\star - (sw)^\star\mathsf{r} + \mathsf{q}$. Obviously, $\det\mathsf{P} = 1$ and thus $\det\mathsf{B} = \det\mathsf{A}$. Consequently, by $\mathrm{len}(t) \leqslant \mathrm{len}(\mathsf{r})$, we have $\mathrm{len}(\tilde{\mathsf{q}}) \leqslant \mathrm{len}(\mathsf{q})$. We are done! □

We shall use $\mathrm{quo}(\mathsf{r}, \mathsf{q}) := sw$ to denote the quotient polynomials obtained in (2.10). After applying the one-step extended Euclidean algorithm to a matrix A and obtaining a matrix B as in (2.11), if we still have $\mathrm{len}(t) \geqslant \mathrm{len}(\tilde{\mathsf{q}})$, then we can continue to apply the same algorithm. Eventually, the matrix must satisfy the condition that the length of the non-diagonal entry is no larger than that of any diagonal entry of the matrix. We say a matrix $\mathsf{A} \in \mathcal{H}_w$ is *irreducible* if

$$\mathrm{len}([\mathsf{A}]_{1,2}) - \mathrm{len}(w) \leqslant \min\{\mathrm{len}([\mathsf{A}]_{1,1}), \mathrm{len}([\mathsf{A}]_{2,2})\} - 2.$$

For any matrix $\mathsf{M} \in \mathcal{H}_w$, up to a finite step of one-step extended Euclidean algorithm, it is necessary complex congruent to an irreducible matrix in $\mathcal{H}_w$. Consequently, we have the following corollary concerning the *extended Euclidean algorithm*; see Algorithm 1 Lines 2 – 12 for its pseudo code.

Corollary 2 (The extended Euclidean algorithm). *For every* $\mathsf{M} \in \mathcal{H}_w$, M *is complex congruent to an irreducible matrix* $\tilde{\mathsf{M}} \in \mathcal{H}_w$*; that is, we can construct an invertible matrix* P *such that* $\tilde{\mathsf{M}} := \mathsf{PMP}^{\star}$ *is irreducible.*

Proof. Define $\mathsf{B}_0 := \mathsf{M}$. Applying Lemma 4 to B_j for $j = 0, 1, \ldots$, We can recursively construct invertible matrix P_j and $\mathsf{B}_{j+1} := \mathsf{P}_j\mathsf{B}_j\mathsf{P}_j^{\star} \in \mathcal{H}_w$ such that

$$\max\{\mathrm{len}([\mathsf{B}_{j+1}]_{1,1}), \mathrm{len}([\mathsf{B}_{j+1}]_{2,2})\} \leqslant \min\{\mathrm{len}([\mathsf{B}_j]_{1,1}), \mathrm{len}([\mathsf{B}_j]_{2,2})\}$$

and

$$\mathrm{len}([\mathsf{B}_{j+1}]_{1,2}) - \mathrm{len}(w) \leqslant \min\{\mathrm{len}([\mathsf{B}_j]_{1,1}), \mathrm{len}([\mathsf{B}_j]_{2,2})\} - 2.$$

By the above recursive construction, we have constructed a sequences of matrices $\mathsf{B}_0, \mathsf{B}_1, \ldots$ Suppose that B_j is not irreducible for $j = 1, \ldots, N$. Then by the definition of irreducibility and the construction of B_j, $j = 1, \ldots, N$, we have

$$\min\{\mathrm{len}([\mathsf{B}_j]_{2,2}), \mathrm{len}([\mathsf{B}_j]_{1,1})\} < \min\{\mathrm{len}([\mathsf{B}_{j-1}]_{2,2}), \mathrm{len}([\mathsf{B}_{j-1}]_{1,1})\}, \quad j = 1, \ldots, N.$$

The above inequality shows that N is bounded above. Thus there exists a $j \in \mathbb{N}_0$ such that B_j is irreducible. Denote J to be the minimum positive integer such that B_J is irreducible. Now define $\mathsf{P} := \mathsf{P}_0\mathsf{P}_1 \cdots \mathsf{P}_{J-1}$ and $\tilde{\mathsf{M}} := \mathsf{PMP}^{\star}$. By the above construction, P and $\tilde{\mathsf{M}}$ are the two required matrices. □

Now, according to the proof of Theorem 1 and the above discussion of the extended Euclidean algorithm, we have a step-by-step algorithm for the matrix splitting with symmetry, see Algorithm 1, which consists of three main steps: taking out common factors, performing the extended Euclidean algorithm, and solving a system of linear equations if necessary.

3. Construction of dyadic algebraic framelet filter banks with symmetry

In this section, we shall discuss the application of our results on the matrix splitting problem with symmetry in Section 2 to the construction of dyadic algebraic framelet filter banks with symmetry in electronic engineering and wavelet analysis.

Before proceeding further, let us review some definitions and notation. Recall that $\mathbb{F}$ always denotes a general subfield of $\mathbb{C}$ satisfying (1.1). A filter $a = \{a(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{F}$ is a finitely supported sequence on $\mathbb{Z}$. The *z-transform or symbol* of the filter a is defined to be

$$\mathsf{a}(z) := \sum_{k\in\mathbb{Z}} a(k)z^k, \quad z \in \mathbb{C} \setminus \{0\},$$

which is a Laurent polynomial in $\mathbb{F}[z, z^{-1}]$. We say that a function ϕ is *refinable* if it satisfies the following *refinement equation*

$$\hat{\phi}(2\xi) = \mathsf{a}(e^{-i\xi})\hat{\phi}(\xi) \tag{3.1}$$

with a being the symbol of some filter $a : \mathbb{Z} \to \mathbb{F}$, which is usually called a *low-pass filter* for ϕ. Here, the *Fourier transform* of a function $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$ and it can be naturally extended to functions in $L_2(\mathbb{R})$.

Define $f_{j,k} := 2^{j/2}f(2^j \cdot -k)$ for a function $f \in L_2(\mathbb{R})$. We say that $\{\phi; \psi^1, \ldots, \psi^s\}$ generates a *(normalized) tight wavelet frame* if

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{0,k}\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} |\langle f, \psi^{\ell}_{j,k}\rangle|^2 = \|f\|_2 \qquad \forall f \in L_2(\mathbb{R}),$$

Algorithm 1 Matrix splitting with symmetry

(a) **Input.** A 2×2 matrix M of Laurent polynomials with symmetry as in Theorem 1.
(b) **Output.** A 2×2 matrix U satisfies items (1) – (3) of Theorem 1.
(c) **Matrix splitting with symmetry.**

Step I. Taking out common factors

$\mathsf{P}_1 \leftarrow \operatorname{diag}\left(\frac{1}{\eta_1(\mathsf{M})}, \frac{1}{\eta_2(\mathsf{M})}\right)$, and $\mathsf{M}_1 \leftarrow \mathsf{P}_1\mathsf{M}\mathsf{P}_1^{\star}$.

Step II. Performing the extended Euclidean algorithm if necessary

$\mathsf{P}_2 \leftarrow \mathsf{I}_2$ and $\mathsf{M}_2 \leftarrow \mathsf{M}_1$.
$\mathsf{p}_2 \leftarrow [\mathsf{M}_2]_{1,1}$, $\mathsf{r}_2 \leftarrow [\mathsf{M}_2]_{1,2}$, $\mathsf{q}_2 \leftarrow [\mathsf{M}_2]_{2,2}$.
while $\operatorname{len}(\mathsf{r}_2) > \min\{\operatorname{len}(\mathsf{p}_2), \operatorname{len}(\mathsf{q}_2)\}$ **do**
  if $\operatorname{len}(\mathsf{p}_2) < \operatorname{len}(\mathsf{q}_2)$ **then**
    $\mathsf{w} \leftarrow \operatorname{quo}(\mathsf{r}_2, \mathsf{p}_2)$ and $\mathsf{Q} \leftarrow \begin{bmatrix} 1 & 0 \\ -\mathsf{w}^{\star} & 1 \end{bmatrix}$.
  else
    $\mathsf{w} \leftarrow \operatorname{quo}(\mathsf{r}_2, \mathsf{q}_2)$ and $\mathsf{Q} \leftarrow \begin{bmatrix} 1 & -\mathsf{w} \\ 0 & 1 \end{bmatrix}$.
  end if
  $\mathsf{M}_2 \leftarrow \mathsf{Q}\mathsf{M}_2\mathsf{Q}^{\star}$ and $\mathsf{P}_2 \leftarrow \mathsf{Q}\mathsf{P}_2$.
  $\mathsf{p}_2 \leftarrow [\mathsf{M}_2]_{1,1}$, $\mathsf{r}_2 \leftarrow [\mathsf{M}_2]_{1,2}$, $\mathsf{q}_2 \leftarrow [\mathsf{M}_2]_{2,2}$.
end while

Step III. Solving a system of linear equations if necessary

$\mathsf{P}_3 \leftarrow \mathsf{I}_2$, $\mathsf{U}_3 \leftarrow \mathsf{I}_2$ and $\mathsf{M}_3 \leftarrow \mathsf{M}_2$.
if M_3 is diagonal **then**
  M_3 must be of the form $\mathsf{M}_3 = \operatorname{diag}(\mathsf{w}_1\mathsf{w}_1^{\star}, \mathsf{w}_2\mathsf{w}_2^{\star})$ for some $\mathsf{w}_1, \mathsf{w}_2$ of Laurent polynomials with symmetry such that $\mathsf{w}_1 = c_1\tilde{\mathsf{w}}_1$ and $\mathsf{w}_2 = c_2\tilde{\mathsf{w}}_2$ with $\tilde{\mathsf{w}}_1$, $\tilde{\mathsf{w}}_2$ being Laurent polynomial in $\mathbb{F}[z, z^{-1}]$ and c_1, c_2 being complex numbers such that $|c_1|^2, |c_2|^2 \in \mathbb{F}$.
  $\mathsf{U}_3 \leftarrow \operatorname{diag}(\mathsf{w}_1, \mathsf{w}_2)$.
else
  $\mathsf{P}_3 \leftarrow \operatorname{diag}\left(\frac{1}{\eta_1(\mathsf{M}_3)}, \frac{1}{\eta_2(\mathsf{M}_3)}\right)$, $\mathsf{M}_3 \leftarrow \mathsf{P}_3\mathsf{M}_3\mathsf{P}_3^{\star}$.
  $\mathsf{p}_3 \leftarrow [\mathsf{M}_3]_{1,1}$, $\mathsf{r}_3 \leftarrow [\mathsf{M}_3]_{1,2}$, $\mathsf{q}_3 \leftarrow [\mathsf{M}_3]_{2,2}$. By Lemmas 2 and 3, we can solve a system of linear equations

$$\begin{cases} \mathsf{r}_3^{\star}(z)\mathsf{u}_1(z) - \mathsf{d}_3(z)\mathsf{v}_1^{\star}(z) - \mathsf{p}_3(z)\mathsf{u}_2(z) = 0 \\ \mathsf{r}_3^{\star}(z)\mathsf{v}_1(z) + \mathsf{d}_3(z)\mathsf{u}_1^{\star}(z) - \mathsf{p}_3(z)\mathsf{v}_2(z) = 0 \end{cases}$$

  to obtain $\mathsf{U}_3 := \begin{bmatrix} \mathsf{u}_1 & \mathsf{v}_1 \\ \mathsf{u}_2 & \mathsf{v}_2 \end{bmatrix}$ such that $\mathsf{M}_3 = \mathsf{U}_3\mathsf{U}_3^{\star}$ with $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ satisfying conditions (2.3), (2.4), and (2.5) for M_3. See (5.4) in Section 5 on how to define $\mathsf{u}_1, \mathsf{v}_1, \mathsf{u}_2, \mathsf{v}_2$.
end if
$\mathsf{U} \leftarrow (\mathsf{P}_3\mathsf{P}_2\mathsf{P}_1)^{-1}\mathsf{U}_3$.

where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}dx$ for all $f, g \in L_2(\mathbb{R})$. In this paper, we are particularly interested in *MRA-based tight wavelet frames*. That is, the *framelet generators* $\psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$ are from the refinable function $\phi \in L_2(\mathbb{R})$ associated with a low-pass filter a by the following way:

$$\hat{\psi}^{\ell}(2\xi) = \mathsf{b}_{\ell}(e^{-i\xi})\hat{\phi}(\xi), \qquad \ell = 1, \ldots, s, \tag{3.2}$$

where $\mathsf{b}_1, \ldots, \mathsf{b}_s$ are symbols for some filters $b_1, \ldots, b_s : \mathbb{Z} \to \mathbb{F}$ and are called *high-pass filters*.

If ϕ is a compactly supported refinable function in $L_2(\mathbb{R})$ associated with a low-pass filter a, then it is well-known (see [5,22]) that $\{\phi; \psi^1, \ldots, \psi^s\}$ associated with the filter system $\{a; b_1, \ldots, b_s\}$ via (3.1) and (3.2) generates a tight wavelet frame if and only if

$$\begin{cases} \mathsf{a}(z)\mathsf{a}^\star(z) + \sum_{\ell=1}^s \mathsf{b}_\ell(z)\mathsf{b}_\ell^\star(z) = 1 \\ \mathsf{a}(z)\mathsf{a}^\star(-z) + \sum_{\ell=1}^s \mathsf{b}_\ell(z)\mathsf{b}_\ell^\star(-z) = 0 \end{cases} \qquad z \in \mathbb{C}\backslash\{0\}, \tag{3.3}$$

which is the so-called *unitary extension principle (UEP)*. A filter system $\{a; b_1, \ldots, b_s\}$ (or $\{\mathsf{a}; \mathsf{b}_1, \ldots, \mathsf{b}_s\}$) satisfies (3.3) is called a *dyadic framelet filter bank (with the perfect reconstruction property)*. By defining

$$\mathsf{U}(z) := \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} \quad \text{and} \quad \mathsf{M}(z) := \begin{bmatrix} 1 - \mathsf{a}(z)\mathsf{a}^\star(z) & -\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\mathsf{a}^\star(z)\mathsf{a}(-z) & 1 - \mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix},$$

one can easily show that (3.3) is equivalent to

$$\mathsf{U}\mathsf{U}^\star = \mathsf{M}. \tag{3.4}$$

According to various requirements of problems in applications, different desired properties of a framelet system are needed, which usually can be characterized by conditions on the low-pass filter a for ϕ and the high-pass filters $b_1, \ldots, b_s$ for $\psi^1, \ldots, \psi^s$. Among all properties of a framelet system, regularity, high order of vanishing moments, and symmetry are highly desirable properties in wavelet and filter bank applications. High order of vanishing moments is crucial for the sparsity representation of a framelet system, which plays an important role in image denoising and compression. Symmetry usually produces better visual effect and less artifact in signal/image processing; not to mention the double reduction of the computational cost for a symmetric system. Moreover, as pointed out in the introduction, it is also desirable to construct framelets whose associated filter bank consists of filters with coefficients in an algebraic number field.

A tight wavelet frame $\{\phi; \psi^1, \ldots, \psi^s\}$ has *vanishing moments* of order n if

$$\int x^j \psi^\ell(x)dx = 0 \quad j = 0, \ldots, n-1; \quad \ell = 1, 2, \ldots, s.$$

In terms of condition on the framelet filter bank, one can show that an MRA-based tight wavelet frame has vanishing moments of order n if and only if

$$(z - 2 + z^{-1})^n \mid [1 - \mathsf{a}\mathsf{a}^\star].$$

The regularity of framelet generators is closely related to polynomial reproducibility of the refinable function ϕ, which can be characterized in terms of sum rule for the low-pass filter a. A low-pass filter a (or a) has *sum rules* of order m if

$$(1 + z)^m \mid \mathsf{a}.$$

A Haar system $\{\frac{1}{2}(1 + z), \frac{1}{2}(1 - z)\}$ is a good example to explain why it is desirable to construct dyadic framelet filter banks with more than one high-pass filters. In this case, it has a low-pass filter $\mathsf{a}(z) = \frac{1}{2}(1 + z)$ that is symmetric about $\frac{1}{2}$ and only one high-pass filter $\mathsf{b}(z) = \frac{1}{2}(1 - z)$ that is antisymmetric about $\frac{1}{2}$. It is a well-known fact that if $\{\mathsf{a}; \mathsf{b}\}$ is a dyadic framelet filter bank with the perfect reconstruction property and with symmetry, then it must be a variant of the Haar type system. However, a Haar type system is lack of regularity and has only one vanishing moment. On the one hand, to have framelet filter bank with symmetry, one necessarily need to consider the construction of a framelet filter bank $\{\mathsf{a}; \mathsf{b}_1, \ldots, \mathsf{b}_s\}$ with more than one high-pass filters. On the other hand, though there is no restriction on the number of filters in a framelet filter bank, in the point of view of

application, too many generators will definitely affect the efficiency of the algorithms employing the framelet transform. Therefore, it is also desirable to construct a framelet filter bank having as fewer number of generators as possible with certain desired properties. Moreover, coefficients of the filters should have certain structure.

From the above discussion, we therefore shall focus our study on the framelet filter system $\{\mathsf{a};\, \mathsf{b}_1,\, \mathsf{b}_2\}$. In this case, equation (3.4) becomes

$$\mathsf{U}\mathsf{U}^{\star} = \mathsf{M} \tag{3.5}$$

with

$$\mathsf{U}(z) := \begin{bmatrix} \mathsf{b}_1(z) & \mathsf{b}_2(z) \\ \mathsf{b}_1(-z) & \mathsf{b}_2(-z) \end{bmatrix} \quad \text{and} \quad \mathsf{M}(z) := \begin{bmatrix} 1 - \mathsf{a}(z)\mathsf{a}^{\star}(z) & -\mathsf{a}(z)\mathsf{a}^{\star}(-z) \\ -\mathsf{a}^{\star}(z)\mathsf{a}(-z) & 1 - \mathsf{a}(-z)\mathsf{a}^{\star}(-z) \end{bmatrix} \tag{3.6}$$

We next provide a step-by-step algorithm for constructing a framelet filter bank $\{\mathsf{a};\, \mathsf{b}_1,\, \mathsf{b}_2\}$ with the perfect reconstruction property and with symmetry. See Algorithm 2 below for a summary of the algorithm, which consists of three main steps: construction of a low-pass filter a satisfying certain sum rule condition, vanishing moment condition, and symmetry condition; application of the matrix splitting with symmetry to the factorization of the matrix M constructed from a; derivation of the high-pass filters b_1, b_2 from the factorization matrix U from M.

Algorithm 2 Construction of dyadic algebraic framelet filter banks with symmetry

(a) **Low-pass filter.** Set up a to satisfy three necessary conditions.

- (a.1) The sum rule condition: $(1+z)^m \mid \mathsf{a}$.
- (a.2) The vanishing moment condition: $(z - 2 + z^{-1})^n \mid [1 - \mathsf{a}\mathsf{a}^{\star}]$.
- (a.3) The splitting condition: $1-\mathsf{a}(z)\mathsf{a}^{\star}(z)-\mathsf{a}(-z)\mathsf{a}^{\star}(-z) = \mathsf{d}(z^2)\mathsf{d}^{\star}(z^2)$ with d being a Laurent polynomial with symmetry.

(b) **Matrix splitting with symmetry.** Application of Algorithm 1.

- (b.1) Define M as in (3.6) and take almost all common row factors out of M to obtain $\mathsf{M}_1 = \mathsf{P}_1\mathsf{M}\mathsf{P}_1^{\star}$ as in (3.8).
- (b.2) Separate polyphase from M_1 to obtain $\mathsf{M}_2 = \mathsf{P}_2\mathsf{M}_1\mathsf{P}_2^{\star} = \tilde{\mathsf{M}}_2(z^2)$ as in (3.9).
- (b.3) Application of Algorithm 1 to $\tilde{\mathsf{M}}_2$ to obtain a matrix U_2 of Laurent polynomials with symmetry such that $\tilde{\mathsf{M}}_2 = \mathsf{U}_2\mathsf{U}_2^{\star}$.

(c) **High-pass filters.** Compute $\mathsf{U}(z) := (\mathsf{P}_2(z)\mathsf{P}_1(z))^{-1}\mathsf{U}_2(z^2)$ and define $\mathsf{b}_1 := [\mathsf{U}]_{1,1}$ and $\mathsf{b}_2 := [\mathsf{U}]_{1,2}$. Then $\{\mathsf{a};\, \mathsf{b}_1,\, \mathsf{b}_2\}$ is a framelet filter bank with the perfect reconstruction property and with symmetry.

In the next three subsections, let us detail the three main steps of Algorithm 2.

3.1. Construction of the low-pass filter

For the construction of a low-pass filter a satisfying the three necessary conditions, we can define a to be

$$\mathsf{a}(z) = z^{-\lfloor m/2 \rfloor} \cdot \left(\frac{1+z}{2}\right)^m \left[1 + \sum_{k=1}^{N} c_k \left(\frac{2 - z - z^{-1}}{4}\right)^k\right] \tag{3.7}$$

with a positive integer m, a nonnegative integer N and some unknowns coefficients $c_1, \ldots, c_N$. The form of a in (3.7) guarantees that a is a low-pass filter having sum rules of order m and having symmetry.

For a given n, the vanishing moment condition $(z - 2 + z^{-1})^n \mid [1 - \mathsf{a}\mathsf{a}^\star]$ together with the splitting condition $1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z) = \mathsf{d}(z^2)\mathsf{d}^\star(z^2)$ is equivalent to a system of quadratic equations on the coefficients $c_1, \ldots, c_N$. Solving such a system of quadratic equations usually gives rise to a low-pass filter a with coefficients in an algebraic number filed $\mathbb{F}$, e.g., $\mathbb{Q}$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6}i)$, etc; see examples in Section 4.

For arbitrary $m, n \in \mathbb{N}$, it is not necessary that the condition—requiring d to be of the form $\mathsf{d} = c_0\mathsf{d}_0$, where $|c_0|^2 \in \mathbb{F}$ and d_0 is a Laurent polynomial with symmetry in $\mathbb{F}[z, z^{-1}]$—is satisfied. However, there do exist families of low-pass filters satisfying the three conditions for certain pairs of (m, n), for more detail, see [25,28].

3.2. Splitting the matrix M

Once we obtain a low-pass filter a satisfying the three necessary conditions as in a.1), a.2), and a.3), we next go over steps b.1) – b.3) on how to split the matrix M as in (3.5). Define M to be the matrix as in (3.6). Since $\det \mathsf{M}(z) = \mathsf{d}(z^2)\mathsf{d}^\star(z^2) = 1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z)$, it is easy to verify that

$$\gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}) = \gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}^\star) = 1 \quad \text{with} \quad \mathsf{M} =: \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^\star & \mathsf{q} \end{bmatrix}.$$

By Theorem 1, we can obtain U as in (3.5) by solving a system of linear equations. However, to further utilize the structure of the matrix M and simplify the complexity of the system of linear equations, we shall perform several steps of preprocessing.

(b.1) Take almost all common row factors out of M to obtain M_1 as in (3.8). In order to utilize the multi-phase separation, we might need to leave one common row factor $(1 - z)$ to make sure $[\mathsf{S}\mathsf{M}_1] = [1, 1]^T[1, 1]$. More specifically, define $\tilde{\theta}_1 := \eta_1(\mathsf{M})$ as in Lemma 1. By that $(1+z)^m \mid \mathsf{a}$, $(z - 2 + z^{-1})^n \mid [1 - \mathsf{a}\mathsf{a}^\star]$, and the definition of $\tilde{\theta}_1$, we have $1 + z \nmid \tilde{\theta}_1$ and there exists a maximal positive number k such that $(1 - z)^k \mid \tilde{\theta}_1$. Define

$$\theta_1(z) := \begin{cases} \tilde{\theta}_1(z) & m - k \text{ even} \\ \frac{\tilde{\theta}_1(z)}{1-z} & m - k \text{ odd}. \end{cases}$$

By this definition and direct calculation, we have

$$[\mathsf{S}\mathsf{a}(\cdot)\mathsf{a}^\star(-\cdot)](z) = (-1)^m = [\mathsf{S}\theta_1(\cdot)\theta_1^\star(-\cdot)](z).$$

Define $\mathsf{P}_1(z) := \operatorname{diag}\left(\frac{1}{\theta_1(z)}, \frac{1}{\theta_1(-z)}\right)$. Let

$$\mathsf{M}_1(z) := \mathsf{P}_1(z)\mathsf{M}_1(z)\mathsf{P}_1^\star(z) = \begin{bmatrix} \dfrac{1 - \mathsf{a}(z)\mathsf{a}^\star(z)}{\theta_1(z)\theta_1^\star(z)} & \dfrac{-\mathsf{a}(z)\mathsf{a}^\star(-z)}{\theta_1(z)\theta_1^\star(-z)} \\ \dfrac{-\mathsf{a}(-z)\mathsf{a}^\star(z)}{\theta_1(-z)\theta_1^\star(z)} & \dfrac{1 - \mathsf{a}(-z)\mathsf{a}^\star(-z)}{\theta_1(-z)\theta_1^\star(-z)} \end{bmatrix}. \tag{3.8}$$

By the definition of θ_1, M_1 is a matrix of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ with symmetry. Furthermore, M_1 is positive semi-definite on $\mathbb{T}$ and $\mathsf{S}\mathsf{M}_1 = [1, 1]^T[1, 1]$.

(b.2) Separate polyphase components from M_1 to obtain M_2 as in (3.9). More precisely, define

$$\mathsf{P}_2(z) := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -z & z \end{bmatrix} \quad \text{and} \quad \mathsf{M}_2(z) := \mathsf{P}_2(z)\mathsf{M}_1(z)\mathsf{P}_2^\star(z). \tag{3.9}$$

It is easily verified that M_2 is a well-defined matrix which is positive semi-definite on $\mathbb{T}$. Note that $\mathsf{SP}_2 = [1, z^2]^T[1, 1]$ and $\mathsf{SM}_1 = [1, 1]^T[1, 1]$. Hence, $\mathsf{SM}_2 = \mathsf{S}(\mathsf{P}_2\mathsf{M}_1\mathsf{P}_2^\star) = [1, z^2]^T[1, z^{-2}]$. Moreover, in view of the polyphase separation, we can show that M_2 is of the form $\mathsf{M}_2(z) = \tilde{\mathsf{M}}_2(z^2)$ for some matrix $\tilde{\mathsf{M}}_2$ of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ with symmetry.

(b.3) Application of Algorithm 1 to the matrix $\tilde{\mathsf{M}}_2$ to obtain U_2 such that $\tilde{\mathsf{M}}_2 = \mathsf{U}_2\mathsf{U}_2^\star$. See details in Algorithm 1.

By utilizing the polyphase separation, we significantly reduce the complexity of the system of linear equations involved in the splitting steps. In some cases, we don't even have to solve a system of linear equations since the extended Euclidean algorithm applying to $\tilde{\mathsf{M}}_2$ already reduces $\tilde{\mathsf{M}}_2$ to be a diagonal matrix, in which case, the splitting is trivial; see examples in Section 4.

3.3. Derivation of high-pass filters

By part b) of Algorithm 2, the matrix M can be represented by $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$ with

$$\mathsf{U}(z) := (\mathsf{P}_2(z)\mathsf{P}_1(z))^{-1}\mathsf{U}_2(z^2) = \begin{bmatrix} \theta_1(z) & 0 \\ 0 & \theta_1(-z) \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} \\ 1 & z^{-1} \end{bmatrix} \begin{bmatrix} \mathsf{u}_1(z^2) & \mathsf{v}_1(z^2) \\ \mathsf{u}_2(z^2) & \mathsf{v}_2(z^2) \end{bmatrix},$$

where $\mathsf{U}_2 = \begin{bmatrix} \mathsf{u}_1 & \mathsf{v}_1 \\ \mathsf{u}_2 & \mathsf{v}_2 \end{bmatrix}$. Consequently, U is of the form

$$\mathsf{U}(z) = \begin{bmatrix} \theta_1(z)(\mathsf{u}_1(z^2) - z^{-1}\mathsf{u}_2(z^2)) & \theta_1(z)(\mathsf{v}_1(z^2) - z^{-1}\mathsf{v}_2(z^2)) \\ \theta_1(-z)(\mathsf{u}_1(z^2) + z^{-1}\mathsf{u}_2(z^2)) & \theta_1(-z)(\mathsf{v}_1(z^2) + z^{-1}\mathsf{v}_2(z^2)) \end{bmatrix}.$$

By defining $\mathsf{b}_1 := [\mathsf{U}]_{1,1}$ and $\mathsf{b}_2 := [\mathsf{U}]_{1,2}$, we have

$$\mathsf{U}(z) = \begin{bmatrix} \mathsf{b}_1(z) & \mathsf{b}_2(z) \\ \mathsf{b}_1(-z) & \mathsf{b}_2(-z) \end{bmatrix} \quad \text{and} \quad \mathsf{U}\mathsf{U}^\star = \mathsf{M}.$$

Noting that $\mathsf{SM}_2 = [1, z^2]^T[1, z^{-2}]$ and $\tilde{\mathsf{M}}_2 = \mathsf{U}_2\mathsf{U}_2^\star$, we have $\mathsf{S}(\mathsf{u}_1\mathsf{u}_2^\star) = \mathsf{S}(\mathsf{v}_1\mathsf{v}_2^\star) = z^{-1}$. Hence $\mathsf{S}(\mathsf{u}_1(z^2)) = \mathsf{S}(\mathsf{u}_2(z^2)) \cdot z^{-2} = \mathsf{S}(z^{-1}\mathsf{u}_2(z^2))$. Similarly, $\mathsf{S}(\mathsf{v}_1(z^2)) = \mathsf{S}(z^{-1}\mathsf{v}_2(z^2))$. Therefore, $\mathsf{Sb}_1 = \mathsf{S}(\theta_1(z)\mathsf{u}_1(z^2))$ and $\mathsf{Sb}_2 = \mathsf{S}(\theta_1(z)\mathsf{v}_1(z^2))$. Consequently, the filter system $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ is a framelet filter bank with the perfect reconstruction property and with symmetry.

3.4. Main theorem

Now, summarizing discussion in the above subsections, we have the following theorem of this section.

Theorem 2. *Let* a *be a Laurent polynomial in* $\mathbb{F}[z, z^{-1}]$ *such that* $\mathsf{Sa} = z^{k_0}$ *for some* $k_0 \in \mathbb{Z}$. *Then, there exist* $\mathsf{b}_1, \mathsf{b}_2$ *such that*

(1) $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ *forms a framelet filter bank with the perfect reconstruction property, i.e.,* (3.5) *holds;*
(2) $\max\{\mathrm{len}(\mathsf{b}_1), \mathrm{len}(\mathsf{b}_2)\} \leqslant \mathrm{len}(\mathsf{a})$;
(3) $\mathsf{Sb}_1 = \epsilon_1 z^{k_1}$ *and* $\mathsf{Sb}_2 = \epsilon_2 z^{k_2}$ *for some* $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ *and some* $k_1, k_2 \in \mathbb{Z}$;
(4) $\mathsf{b}_1 = c_1\tilde{\mathsf{b}}_1$, $\mathsf{b}_2 = c_2\tilde{\mathsf{b}}_2$ *for some Laurent polynomials* $\tilde{\mathsf{b}}_1, \tilde{\mathsf{b}}_2 \in \mathbb{F}[z, z^{-1}]$ *with symmetry and some constants* c_1, c_2 *satisfying* $|c_1|^2, |c_2|^2 \in \mathbb{F}$;

if and only if,

$$1 - \mathsf{a}(z)\mathsf{a}^*(z) - \mathsf{a}(-z)\mathsf{a}^*(-z) = c_0\mathsf{d}(z^2)\mathsf{d}^*(z^2) \tag{3.10}$$

for some Lauent polynomial $\mathsf{d} \in \mathbb{F}[z, z^{-1}]$ *with symmetry and some non-negative constant* $c \in \mathbb{F}$. *Furthermore, if* $\mathsf{a}(0) = 1$, *then* $\{\phi; \psi^1, \psi^2\}$ *associated with* $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ *via* (3.1) *and* (3.2) *generates a tight wavelet frame in* $L_2(\mathbb{R})$. *Moreover, if* $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ *has symmetry as above, then* $\{\phi; \psi^1, \psi^2\}$ *has the following symmetry property:*

$$\phi = \phi(k_0 - \cdot), \quad \psi^1 = \epsilon_1\psi^1(\tfrac{k_1+k_0}{2} - \cdot), \quad \psi^2 = \epsilon_2\psi^2(\tfrac{k_2+k_0}{2} - \cdot). \tag{3.11}$$

Proof. The necessity is obvious. Define M to be the matrix as in (3.3). Since $\det(\mathsf{M}) = \mathsf{d}(z^2)\mathsf{d}^*(z^2) = 1 - \mathsf{a}(z)\mathsf{a}^*(z) - \mathsf{a}(-z)\mathsf{a}^*(-z)$, it is easy to verify that

$$\gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}) = \gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}^*) = 1 \quad \text{with} \quad \mathsf{M} =: \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^* & \mathsf{q} \end{bmatrix}.$$

Now the sufficiency part follows from Theorem 1. Since $\mathsf{a}(0) = 1$, by [9, Lemma 2.1 and Theorem 2.3], ϕ is a compactly supported function in $L_2(\mathbb{R})$. Consequently, together with the perfect reconstruction property of the framelet filter bank $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$, $\{\phi; \psi^1, \psi^2\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. The symmetry property of $\{\phi; \psi^1, \psi^2\}$ can be checked by direct computation. We are done. □

4. Illustrative examples

In this section, we shall present several examples to illustrate our algorithms for the construction of dyadic algebraic framelet filter bank with the perfect reconstruction property and with symmetry. For $\mathbb{F} = \mathbb{Q}$, in [19], there are two examples with low-pass filters given by $\mathsf{a}(z) := \frac{1}{10}(1+z)(3+z+z^{-1})$ and $\mathsf{a}(z) = \frac{1}{4}(1+z)(z+z^{-1})$, respectively. For these two low-pass filters, they both satisfy $1 - \mathsf{a}(z)\mathsf{a}^*(z) - \mathsf{a}(-z)\mathsf{a}^*(-z) = \mathsf{d}(z)\mathsf{d}^*(z)$ with d being some Laurent polynomials in $\mathbb{Q}[z, z^{-1}]$. Applying our Algorithm 2 to these two examples, we recover the results as in [19]; that is, all filters in the framelet filter bank from each of these two low-pass filters are with coefficients in the rational number field $\mathbb{Q}$. But as pointed out in the introduction, the condition requiring the high-pass filters to be also with coefficients in $\mathbb{Q}$ is too restricted to have high-pass filters having high order of vanishing moments and high order of regularity. In fact, the framelet filter banks from these two low-pass filters only have vanishing moment of order 1.

We next provide several examples to show that high order of vanishing moments and regularity can be indeed achieved under our setting as stated in Theorem 2. To quantify the order of regularity, let us first introduce an important quantity $\nu_p(\mathsf{a}, 2)$, $1 \leqslant p \leqslant \infty$ to characterize the smoothness of a filter and its associated refinable function.

For $0 < \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$, we say that $f \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}))$ if there is a constant C such that $\|f - f(\cdot - h)\|_{L_p(\mathbb{R})} \leqslant Ch^\alpha$ for all $h > 0$. The smoothness of a function f in $L_p(\mathbb{R})$ is measured by

$$\nu_p(f) := \sup\{n + \alpha \mid n \in \mathbb{N}_0, 0 < \alpha \leqslant 1, f^{(n)} \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}))\},$$

where $f^{(n)}$ denotes the nth derivative of f. Since the symbol a of a is a Laurent polynomial, we can write $\mathsf{a}(z) = (1+z)^m\mathsf{Q}(z)$ for some Laurent polynomial Q such that $(1+z) \nmid \mathsf{Q}(z)$. Following [8, p. 61 and Proposition 7.2], we may define

$$\nu_p(\mathsf{a}, 2) := 1/p - 1 - \log_2\left(\limsup_{n\to\infty} \|Q_n\|^{1/n}_{\ell_p(\mathbb{Z})}\right), \quad 1 \leqslant p \leqslant \infty,$$

where $\|Q_n\|^p_{\ell_p(\mathbb{Z})} := \sum_{k\in\mathbb{Z}} |Q_n(k)|^p$ and $\sum_{k\in\mathbb{Z}} Q_n(k)z^k := \mathsf{Q}(z)\mathsf{Q}(z^2)\cdots\mathsf{Q}(z^{2^{n-1}})$. It has been proved in [8, Theorem 4.3] that the cascade algorithm with some mask (low-pass filter) a converges in $L_p(\mathbb{R})$ (as well as $C(\mathbb{R})$ when $p=\infty$) if and only if $\nu_p(\mathsf{a},2) > 0$. Let ϕ be the compactly supported normalized refinable distribution with the low-pass filter a such that $\hat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathsf{a}(e^{-i2^{-j}\xi})$. In general, we have $\nu_p(\mathsf{a},2) \leqslant \nu_p(\phi)$. If the integer shifts of ϕ form a Riesz system, then $\nu_p(\mathsf{a},2) = \nu_p(\phi)$. Also, $\nu_\infty(\mathsf{a},2) \geqslant \nu_2(\mathsf{a},2) - 1/2$. Hence, if $\nu_2(\mathsf{a},2) > k + 1/2$ for some positive integer k, then the 2-refinable function ϕ associated with a is at least kth order differentiable. The quantity $\nu_p(\mathsf{a},2)$ plays an important role in the study of the convergence of cascade algorithms and smoothness of refinable functions, see [8] and the references therein on these topics. Moreover, when $p=2$, we can compute $\nu_2(\mathsf{a},2)$ through

$$\nu_2(\mathsf{a},2) = -1/2 - \log_2\sqrt{\rho(\mathsf{a},2)}, \tag{4.1}$$

where $\rho(\mathsf{a},2)$ denotes the spectral radius of the square matrix $(u(2j-k))_{-N\leqslant j,k\leqslant N}$, where $\mathsf{Q}(z)\mathsf{Q}(z)^\star =: \sum_{k=-N}^{k=N} u(k)z^k$ (see [7, Theorem 2.1]).

Now, we are ready to present our examples. In Examples 1 – 3, the subfield is $\mathbb{F} = \mathbb{Q}$, while $\mathbb{F} = \mathbb{Q}(\sqrt{6}i)$ for Example 4.

Example 1. We consider $\mathbb{F} = \mathbb{Q}$ and follow the steps of Algorithm 2.

First, construct the low-pass filter. Let $m = n = 3$. We can obtain a low-pass filter a as follow:

$$\mathsf{a}(z) = \frac{1}{z}\cdot\left(\frac{1+z}{2}\right)^3\cdot\left(\frac{5}{128}(z^{-3}+z^3) - \frac{15}{128}(z^{-2}+z^2) - \frac{33}{128}(z^{-1}+z) + \frac{107}{64}\right).$$

Then a satisfies the sum rules of order 3 and we have $(2-z-z^{-1})^3 \mid [1-\mathsf{a}\mathsf{a}^\star]$, which guarantees that the high-pass filters b_1, b_2 deduced from a will have vanishing moments of order 3. Moreover, we have

$$1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z) = \frac{(5\sqrt{15})^2}{512^2}(z^2-1)^3(z^{-2}-1)^3 =: \mathsf{d}(z^2)\mathsf{d}^\star(z^2)$$

with $\mathsf{d}(z) = \frac{5\sqrt{15}}{512}(z-1)^3$.

Second, perform the algorithm of matrix splitting with symmetry. Define

$$\mathsf{M}(z) := \begin{bmatrix} 1-\mathsf{a}(z)\mathsf{a}^\star(z) & -\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\mathsf{a}^\star(z)\mathsf{a}(-z) & 1-\mathsf{a}(-z)\mathsf{a}^\star(-z)\end{bmatrix} =: \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^\star & \mathsf{q}\end{bmatrix},$$

where $\mathsf{q}(z) = \mathsf{p}(-z)$ and

$$\begin{aligned}\mathsf{p}(z) = {}& \frac{1}{1024^2}(z-2+z^{-1})^3(-25(z^6+z^{-6}) - 150(z^5+z^{-5}) + 105(z^4+z^{-4}) \\ & + 1630(z^3+z^{-3}) - 3339(z^2+z^{-2}) - 35784(z+z^{-1}) - 78474); \\ \mathsf{r}(z) = {}& \frac{1}{1024^2}(1+z)^3(1-z^{-1})^3(-25(z^6+z^{-6}) + 555(z^4+z^{-4}) \\ & - 7179(z^2+z^{-2}) + 44018).\end{aligned}$$

We have $\gcd(\mathsf{p},\mathsf{r}\mathsf{r}^\star) = \frac{(1-z)^3(1-z^{-1})^3}{1024^2}$ and $\gcd(\mathsf{q},\mathsf{r}\mathsf{r}^\star) = \frac{(1+z)^3(1+z^{-1})^3}{1024^2}$. Hence, we can define

$$\mathsf{P}_1(z) := \begin{bmatrix} \frac{1024}{(1-z)^3} & 0 \\ 0 & \frac{1024}{(1+z)^3)}\end{bmatrix}.$$

Then use P_1 to take out the common factor of M and define

$$M_1 := P_1 M P_1^{\star} = \begin{bmatrix} p_1 & r_1 \\ r_1^{\star} & q_1 \end{bmatrix},$$

where $q_1(z) = p_1(-z)$ and

$$\begin{aligned} p_1(z) &= -25(z^6 + z^{-6}) - 150(z^5 + z^{-5}) + 105(z^4 + z^{-4}) + 1630(z^3 + z^{-3}) \\ &\quad - 3339(z^2 + z^{-2}) - 35784(z + z^{-1}) - 78474; \\ r_1(z) &= 25(z^6 + z^{-6}) - 555(z^4 + z^{-4}) + 7179(z^2 + z^{-2}) - 44018. \end{aligned}$$

Define P_2 to be the polyphase generating matrix as in (3.9) and define

$$M_2 := P_2 M_1 P_2^{\star} =: \begin{bmatrix} p_2 & r_2 \\ r_2^{\star} & q_2 \end{bmatrix}.$$

Then, we have

$$\begin{aligned} p_2(z) &= 25z^6 - 330z^4 + 5259z^2 + 17228 + 5259z^{-2} - 330z^{-4} + 25z^{-6}; \\ r_2(z) &= -75z^4 + 815z^2 - 17892 - 17892z^{-2} + 815z^{-4} - 75z^{-6}; \\ q_2(z) &= 225z^4 - 1920z^2 + 61246 - 1920z^{-2} + 225z^{-4}. \end{aligned}$$

Now, we shall factor M_3 using the extended Euclidean algorithm. Note that $\operatorname{quo}(r_2, q_2) = -\frac{1}{3}(1 + z^{-2})$. Define P_3 and M_3 as follows:

$$P_3(z) := \begin{bmatrix} 1 & \frac{1}{3}(z^{-1} + 1) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_3 := P_3 M_2 P_3^{\star} =: \begin{bmatrix} p_3 & r_3 \\ r_3^{\star} & q_3 \end{bmatrix}.$$

Then, we have

$$M_3(z) = \begin{bmatrix} \frac{59000}{9} + \frac{2500}{9}z^2 + \frac{2500}{9}z^{-2} & \frac{5650}{3} + 250z^2 + \frac{5650}{3}z^{-2} + 250z^{-4} \\ \frac{5650}{3} + 250z^4 + \frac{5650}{3}z^2 + 250z^{-2} & 61246 + 225z^4 - 1920z^2 - 1920z^{-2} + 225z^{-4} \end{bmatrix}.$$

Again, we have $\operatorname{quo}(r_3, p_3) = \frac{9}{10}(1 + z^2)$. Define P_4 and M_4 as follows:

$$P_4(z) := \begin{bmatrix} 1 & 0 \\ -\frac{9}{10}(z^2 + 1) & 1 \end{bmatrix} \quad \text{and} \quad M_4 := P_4 M_3 P_4^{\star} =: \begin{bmatrix} p_4 & r_4 \\ r_4^{\star} & q_4 \end{bmatrix}.$$

Then, we have

$$M_4 = \begin{bmatrix} \frac{59000}{9} + \frac{2500}{9}z^2 + \frac{2500}{9}z^{-2} & -\frac{12800}{3} - \frac{12800}{3}z^{-2} \\ -\frac{12800}{3}z^2 - \frac{12800}{3} & 65536 \end{bmatrix}.$$

Now q_4 is a constant. We can define P_5 and M_5 as follows:

$$P_5(z) := \begin{bmatrix} 1 & \frac{25}{284}(z^{-2} + 1) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_5 := P_5 M_4 P_5^{\star} =: \begin{bmatrix} p_5 & r_5 \\ r_5^{\star} & q_5 \end{bmatrix}.$$

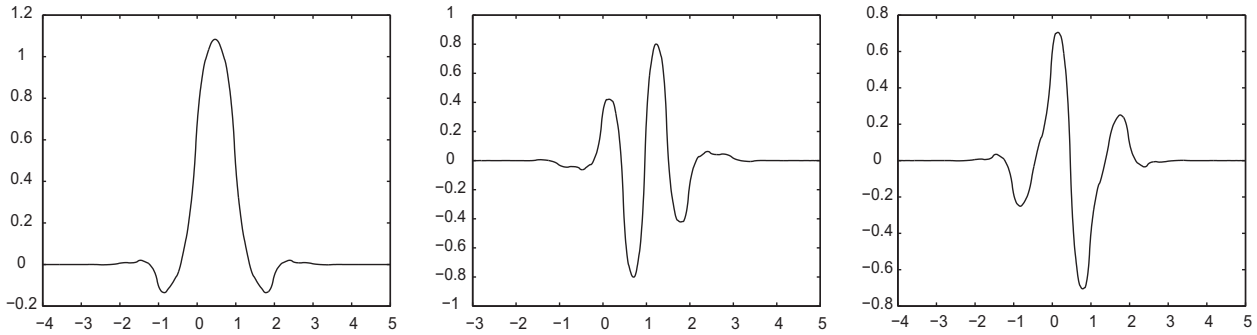

Fig. 1. Graphs of ϕ, ψ^1, ψ^2 (left to right) in Example 1.

Then, we have $\mathsf{M}_5 = \mathrm{diag}(6000, 65536)$. Define $\mathsf{P}_6 := \mathrm{diag}\left(\frac{\sqrt{15}}{300}, \frac{1}{256}\right)$, we obtain $\mathsf{P}_6\mathsf{M}_5\mathsf{P}_6^\star = \mathsf{I}_2$, the 2×2 identity matrix. Consequently, we have $\mathsf{P}_6 \cdots \mathsf{P}_1\mathsf{M}\mathsf{P}_1^\star \cdots \mathsf{P}_6^\star = \mathsf{I}_2$.

Finally, derive the high-pass filters from the factorization matrix. From the second step, we have $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$ with

$$\mathsf{U} = (\mathsf{P}_6 \cdots \mathsf{P}_1)^{-1} = \begin{bmatrix} \mathsf{b}_1(z) & \mathsf{b}_2(z) \\ \mathsf{b}_1(-z) & \mathsf{b}_2(-z) \end{bmatrix},$$

where

$$\mathsf{b}_1(z) = \frac{\sqrt{15}}{512}(3z^5 - 22z^3 + 45z^2 - 45z + 22 - 3z^{-2}),$$
$$\mathsf{b}_2(z) = \frac{1}{1024}(-5z^5 + 117z^3 - 75z^2 - 315z + 315 + 75z^{-1} - 117z^{-2} + 5z^{-4}).$$

Note that $\mathsf{S}\mathsf{a} = z$, $\mathsf{S}\mathsf{b}_1 = -z^3$, and $\mathsf{S}\mathsf{b}_2 = -z$. The filter system $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ forms a dyadic framelet filter bank with the perfect reconstruction property and with symmetry over the algebraic number field $\mathbb{Q}(\sqrt{15})$.

By Theorem 2, the system $\{\phi; \psi^1, \psi^2\}$ associated with $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (3.11) with $k_0 = 1, \epsilon_1 = -1, k_1 = 3$, and $\epsilon_2 = -1, k_2 = 1$. That is, $\phi = \phi(-\cdot)$, $\psi^1 = -\psi^1(2 - \cdot)$, and $\psi^2 = -\psi^2(1 - \cdot)$. By calculation, we have $\nu_2(\mathsf{a}, 2) \approx 1.6785$. Hence $\nu_\infty(\mathsf{a}, 2) > 1$, which means ϕ, ψ^1, ψ^2 are differentiable. See Fig. 1 for graphs of ϕ, ψ^1, ψ^2.

Example 2. Consider $\mathbb{F} = \mathbb{Q}$ and let $m = n = 3$. We can also obtain a low-pass filter a as follows:

$$\mathsf{a}(z) = \frac{1}{640z} \cdot \left(\frac{1+z}{2}\right)^3 \Big(63(z^{-4} + z^4) - 189(z^{-3} + z^3) - 51(z^{-2} + z^2) + 657(z^{-1} + z) - 320\Big).$$

Then a satisfies the sum rules of order 3 and we have $(2 - z - z^{-1})^3 \mid [1 - \mathsf{a}\mathsf{a}^\star]$. Moreover, we have

$$1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z) = \frac{(21\sqrt{231})^2}{2560^2}(z^2 - 1)^3(z^{-2} - 1)^3 =: \mathsf{d}(z^2)\mathsf{d}^\star(z^2)$$

with $\mathsf{d}(z) = \frac{21\sqrt{231}}{2560}(z - 1)^3$.

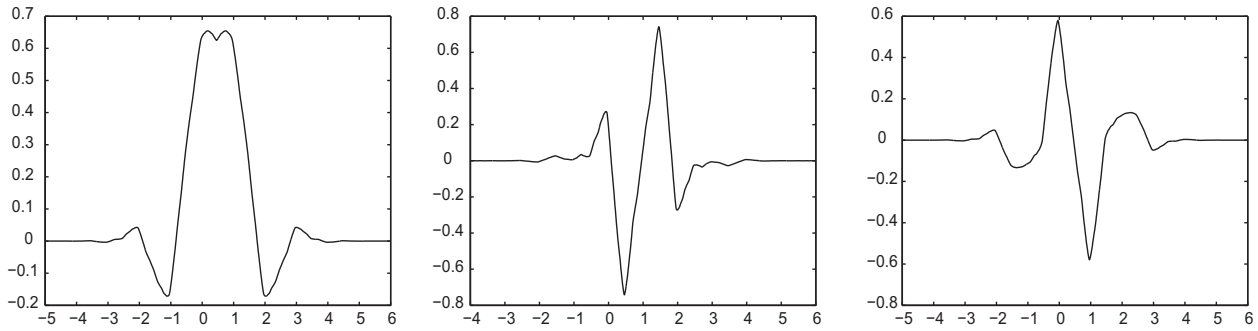

Fig. 2. Graphs of ϕ, ψ^1, ψ^2 (left to right) in Example 2.

Define

$$\mathsf{M}(z) := \begin{bmatrix} 1-\mathsf{a}(z)\mathsf{a}^{\star}(z) & -\mathsf{a}(z)\mathsf{a}^{\star}(-z) \\ -\mathsf{a}^{\star}(z)\mathsf{a}(-z) & 1-\mathsf{a}(-z)\mathsf{a}^{\star}(-z) \end{bmatrix} =: \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^{\star} & \mathsf{q} \end{bmatrix},$$

where we have $\mathsf{q}(z) = \mathsf{p}(-z)$ and

$$\begin{aligned} \mathsf{p}(z) = {} & \frac{1}{5120^2}(z-2+z^{-1})^3(-3969(z^8+z^{-8}) - 23814(z^7+z^{-7}) - 29295(z^6+z^{-6}) \\ & - 102060(z^5+z^{-5}) + 286065(z^4+z^{-4}) - 270756(z^3+z^{-3}) - 2654421(z^2+z^{-2}) \\ & - 6604290(z+z^{-1}) - 9199960); \\ \mathsf{r}(z) = {} & \frac{1}{5120^2}(1+z)^3(1-z^{-1})^3(3969(z^8+z^{-8}) - 42147(z^6+z^{-6}) + 210627(z^4+z^{-4}) \\ & - 157089(z^2+z^{-2}) - 819200); \end{aligned}$$

Applying matrix splitting with symmetry as state in Algorithm 2, we can decompose M to be $\mathsf{M} = \mathsf{U}\mathsf{U}^{\star}$ with

$$\mathsf{U} = \begin{bmatrix} \mathsf{b}_1(z) & \mathsf{b}_2(z) \\ \mathsf{b}_1(-z) & \mathsf{b}_2(-z) \end{bmatrix},$$

where

$$\begin{aligned} \mathsf{b}_1(z) &= \frac{\sqrt{231}}{2560}(-3z^6 - 10z^4 + 77z^2 - 77z + 10z^{-1} + 3z^{-3}); \\ \mathsf{b}_2(z) &= \frac{1}{5120}(-63z^6 + 495z^4 - 385z^2 - 1617z + 1617 + 385z^{-1} - 495z^{-3} + 63z^{-5}). \end{aligned}$$

Note that $\mathsf{Sa} = z$, $\mathsf{Sb}_1 = -z^3$, and $\mathsf{Sb}_2 = -z$. The filter system $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ forms a dyadic framelet filter bank with the perfect reconstruction property and with symmetry over the algebraic number field $\mathbb{Q}(\sqrt{231})$.

By Theorem 2, the system $\{\phi; \psi^1, \psi^2\}$ associated with $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (3.11) with $k_0 = 1, \epsilon_1 = -1, k_1 = 3$, and $\epsilon_2 = -1, k_2 = 1$. That is, $\phi = \phi(-\cdot)$, $\psi^1 = -\psi^1(2-\cdot)$, and $\psi^2 = -\psi^2(1-\cdot)$. By calculation, we have $\nu_2(\mathsf{a}, 2) \approx 1.8198$. Hence $\nu_\infty(\mathsf{a}, 2) > 1$, which means ϕ, ψ^1, ψ^2 are differentiable. See Fig. 2 for graphs of ϕ, ψ^1, ψ^2.

The above two examples do not involve solving any system of linear equations in the step of matrix splitting due to the simplicity of M. The extended Euclidean algorithm is enough for factoring M to

be of diagonal form. Then next two examples show that we might need to solve a system of linear equations when the structure of M is a little bit more complicated.

Example 3. We consider $\mathbb{F} = \mathbb{Q}$ and follow the steps of Algorithm 2.

First, construct the low-pass filter. Let $m = 5$ and $n = 3$. We can obtain a low-pass filter a as follows:

$$\mathsf{a}(z) = \frac{1}{z^2} \cdot \left(\frac{1+z}{2}\right)^5 \cdot \frac{1}{2048}\left(21(z^3+z^{-3}) + 434(z^2+z^{-2}) - 3205(z+z^{-1}) + 7548\right).$$

Then a satisfies the sum rules of order 5 and $(2 - z - z^{-1})^3 \mid [1 - \mathsf{a}\mathsf{a}^\star]$. Moreover,

$$1-\mathsf{a}(z)\mathsf{a}^\star(z)-\mathsf{a}(-z)\mathsf{a}^\star(-z)=\frac{(\sqrt{231})^2}{32768^2}(7z^2-142+7z^{-2})^2(z^2-1)^3(z^{-2}-1)^3 =: \mathsf{d}(z^2)\mathsf{d}^\star(z^2)$$

with $\mathsf{d}(z) = \frac{\sqrt{231}}{32768}(7z - 142 + 7z^{-1})(z-1)^3$.

Second, perform the algorithm of matrix splitting with symmetry. Define

$$\mathsf{M}(z) := \begin{bmatrix} 1 - \mathsf{a}(z)\mathsf{a}^\star(z) & -\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\mathsf{a}^\star(z)\mathsf{a}(-z) & 1 - \mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix} =: \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^\star & \mathsf{q} \end{bmatrix},$$

where $\mathsf{q}(z) = \mathsf{p}(-z)$ and

$$\begin{aligned}
\mathsf{p}(z) = {} & \frac{1}{65536^2}(z-2+z^{-1})^3(-441(z^8+z^{-8}) - 25284(z^7+z^{-7}) - 400960(z^6+z^{-6}) \\
& - 981036(z^5+z^{-5}) + 2890740(z^4+z^{-4}) + 9206044(z^3+z^{-3}) \\
& - 39502848(z^2+z^{-2}) - 233643564(z+z^{-1}) - 443969526); \\
\mathsf{r}(z) = {} & \frac{1}{65536^2}(1+z)^3(1-z^{-1})^3(441(z^8+z^{-8}) - 323848(z^6+z^{-6}) \\
& + 4232124(z^4+z^{-4}) - 44298552(z^2+z^{-2}) + 80779670).
\end{aligned}$$

By that $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^\star) = \frac{(1-z)^3(1-z^{-1})^3}{65536^2}$ and $\gcd(\mathsf{q}, \mathsf{r}\mathsf{r}^\star) = \frac{(1+z)^3(1+z^{-1})^3}{65536^2}$, we can define P_1 and M_1 to be

$$\mathsf{P}_1(z) := \begin{bmatrix} \frac{6536}{(1-z)^3} & 0 \\ 0 & \frac{65536}{(1+z)^3)} \end{bmatrix}, \quad \mathsf{M}_1 := \mathsf{P}_1\mathsf{M}\mathsf{P}_1^\star = \begin{bmatrix} \mathsf{p}_1 & \mathsf{r}_1 \\ \mathsf{r}_1^\star & \mathsf{q}_1 \end{bmatrix},$$

where $\mathsf{q}_1(z) = \mathsf{p}_1(z)$ and

$$\begin{aligned}
\mathsf{p}_1(z) = {} & 441(z^8+z^{-8}) + 25284(z^7+z^{-7}) + 400960(z^6+z^{-6}) \\
& + 981036(z^5+z^{-5}) - 2890740(z^4+z^{-4}) - 9206044(z^3+z^{-3}) \\
& + 39502848(z^2+z^{-2}) + 233643564(z+z^{-1}) + 443969526); \\
\mathsf{r}_1(z) = {} & -(441(z^8+z^{-8}) - 323848(z^6+z^{-6}) + 4232124(z^4+z^{-4}) \\
& - 44298552(z^2+z^{-2}) + 80779670);
\end{aligned}$$

Define P_2 to be the polyphase generating matrix and M_2 as follows:

$$\mathsf{P}_2(z) := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -z & z \end{bmatrix} \quad \text{and} \quad \mathsf{M}_2 := \mathsf{P}_2\mathsf{M}_1\mathsf{P}_2^\star =: \begin{bmatrix} \mathsf{p}_2 & \mathsf{r}_2 \\ \mathsf{r}_2^\star & \mathsf{q}_2 \end{bmatrix}.$$

Then, we have

$$\begin{aligned}
\mathsf{p}_2(z) &= 362404(z^6+z^{-6}) - 3561432(z^4+z^{-4}) + 41900700(z^2+z^{-2}) + 181594928;\\
\mathsf{r}_2(z) &= -12642(z^6+z^{-8}) - 490518(z^4+z^{-6}) + 4603022(z^2+z^{-4}) - 116821782(1+z^{-2});\\
\mathsf{q}_2(z) &= 441(z^8+z^{-8}) + 38556(z^6+z^{-6}) + 670692(z^4+z^{-4}) - 2397852(z^2+z^{-2})\\
&\quad + 262374598;
\end{aligned}$$

Now, applying the extended Euclidean algorithm to M_2 with P_3, P_4, P_5 defined to be

$$\mathsf{P}_3(z) := \begin{bmatrix} 1 & 0 \\ \frac{3}{86}(z^2+1) & 1 \end{bmatrix}, \mathsf{P}_4(z) := \begin{bmatrix} 1 & \frac{1849}{3072}(z^2+1) \\ 0 & 1 \end{bmatrix}, \mathsf{P}_5(z) := \begin{bmatrix} \frac{3}{8} & 0 \\ -\frac{9}{1376}(1+z^2) & \frac{31}{8192} \end{bmatrix},$$

we obtain $\mathsf{M}_5 := \mathsf{P}_5\mathsf{P}_4\mathsf{P}_3\mathsf{M}_2\mathsf{P}_3^{\star}\mathsf{P}_4^{\star}\mathsf{P}_5^{\star}$ as follows:

$$\mathsf{M}_5 = \begin{bmatrix} 12742814 + 47089\,z^2 + 47089\,z^{-2} & -175616 - 175616\,z^{-2} \\ -175616\,z^2 - 175616 & 8002 + 2079\,z^2 + 2079\,z^{-2} \end{bmatrix}$$

At this point, we can no longer use the extended Euclidean algorithm to lower the degree of the matrix M_5 since M_5 is irreducible. However, we now have $\gcd(\mathsf{p}_5, \mathsf{r}_5\mathsf{r}_5^{\star}) = 7$, which is a constant. Hence, by Theorem 1 (also see proof of Lemma 3), we can factor $\mathsf{M}_5 := \mathsf{U}_5\mathsf{U}_5^{\star}$ with U_5 satisfies (2.2) and (2.4). In fact, let

$$\mathsf{U}_5 := \begin{bmatrix} c_1(1+z^{-2}) & c_2 z^{-2} \\ c_3 & c_4(1+z^{-2}) \end{bmatrix}$$

and solve a linear system of equation as in (2.7), we can obtain

$$\mathsf{U}_5 = \begin{bmatrix} 217(1+z^{-2}) & -234\sqrt{231}z^{-2} \\ -62 & 3\sqrt{231}(1+z^{-2}) \end{bmatrix}.$$

Finally, derive the high-pass filters from the factorization matrix. From the second step, we have $\mathsf{M} = \mathsf{U}\mathsf{U}^{\star}$ with

$$\mathsf{U} = (\mathsf{P}_5\cdots\mathsf{P}_1)^{-1}\mathsf{U}_5 = \begin{bmatrix} \mathsf{b}_1(z) & \mathsf{b}_2(z) \\ \mathsf{b}_1(-z) & \mathsf{b}_2(-z) \end{bmatrix},$$

where

$$\begin{aligned}
\mathsf{b}_1(z) &= \frac{1}{65536}(21z^6 + 539z^5 - 1023z^4 - 9009z^3 + 12474z^2 + 9702z - 9702\\
&\quad - 12474z^{-1} + 9009z^{-2} + 1023z^{-3} - 539z^{-4} - 21z^{-5});\\
\mathsf{b}_2(z) &= \frac{\sqrt{231}}{32768}(-3z^4 - 77z^3 + 108z^2 + 308z - 898 + 898z^{-1} - 308z^{-2}\\
&\quad - 108z^{-3} + 77z^{-4} + 3z^{-5}).
\end{aligned}$$

Note that $\mathsf{Sa} = z$, $\mathsf{Sb}_1 = -z$, and $\mathsf{Sb}_2 = -z^{-1}$. The filter system $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ forms a dyadic framelet filter bank with the perfect reconstruction property and with symmetry over the algebraic number field $\mathbb{Q}(\sqrt{231})$.

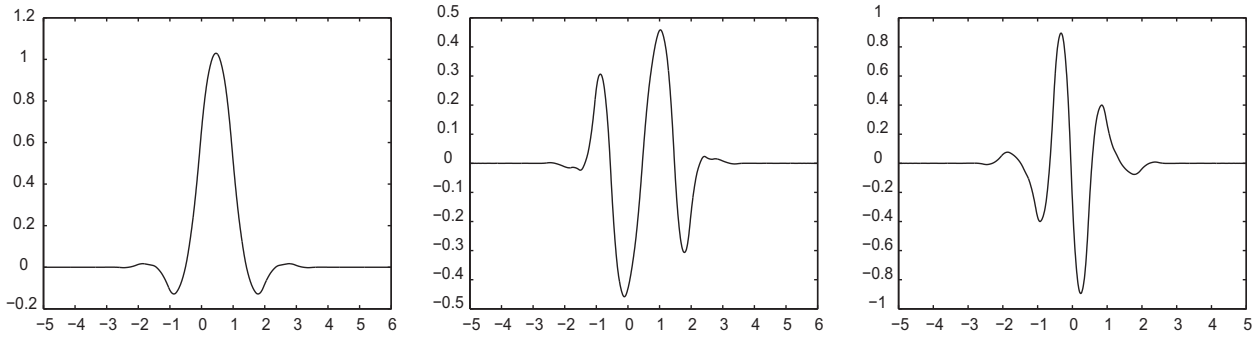

Fig. 3. Graphs of ϕ, ψ^1, ψ^2 (left to right) in Example 3.

By Theorem 2, the system $\{\phi; \psi^1, \psi^2\}$ associated with $\{\mathsf{a}; \mathsf{b}_1, \mathsf{b}_2\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (3.11) with $k_0 = 1, \epsilon_1 = -1, k_1 = 1$, and $\epsilon_2 = -1, k_2 = -1$. That is, $\phi = \phi(-\cdot)$, $\psi^1 = -\psi^1(1-\cdot)$, and $\psi^2 = -\psi^2(-\cdot)$. By calculation, we have $\nu_2(\mathsf{a}, 2) \approx 2.5395$. Hence $\nu_\infty(\mathsf{a}, 2) > 2$, which means ϕ, ψ^1, ψ^2 are at least twice differentiable. See Fig. 3 for graphs of ϕ, ψ^1, ψ^2.

We finally present an example with coefficient in the algebraic number field $\mathbb{Q}(\sqrt{6}i)$.

Example 4. Let the low-pass filter a be given by

$$\mathsf{a}(z) = \frac{1}{z^2}\left(\frac{1+z}{2}\right)^4 \cdot \left((-\frac{1}{2} - \frac{\sqrt{6}}{4}i)(z + z^{-1}) + 2 - \frac{\sqrt{6}}{2}i\right).$$

Then $\mathsf{a} \in \mathbb{F}[z, z^{-1}]$ with $\mathbb{F} = \mathbb{Q}(\sqrt{6}i)$ satisfies the sum rules of order 4 and $(2 - z - z^{-1})^3 \mid [1 - \mathsf{a}\mathsf{a}^\star]$. Moreover,

$$1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z) = \frac{(\sqrt{5})^2}{32^2}(z^2-1)^3(z^{-2}-1)^3 =: \mathsf{d}(z^2)\mathsf{d}^\star(z^2)$$

with $\mathsf{d}(z) = \frac{\sqrt{5}}{32}(z-1)^3$. Define

$$\mathsf{M}(z) = \begin{bmatrix} 1 - \mathsf{a}(z)\mathsf{a}^\star(z) & -\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\mathsf{a}^\star(z)\mathsf{a}(-z) & 1 - \mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix} := \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^\star & \mathsf{q} \end{bmatrix},$$

where $\mathsf{q}(z) = \mathsf{p}(-z)$ and

$$\mathsf{p}(z) = \frac{1}{2048}(z - 2 + z^{-1})^3(-5(z^3 + z^{-3}) - 42(z^2 + z^{-2}) - 147(z + z^{-1}) - 252);$$
$$\mathsf{r}(z) = \frac{1}{10240}(1+z)^4(1-z^{-1})^4(25(z^2 + z^{-2}) + 20\sqrt{6}i(z + z^{-1}) - 170).$$

We next provide two different approaches to factor $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$. One approach obtains U by solving a system of linear equations. The other only employs the extended Euclidean algorithm.

Approach 1: By that $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^\star) = (1-z)^3(1-z^{-1})^3$, we can define P_1 and M_1 to be

$$\mathsf{P}_1(z) := \begin{bmatrix} \frac{1}{(1-z)^3} & 0 \\ 0 & \frac{1}{(1+z)^3)} \end{bmatrix}, \quad \mathsf{M}_1 := \mathsf{P}_1\mathsf{M}\mathsf{P}_1^\star = \begin{bmatrix} \mathsf{p}_1 & \mathsf{r}_1 \\ \mathsf{r}_1^\star & \mathsf{q}_1 \end{bmatrix}$$

with $q_1(z) = p_1(-z)$ and

$$p_1(z) = \frac{1}{2048}(5(z^3 + z^{-3}) + 42(z^2 + z^{-2}) + 147(z + z^{-1}) + 252);$$
$$r_1(z) = -\frac{(1+z)(1-z^{-1})}{10240}(25(z^2 + z^{-2}) + 20\sqrt{6}i(z + z^{-1}) - 170).$$

We have $\gcd(p_1, r_1 r_1^{\star}) = 1$, $\mathrm{fsupp}(p_1) = [-3, 3]$, and $SM_1 = [1, -1]^T[1, -1]$. By Theorem 1 (see proof of Lemma 3), we can define

$$U_1 := \begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix}$$

with

$$u_1(z) = (u_{10} + u_{11}(z + z^{-1}))(1 + z); \qquad v_1(z) = (v_{10} + v_{11}(z + z^{-1}))(1 - z);$$
$$u_2(z) = (u_{20} + u_{21}(z + z^{-1}))(1 - z^{-1}); \quad v_2(z) = (v_{20} + v_{21}(z + z^{-1}))(1 + z^{-1}).$$

Now, by solving the system of linear equations

$$\begin{cases} r_1^{\star}(z)\tilde{u}_1(z) - v_1^{\star}(z) - p_1(z)\tilde{u}_2(z) = 0 \\ r_1^{\star}(z)v_1(z) + \frac{5}{1024}\tilde{u}_1^{\star}(z) - p_1(z)v_2(z) = 0 \end{cases},$$

we can obtain solution

$$u_1(z) = \frac{-4\sqrt{2} + \sqrt{3}i}{560}(5(z + z^{-1}) + 16 + 2\sqrt{6}i)(1 + z); \qquad u_2(z) = u_1(-z);$$
$$v_1(z) = \frac{-13\sqrt{10} + \sqrt{60}i}{11200}(5(z + z^{-1}) + 26 + 2\sqrt{6}i)(1 + z); \quad v_2(z) = -v_1(-z).$$

Note that $[(P_1)^{-1}U_1]_{2,2}(z) \neq [(P_1)^{-1}U_1]_{1,2}(-z)$. However, we can define

$$U := (P_1)^{-1}U_1\mathrm{diag}(z^{-2}, z^{-1}).$$

Then we have $UU^{\star} = M$ and

$$U := \begin{bmatrix} b_1(z) & b_2(z) \\ b_1(-z) & b_2(-z) \end{bmatrix}$$

with

$$b_1(z) = \frac{1}{z^2}\frac{4\sqrt{2} - \sqrt{3}i}{560}(5(z + z^{-1}) + 16 + 2\sqrt{6}i)(1 + z)(z - 1)^3;$$
$$b_2(z) = \frac{1}{z^2}\frac{-13\sqrt{10} + \sqrt{60}i}{11200}(5(z + z^{-1}) + 26 + 2\sqrt{6}i)(z - 1)^4.$$

Approach 2: We utilize the polyphase factorization technique. We define P_1 and M_1 as follows:

$$P_1(z) := \begin{bmatrix} \frac{1}{(1-z)^2} & 0 \\ 0 & \frac{1}{(1+z)^2} \end{bmatrix}, \quad M_1 := P_1 M P_1^{\star} = \begin{bmatrix} p_1 & r_1 \\ r_1^{\star} & q_1 \end{bmatrix}$$

with $q_1(z) = p_1(-z)$ and

$$p_1(z) = -\frac{(z-1)^2}{2048z}(5(z^3+z^{-3}) + 42(z^2+z^{-2}) + 147(z+z^{-1}) + 252);$$
$$r_1(z) = -\frac{(1+z)^2(1-z^{-1})^2}{10240}(25(z^2+z^{-2}) + 20\sqrt{6}i(z+z^{-1}) - 170).$$

Then define P_2 to be the polyphase generating matrix and M_2 as follows:

$$P_2(z) := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -z & z \end{bmatrix} \quad \text{and} \quad M_2 := P_2 M_1 P_2^{\star} =: \begin{bmatrix} p_2 & r_2 \\ r_2^{\star} & q_2 \end{bmatrix},$$

where

$$p_2(z) = \frac{1}{256}(-7(z^2+z^{-2}) + 18);$$
$$r_2(z) = \frac{1}{1024}((8+\sqrt{6}i)(z^2+z^{-4}) - \sqrt{6}i(1+z^{-2}));$$
$$q_2(z) = \frac{1}{2048}(-5(z^4+z^{-4}) - 12(z^2+z^{-2}) + 66).$$

Now, applying the extended Euclidean algorithm with P_3, P_4 defined to be

$$P_3(z) := \begin{bmatrix} 1 & 0 \\ \frac{8-\sqrt{6}i}{28}(z^2+1) & 1 \end{bmatrix}, \quad P_4(z) := \begin{bmatrix} 1 & -\frac{22+\sqrt{6}i}{70}(z^{-2}+1) \\ 0 & 1 \end{bmatrix}.$$

Then, we have $M_4 := P_4 P_3 M_2 P_3^{\star} P_4^{\star} = \operatorname{diag}\left(\frac{(1-z^2)(1-z^{-2})}{32}, \frac{5}{128}\right)$. By defining $P_5 := \operatorname{diag}\left(\frac{\sqrt{2}(1-z^{-1})}{8}, \frac{10}{16}\right)$, we obtain $P_5 M_4 P_5^{\star} = I_2$. Consequently, we have $M = UU^{\star}$ with $U = (P_5 \cdots P_1)^{-1}$, where

$$U = \begin{bmatrix} b_1(z) & b_2(z) \\ b_1(-z) & b_2(-z) \end{bmatrix}$$

with

$$b_1(z) = \frac{1}{z^2}\frac{4\sqrt{2}-\sqrt{3}i}{560}(5(z+z^{-1}) + 16 + 2\sqrt{6}i)(1+z)(z-1)^3;$$
$$b_2(z) = \frac{1}{z^2}\frac{-13\sqrt{10}+\sqrt{60}i}{11200}(5(z+z^{-1}) + 26 + 2\sqrt{6}i)(z-1)^4.$$

We get the same results as in Approach 1.

Note that $Sa = 1$, $Sb_1 = -1$, and $Sb_2 = 1$. The filter system $\{a; b_1, b_2\}$ forms a dyadic framelet filter bank with the perfect reconstruction property and with symmetry over the algebraic number field $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, i)$.

By Theorem 2, the system $\{\phi; \psi^1, \psi^2\}$ associated with $\{a; b_1, b_2\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (3.11) with $k_0 = 0, \epsilon_1 = -1, k_1 = 0$, and $\epsilon_2 = 1, k_2 = 0$. That is, $\phi = \phi(-\cdot)$, $\psi^1 = -\psi^1(-\cdot)$, and $\psi^2 = \psi^2(-\cdot)$. By calculation, we have $\nu_2(a, 2) \approx 2.1319$. Hence $\nu_\infty(a, 2) > 1.5$, which means ϕ, ψ^1, ψ^2 are at least first order differentiable. See Fig. 4 for graphs of ϕ, ψ^1, ψ^2.

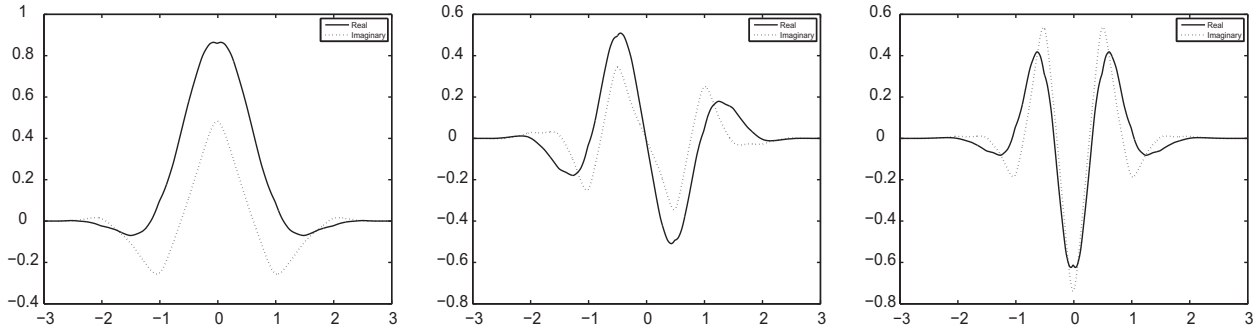

Fig. 4. Graphs of ϕ, ψ^1, ψ^2 (left to right) in Example 4.

5. Proofs of Lemmas 2 and 3

Proof of Lemma 2. We first show that $\mathsf{p}(z) > 0$ for all $z \in \mathbb{T}$. Since $\mathsf{M}(z) \geqslant 0$ for all $z \in \mathbb{T}$, if $\mathsf{p}(z_0) = 0$ for some $z_0 \in \mathbb{T}$, then by the condition $\mathsf{M}(z_0) \geqslant 0$, we have

$$0 \leqslant \det \mathsf{M}(z_0) = \mathsf{p}(z_0)\mathsf{q}(z_0) - \mathsf{r}(z_0)\mathsf{r}^\star(z_0) = -|\mathsf{r}(z_0)|^2.$$

Hence $\mathsf{r}(z_0) = 0$. Therefore, $(z - z_0) \mid \mathsf{r}$, which contradicts to $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^\star) = 1$. Consequently, we must have $\mathsf{p}(z) > 0$ for all $z \in \mathbb{T}$.

Necessity. Suppose (2.2) holds; that is $\mathsf{M} = \mathsf{U}\mathsf{U}^\star$ with U being defined as in (2.2). Without loss of generality, we can assume that $\mathsf{d} = \det \mathsf{U}$. By $\mathsf{U}\mathsf{U}^\star = \mathsf{M}$, we have $|\mathsf{u}_1(1)|^2 + |\mathsf{v}_1(1)|^2 = \mathsf{p}(1)$ and therefore (2.8) holds. Since $\mathsf{d} \not\equiv 0$ and $\mathsf{d}\mathsf{U}^{-1} = \mathrm{adj}(\mathsf{U})$ with $\mathrm{adj}(\mathsf{U})$ being the adjugate matrix of U, it follows from $\mathsf{U}\mathsf{U}^\star = \mathsf{M}$ that

$$\mathsf{d}(z)\mathsf{U}^\star(z) = \mathsf{d}(z)\mathsf{U}^{-1}(z)\mathsf{M}(z) = [\mathrm{adj}\mathsf{U}(z)]\mathsf{M}(z) = \begin{bmatrix} \mathsf{v}_2(z) & -\mathsf{v}_1(z) \\ -\mathsf{u}_2(z) & \mathsf{u}_1(z) \end{bmatrix} \begin{bmatrix} \mathsf{p}(z) & \mathsf{r}(z) \\ \mathsf{r}^\star(z) & \mathsf{q}(z) \end{bmatrix}.$$

Comparing (1,1)- and (2,1)-entries of the above matrices, we see that (2.7) holds.

Sufficiency. Conversely, let u_1, u_2, v_1, v_2 be Laurent polynomials with symmetry in $\mathbb{C}[z, z^{-1}]$ such that (2.4) and (2.5) hold. If $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ is a solution of the linear system of equations in (2.7) and satisfies the normalization condition in (2.8), then we show that (2.2) must be true.

Multiplying $\mathsf{u}_1^\star$ with the first equation and $\mathsf{v}_1^\star$ with the second equation in (2.7), by adding them together, we have

$$\mathsf{r}^\star(z)[\mathsf{u}_1(z)\mathsf{u}_1^\star(z) + \mathsf{v}_1(z)\mathsf{v}_1^\star(z)] = \mathsf{p}(z)[\mathsf{u}_2(z)\mathsf{u}_1^\star(z) + \mathsf{v}_2(z)\mathsf{v}_1^\star(z)]. \tag{5.1}$$

Since $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^\star) = 1$, we must have $\mathsf{u}_1\mathsf{u}_1^\star + \mathsf{v}_1\mathsf{v}_1^\star = \alpha\mathsf{p}$ for some Laurent polynomial α. By the support control property as in (2.5), we conclude that α must be a constant. By (2.8) and $\mathsf{p}(1) > 0$, we must further have $\alpha \equiv 1$. Therefore, $\mathsf{p} = \mathsf{u}_1\mathsf{u}_1^\star + \mathsf{v}_1\mathsf{v}_1^\star$. It follows from (5.1) that $\mathsf{r}^\star = \mathsf{u}_1^\star\mathsf{u}_2 + \mathsf{v}_1^\star\mathsf{v}_2$ and $\mathsf{r} = \mathsf{u}_1\mathsf{u}_2^\star + \mathsf{v}_1\mathsf{v}_2^\star$. In other word, we have $[\mathsf{U}\mathsf{U}^\star]_{j,k} = [\mathsf{M}]_{j,k}$ for all $1 \leqslant j, k \leqslant 2$ except $j = k = 2$.

Multiplying v_2 with the first equation and u_2 with the second equation in (2.7), by subtracting the second one from the first one, we get $\mathsf{r}^\star(\mathsf{u}_1\mathsf{v}_2 - \mathsf{u}_2\mathsf{v}_1) = \mathsf{d}(\mathsf{u}_1^\star\mathsf{u}_2 + \mathsf{v}_1^\star\mathsf{v}_2) = \mathsf{d}\mathsf{r}^\star$. By $\mathsf{r} \not\equiv 0$, we obtain $\mathsf{d} = \mathsf{u}_1\mathsf{v}_2 - \mathsf{u}_2\mathsf{v}_1$. Consequently,

$$\det(\mathsf{U}\mathsf{U}^\star) = \mathsf{d}\mathsf{d}^\star = \det \mathsf{M}.$$

Now it is easy to deduce that $[\mathsf{U}\mathsf{U}^\star]_{j,k} = [\mathsf{M}]_{j,k}$ for all $1 \leqslant j, k \leqslant 2$ from the fact that $\det(\mathsf{U}\mathsf{U}^\star) = \det \mathsf{M}$ and $[\mathsf{U}\mathsf{U}^\star]_{j,k} = [\mathsf{M}]_{j,k}$ for $(j, k) \in \{(1, 1), (1, 2), (2, 1)\}$. So, (2.2) holds. □

Proof of Lemma 3. First, we demonstrate that there are Laurent polynomials u_1 and v_1 in $\mathbb{C}[z, z^{-1}]$ satisfying

$$\mathsf{p} \mid \mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star} \tag{5.2}$$

and

$$(\mathsf{S}\mathsf{u}_1)(\mathsf{S}\mathsf{v}_1) = (\mathsf{S}\mathsf{r})(\mathsf{S}\mathsf{d}). \tag{5.3}$$

Let u_0 and v_0 be Laurent polynomials in the following parametric forms:

$$\mathsf{u}_0(z) = u_0 + \sum_{j=1}^{h_b} u_j(z^j + z^{-j}) \quad \text{and} \quad \mathsf{v}_0(z) = v_0 + \sum_{k=1}^{h_c} v_k(z^k + z^{-k}), \tag{5.4}$$

where h_b, h_c are nonnegative integers and u_j's, v_k's are constants in $\mathbb{C}$ which are to be determined later. Suppose $\mathrm{fsupp}(\mathsf{p}) = [-N, N]$ with $N \in \mathbb{N} \cup \{0\}$. Let us consider the following four cases.

(1) $(\mathsf{S}\mathsf{r})(\mathsf{S}\mathsf{d}) = z^{2n}$ for some $n \in \mathbb{Z}$. We choose $\mathsf{u}_1(z) = z^n\mathsf{u}_0(z)$ and $\mathsf{v}_1(z) = \mathsf{v}_0(z)$. When N is even, set $h_b = h_c = N/2$; when N is odd, set $h_b = h_c = (N-1)/2$.
(2) $(\mathsf{S}\mathsf{r})(\mathsf{S}\mathsf{d}) = z^{2n+1}$ for some $n \in \mathbb{Z}$. We choose $\mathsf{u}_1(z) = z^n(1+z)\mathsf{u}_0(z)$ and $\mathsf{v}_1(z) = \mathsf{v}_0(z)$. When N is even, set $h_b = N/2 - 1$ and $h_c = N/2$; when N is odd, set $h_b = h_c = (N-1)/2$.
(3) $(\mathsf{S}\mathsf{r})(\mathsf{S}\mathsf{d}) = -z^{2n}$ for some $n \in \mathbb{Z}$. When N is even, we choose $\mathsf{u}_1(z) = z^n(z - z^{-1})\mathsf{u}_0(z)$, $\mathsf{v}_1(z) = \mathsf{v}_0(z)$, and set $h_b = N/2 - 1$, $h_c = N/2$; when N is odd we choose $\mathsf{u}_1(z) = z^n(1+z)\mathsf{u}_0(z)$, $\mathsf{v}_1(z) = (1 - z^{-1})\mathsf{v}_0(z)$, and set $h_b = h_c = (N-1)/2$;
(4) $(\mathsf{S}\mathsf{r})(\mathsf{S}\mathsf{d}) = -z^{2n+1}$ for some $n \in \mathbb{Z}$. We choose $\mathsf{u}_1(z) = z^n(1-z)\mathsf{u}_0(z)$ and $\mathsf{v}_1(z) = \mathsf{v}_0(z)$. When N is even, set $h_b = N/2 - 1$ and $h_c = N/2$; when N is odd, set $h_b = h_c = (N-1)/2$.

It is easy to see that both u_1 and v_1 are Laurent polynomials such that (5.3) holds. Moreover, $\max(\mathrm{len}(\mathsf{u}_1\mathsf{u}_1^{\star}), \mathrm{len}(\mathsf{v}_1\mathsf{v}_1^{\star})) \leqslant \mathrm{len}(\mathsf{p})$ and it is easy to verify that $h_b + h_c + 2 > N$. Since $\mathsf{p}(z) > 0$ for all $z \in \mathbb{T}$, by Fejér-Riesz Lemma, we have $\mathsf{p} = \tilde{\mathsf{p}}\tilde{\mathsf{p}}^{\star}$ for some Laurent polynomial $\tilde{\mathsf{p}}$ such that all the roots of $\tilde{\mathsf{p}}$ are contained inside $\{z \in \mathbb{C} : |z| < 1\}$. Therefore, $\tilde{\mathsf{p}}$ and $\tilde{\mathsf{p}}^{\star}$ have no common zeros in $\mathbb{C} \setminus \{0\}$. Since $\mathsf{p}(z) = p_0 + \sum_{k=1}^{N} p_k(z^k + z^{-k})$, $\tilde{\mathsf{p}}$ can have at most N zeros in $\mathbb{C} \setminus \{0\}$, say, $\{z_1, \ldots, z_{N'}\}$ are all of the distinct roots of the Laurent polynomial $\tilde{\mathsf{p}}$ in $\mathbb{C} \setminus \{0\}$ such $Z(\tilde{\mathsf{p}}, z_1) + \ldots + Z(\tilde{\mathsf{p}}, z_{N'}) = N$, where $Z(\tilde{\mathsf{p}}, z_j)$ denotes the multiplicity of the root z_j of $\tilde{\mathsf{p}}$.

Define $F(z) := \mathsf{r}^{\star}(z)\mathsf{u}_1(z) - \mathsf{d}(z)\mathsf{v}_1^{\star}(z), z \in \mathbb{C}\setminus\{0\}$. Now we have the following system of homogeneous linear equations:

$$F^{(j)}(z_k) = 0, \quad k = 0, \ldots, N'; j = 0, \ldots, Z(\tilde{\mathsf{p}}, z_k) - 1. \tag{5.5}$$

Since the number of free parameters in $\{u_j, v_k : j = 0, \ldots, h_b; k = 0, \ldots, h_c\}$ is $h_b + h_c + 2 > N$ and we have N homogeneous linear equations, there must be a nonzero solution $\{u_j, v_k : j = 0, \ldots, h_b; k = 0, \ldots, h_c\}$ to the system of homogeneous linear equations in (5.5). So there exist u_1 and v_1 satisfying (5.5) with at least one of them nonzero. In other words, we deduce from (5.5) that

$$\tilde{\mathsf{p}} \mid [\mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star}]. \tag{5.6}$$

Noting that $\mathsf{p} = \tilde{\mathsf{p}}\tilde{\mathsf{p}}^{\star}$ with $\tilde{\mathsf{p}}$ having all roots contained in $\{z \in \mathbb{C} : |z| < 1\}$, we can deduce that $\tilde{\mathsf{p}}^{\star} \mid [\mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star}]$. In fact, if $(z - z_j) \mid \tilde{\mathsf{p}}$ with $|z_j| < 1$, then by the symmetry of p, we must have $(z - \frac{1}{z_j}) \mid \mathsf{p}$, which implies $(z - \frac{1}{z_j}) \mid \tilde{\mathsf{p}}^{\star}$. But then $(z - \bar{z}_j) \mid \tilde{\mathsf{p}}$. And by (5.6), we have $(z - \bar{z}_j) \mid [\mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star}]$. By the symmetry of $\mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star}$, we must have $\left(\frac{1}{z} - \bar{z}_j\right) \mid [\mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star}]$. Obviously, $\left(\frac{1}{z} - \bar{z}_j\right) = (z - z_j)^{\star} \mid \tilde{\mathsf{p}}^{\star}$. Hence, $\tilde{\mathsf{p}}^{\star} \mid [\mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star}]$. Consequently, $\mathsf{p} \mid [\mathsf{r}^{\star}\mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^{\star}]$.

Since $\mathsf{p} \not\equiv 0$, we can define

$$\mathsf{u}_2 = \frac{\mathsf{r}^\star \mathsf{u}_1 - \mathsf{d}\mathsf{v}_1^\star}{\mathsf{p}} \quad \text{and} \quad \mathsf{v}_2 = \frac{\mathsf{d}\mathsf{u}_1^\star + \mathsf{r}^\star \mathsf{v}_1}{\mathsf{p}}. \tag{5.7}$$

By (5.2) and the fact that $\mathsf{p}\mathsf{q} - \mathsf{r}\mathsf{r}^\star = \mathsf{d}\mathsf{d}^\star$, we see that u_2 has symmetry and

$$\max(\operatorname{len}(\mathsf{u}_2\mathsf{u}_2^\star), \operatorname{len}(\mathsf{v}_2\mathsf{v}_2^\star)) \leqslant \operatorname{len}(\mathsf{q}).$$

Now we show that $\mathsf{p} \mid [\mathsf{d}\mathsf{u}_1^\star + \mathsf{r}^\star \mathsf{v}_1]$, which together with (5.2) implies v_2 also has symmetry. By definition of u_2 and the fact that $\mathsf{d}\mathsf{d}^\star = \det \mathsf{M} = \mathsf{p}\mathsf{q} - \mathsf{r}\mathsf{r}^\star$, we have

$$\mathsf{p}\mathsf{d}^\star \mathsf{u}_2 = \mathsf{r}^\star \mathsf{d}^\star \mathsf{u}_1 - (\det \mathsf{M})\mathsf{v}_1^\star = \mathsf{r}^\star \mathsf{d}^\star \mathsf{u}_1 - \mathsf{p}\mathsf{q}\mathsf{v}_1^\star + \mathsf{r}\mathsf{r}^\star \mathsf{v}_1^\star.$$

From the above identity, we have $\mathsf{p}(\mathsf{d}^\star \mathsf{u}_2 + \mathsf{q}\mathsf{v}_1^\star) = \mathsf{r}^\star(\mathsf{d}^\star \mathsf{u}_1 + \mathsf{r}\mathsf{v}_1^\star)$. Since $\gcd(\mathsf{p}, \mathsf{r}\mathsf{r}^\star) = 1$ and $\mathsf{p} = \mathsf{p}^\star$, we conclude that $\mathsf{p} \mid [\mathsf{d}\mathsf{u}_1^\star + \mathsf{r}^\star \mathsf{v}_1]$. Therefore, we see that v_2 has symmetry.

We finally show that $|\mathsf{u}_1(1)|^2 + |\mathsf{v}_1(1)|^2 \neq 0$. Since both (2.4) and (2.7) are satisfied, as we demonstrated in the proof of Lemma 2, we must have $\mathsf{u}_1\mathsf{u}_1^\star + \mathsf{v}_1\mathsf{v}_1^\star = c_0\mathsf{p}$ for some constant c_0. If $\mathsf{u}_1(1) = \mathsf{v}_1(1) = 0$, by $\mathsf{p}(1) > 0$, then we must have $c_0 = 0$. That is $|\mathsf{u}_1(z)|^2 + |\mathsf{v}_1(z)|^2 = 0$ for all $z \in \mathbb{T}$. So, u_1 and v_1 must be identically zero, which contradicts to our choice of u_1 and v_1 since one of them must be nonzero. So $|\mathsf{u}_1(1)|^2 + |\mathsf{v}_1(1)|^2 \neq 0$. Now replacing u_1 and v_1 by $c\mathsf{u}_1$ and $c\mathsf{v}_1$ with $c = \sqrt{\mathsf{p}(1)/(|\mathsf{u}_1(1)|^2 + |\mathsf{v}_1(1)|^2)}$ in the above proof, we see that (2.4), (2.3), and (2.7) still hold. Moreover, we have $|\mathsf{u}_1(1)|^2 + |\mathsf{v}_1(1)|^2 = \mathsf{p}(1)$, which completes our proof. □

6. Final remarks

(1) Algorithm given in [19] cannot be applied to matrices that are irreducible, e.g.,

$$\mathsf{M} := \begin{bmatrix} 3 + z + z^{-1} & z - z^{-1} \\ z^{-1} - z & 3 - z - z^{-1} \end{bmatrix}.$$

In certain case, a step of applying a rotation matrix is needed in order to diagonalize the input matrix. However, such a step could cause the high-pass filters to have support length longer than that of the input low-pass filter. Moreover, too many cases need to be considered concerning the symmetry in many steps of the algorithm. Here, in this paper, our algorithms are more concise and we can guarantee that the supports of the high-pass filters are no larger than that of the low-pass filter.

(2) In Theorem 1, the condition $\gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}) = \gcd(\mathsf{p}, \mathsf{q}, \mathsf{r}^\star) = 1$ can be removed by requiring the common factor $\theta := \gcd(\mathsf{p}, \mathsf{q}, \mathsf{r})$ satisfying certain constrains as in [13, Theorem 2.3]. In practice, it is not difficult to satisfy such condition.

(3) Algorithms for solving the general matrix splitting problem stated in the introduction part for any $2 \leqslant r \leqslant s$ remain open to our best knowledge especially when we require the matrix to be of Laurent polynomials with coefficients in number fields with certain structures. We expect that similar results hold as in this paper for a special case $r = s$ for any integer $r \geqslant 2$, which shall be addressed elsewhere in future.

(4) It would be even more interesting to consider the matrix splitting problem in higher dimensions.

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References

[1] T. Beth, A. Klappenecker, A. Nückel, Construction of algebraic wavelet coefficients, in: Proceedings of Internationl Symposium on Information Theory and its Applications 1994, ISITA'94, Sydney, 1994, pp. 341–344.
[2] C.K. Chui, W. He, Compactly supported tight frames associated with refinable functions, Appl. Comput. Harmon. Anal. 8 (2000) 293–319.
[3] C.K. Chui, W. He, J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal. 13 (2002) 224–262.
[4] I. Daubechies, Ten Lectures on Wavelets, in: CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61, SIAM, Philadelphia, PA, 1992.
[5] I. Daubechies, B. Han, A. Ron, Z.W. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (2003) 1–46.
[6] B. Han, On dual wavelet tight frames, Appl. Comput. Harmon. Anal. 4 (1997) 380–413.
[7] B. Han, Symmetric orthonormal scaling functions and wavelets with dilation factor 4, Adv. Comput. Math. 8 (1998) 221–247.
[8] B. Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, J. Approx. Theory 124 (2003) 44–88.
[9] B. Han, Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix, J. Comput. Appl. Math. 155 (1) (2003) 43–67.
[10] B. Han, Matrix extension with symmetry and applications to symmetric orthonormal complex M-wavelets, J. Fourier Anal. Appl. 15 (2009) 684–705.
[11] B. Han, Symmetric orthogonal filters and wavelets with linear-phase moments, J. Comput. Appl. Math. 236 (4) (2011) 482–503.
[12] B. Han, Q. Mo, Tight wavelet frames generated by three symmetric B-spline functions with high vanishing moments, Proc. Amer. Math. Soc. 132 (2004) 77–86.
[13] B. Han, Q. Mo, Splitting a matrix of Laurent polynomials with symmetry and its application to symmetric framelet filter banks, SIAM J. Matrix Anal. Appl. 26 (1) (2004) 97–124.
[14] B. Han, X. Zhuang, Matrix extension with symmetry and its application to symmetric orthonormal multiwavelets, SIAM J. Math. Anal. 42 (5) (2010) 2297–2317.
[15] B. Han, X. Zhuang, Algorithm for matrix extension and orthogonal wavelet filter banks over algebraic number fields, Math. Comp., 2012 in press.
[16] D.P. Hardin, T.A. Hogan, Q.Y. Sun, The matrix-valued Riesz lemma and local orthonormal bases in shift-invariant spaces, Adv. Comput. Math. 20 (2004) 367–384.
[17] Q.T. Jiang, Parameterizations of masks for tight affine frames with two symmetric/antisymmetric generators, Adv. Comput. Math. 18 (2003) 247–268.
[18] A. Klappenecker, Algebraic wavelet filters, Int. J. Imag. Syst. Tech. 7 (3) (1996) 166–169.
[19] Q. Mo, S. Li, Symmetric tight wavelet frames with rational coefficients, Appl. Comput. Harmon. Anal. 31 (2) (2011) 249–263.
[20] A. Petukhov, Symmetric framelets, Constr. Approx. 19 (2003) 309–328.
[21] A. Petukhov, Construction of symmetric orthogonal bases of wavelets and tight wavelet frames with integer dilation factor, Appl. Comput. Harmon. Anal. 17 (2004) 198–210.
[22] A. Ron, Z. Shen, Affine systems in $L_2(R^d)$: the analysis of the analysis operator, J. Funct. Anal. 148 (2) (1997) 408–447.
[23] I.W. Selesnick, Smooth wavelet tight frames with zero moments, Appl. Comput. Harmon. Anal. 10 (2000) 163–181.
[24] I.W. Selesnick, A.F. Abdelnour, Symmetric wavelet tight frames with two generators, Appl. Comput. Harmon. Anal. 17 (2004) 211–225.
[25] Y. Shen, S. Li, Q. Mo, Complex wavelets and framelets from pseudo splines, J. Fourier Anal. Appl. 16 (2010) 885–900.
[26] Z. Shen, Wavelet Frames and Image Restorations, Proceedings of the International Congress of Mathematicians 4 (2010) 2834–2863., Univ. Press, 2001.
[27] P.P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice-Hall, Englewood Cliffs, 1993.
[28] X. Zhuang, Matrix extension with symmetry and its applications, in: M. Neamtu, L.L. Schumaker (Eds.), Approximation Theory XIII: San Antonio 2010, Springer, 2012.