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Matrix extension with symmetry and construction of biorthogonal multiwavelets with any integer dilation

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ABSTRACT

In this paper, we investigate the biorthogonal matrix extension problem with symmetry and its application to construction of biorthogonal multiwavelets. Given a pair of biorthogonal matrices (P, \tilde{P}) , the biorthogonal matrix extension problem is to find a pair of extension matrices (P_e, \tilde{P}_e) of Laurent polynomials with symmetry such that the submatrix of the first r rows of P_e, \tilde{P}_e is the given matrix P, \tilde{P} , respectively; P_e and \tilde{P}_e are biorthogonal satisfying $P_e \tilde{P}_e^* = I_s$; and P_e and \tilde{P}_e have the same compatible symmetry. We satisfactorily solve the biorthogonal matrix extension problem with symmetry and provide a step-by-step algorithm for constructing the desired pair of extension matrices (P_e, \tilde{P}_e) from the given pair of matrices (P, \tilde{P}) . Moreover, our results cover the case for paraunitary matrix extension with symmetry (i.e., the given pair satisfies $P = \tilde{P}$). Matrix extension plays an important role in many areas such as wavelet analysis, electronic engineering, system sciences, and so on. As an application of our general results on biorthogonal matrix extension with symmetry, we obtain a satisfactory algorithm for constructing univariate biorthogonal multiwavelets with symmetry for any dilation factor d from a given pair of biorthogonal d -refinable function vectors with symmetry. Correspondingly, pairs of d -dual filter banks with the perfect reconstruction property and with symmetry can be derived by applying our algorithm to a given pair of d -dual low-pass filters with symmetry. Several examples of symmetric biorthogonal multiwavelets are provided to illustrate our results in this paper.

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1. Introduction

The multiresolution analysis (MRA) plays a fundamental role in wavelet analysis, which provides not only a framework unifying both the continuous and digital realms, but also a systematic approach of constructing wavelets in terms of wavelet masks. Under the framework of MRA, the construction of multiwavelet systems having some desirable properties – say, (bi)orthogonality, symmetry, regularity, and so on – can be reduced to two main parts. One part is on the construction of refinable function vectors that satisfy certain desired properties. Another part is on the derivation of multiwavelet generators, which should be able to inherit certain properties similar to those of their refinable function vectors. From the point of view of electronic engineering, such multiwavelet systems are associated with filter banks with the perfect reconstruction property. The first part corresponds to the design of low-pass filters (scaling masks), while the second part is to derive high-pass filters (wavelet masks) so that together they form filter banks with the perfect reconstruction property. It is well known that the second part can be formulated as a *matrix extension* problem, see [31,35].

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Since the introduction of biorthogonal wavelets by Cohen et al. in [4], biorthogonal wavelets have been widely used in many applications, especially in signal/image processing. For example, the famous (9, 7)-biorthogonal wavelet system has been shown to be a very effective system for image compression and is now implemented in part of the standard JPEG 2000. On the other hand, among many desirable properties, *symmetry* property is one of the most desirable properties for multiwavelet systems in wavelet analysis or filter banks in electronic engineering. Symmetry plays a very important and sometimes crucial role in applications. The ‘linear phase’ property and the opportunity not to increase the dimension of data for encoding finite data vectors with symmetric wavelets cannot be compensated by any other properties, not to mention the reduction of the computational cost using a symmetric system. In this paper, we are interested in the matrix extension problem associated with the construction of univariate biorthogonal multiwavelets with symmetry for *any dilation factor* d . More precisely, in terms of filter banks, we are interested in deriving high-pass filters from low-pass filters so that together they form filter banks with the perfect reconstruction property and with the symmetry property simultaneously. For the case $d = 2$, it is well known that the construction of biorthogonal wavelets is trivial once a pair of biorthogonal 2-refinable functions (scaling functions) is given, see [4,7]. However, how to construct biorthogonal multiwavelets with symmetry from a pair of biorthogonal d -refinable function vectors satisfying certain symmetry pattern for any dilation factor $d > 2$ remains *open*. One of the main reasons is because its associated matrix extension problem becomes far more complicated when integrated with symmetry. Extra effort is needed to guarantee symmetry property of extension matrices from which multiwavelets can possess the symmetry property as well. The objective of this paper is to present a systematic approach along with a step-by-step algorithm for this problem, which gives an *affirmative answer* to the matrix extension problem with symmetry for the construction of biorthogonal multiwavelets with symmetry.

1.1. The matrix extension problem

To facilitate and simplify our presentation of the matrix extension problem, let us introduce some notation and definitions. In short, we are going to investigate how to extend vectors or submatrices of Laurent polynomials with coefficients in a field to square matrices of Laurent polynomials with some desirable properties.

In the biorthogonal setting, the most interesting coefficient field is of course the rational number field \mathbb{Q} . Our results in this paper apply not only for the rational number field \mathbb{Q} but also for any subfield \mathbb{F} of the complex number field \mathbb{C} that is closed under complex conjugate; that is, $\bar{x} \in \mathbb{F}$ provided $x \in \mathbb{F}$. Several particular examples of such subfields \mathbb{F} are $\mathbb{F} = \mathbb{Q}(\sqrt{t})$ (the field of algebraic numbers), $\mathbb{F} = \mathbb{R}$ (the field of real numbers), and $\mathbb{F} = \mathbb{C}$ (the field of complex numbers). Throughout the paper, \mathbb{F} denotes any such a subfield of \mathbb{C} .

Let $p(z) = \sum_{k \in \mathbb{Z}} p_k z^k$, $z \in \mathbb{C} \setminus \{0\}$ be a Laurent polynomial with coefficients $p_k \in \mathbb{F}$ for all $k \in \mathbb{Z}$. We say that $p(z)$ has *symmetry* if its coefficient sequence $\{p_k\}_{k \in \mathbb{Z}}$ has symmetry; more precisely, there exist $\varepsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ such that

$$p_{c-k} = \varepsilon p_k \quad \forall k \in \mathbb{Z}. \tag{1.1}$$

If $\varepsilon = 1$, then $p(z)$ is *symmetric* about the point $c/2$; if $\varepsilon = -1$, then $p(z)$ is *antisymmetric* about the point $c/2$. Symmetry of a Laurent polynomial can be conveniently expressed using a symmetry operator S defined by

$$Sp(z) := \frac{p(z)}{p(z^{-1})}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{1.2}$$

When $p(z)$ is not identically zero, it is evident that (1.1) holds if and only if $Sp = \varepsilon z^c$, i.e., Sp is a monomial. For the zero polynomial, it is very natural that $S0$ can be assigned any symmetry pattern. By shifting the symmetry center so that $c \in \{0, -1\}$, there are at most *four basic symmetry types* for a Laurent polynomial $p(z)$ having symmetry in terms of S : $Sp \in \{1, -1, z^{-1}, -z^{-1}\}$. The prototype polynomials for these four symmetry types are: $z + z^{-1}$, $z - z^{-1}$, $1 + z^{-1}$, and $1 - z^{-1}$.

For an $r \times s$ matrix $P(z) = \sum_{k \in \mathbb{Z}} P_k z^k$ with $P_k \in \mathbb{F}^{r \times s}$ for all $k \in \mathbb{Z}$, we denote

$$P^*(z) := \sum_{k \in \mathbb{Z}} P_k^* z^{-k} \quad \text{with } P_k^* := \overline{P_k}^T, \quad k \in \mathbb{Z}, \tag{1.3}$$

where $\overline{P_k}^T$ denotes the transpose of the complex conjugate of the constant matrix P_k in $\mathbb{F}^{r \times s}$. Note that with the \star notation, we are actually considering z in the unit circle $|z| = 1$, $z \in \mathbb{C}$. If $P(z)$ is an $r \times s$ matrix of Laurent polynomials with symmetry, then we can apply the operator S to each entry of $P(z)$; that is, SP is an $r \times s$ matrix such that $[SP]_{j,k} := S([P]_{j,k})$, where $[P]_{j,k}$ is the (j, k) -entry of the matrix $P(z)$.

For a matrix of Laurent polynomials, another important property is the support of its coefficient sequence. For $P(z) = \sum_{k \in \mathbb{Z}} P_k z^k$ such that $P_k = \mathbf{0}$ for all $k \in \mathbb{Z} \setminus [m, n]$ with $P_m \neq \mathbf{0}$ and $P_n \neq \mathbf{0}$, we define its *coefficient support* to be $\text{csupp}(P) := [m, n]$ and the length of its coefficient support to be $|\text{csupp}(P)| := n - m$. In particular, we define $\text{csupp}(\mathbf{0}) := \emptyset$, the empty set, and $|\text{csupp}(\mathbf{0})| := -\infty$. Also, we use $\text{coeff}(P, k) := P_k$ to denote the coefficient matrix (vector) P_k of z^k in $P(z)$.

Throughout this paper, $\mathbf{0}$ always denotes a general zero matrix whose size can be determined in the context. r and s denote two positive integers such that $1 \leq r \leq s$. I_n is the identity matrix of size $n \times n$. Now the matrix extension problem we consider in this paper can be stated as follows.

Problem (*Biorthogonal matrix extension problem*). Let $(P(z), \tilde{P}(z))$ be a pair of $r \times s$ matrices of Laurent polynomials with coefficients in a subfield \mathbb{F} of \mathbb{C} such that $P(z)\tilde{P}^*(z) = I_r$ for all $z \in \mathbb{C} \setminus \{0\}$. Find a pair $(P_e(z), \tilde{P}_e(z))$ of $s \times s$ extension matrices of Laurent polynomials with coefficients in \mathbb{F} as well and with the following desiderata.

- (D1) Extension: $[I_r, \mathbf{0}]P_e(z) = P(z)$ and $[I_r, \mathbf{0}]\tilde{P}_e(z) = \tilde{P}(z)$; that is, the submatrices of the first r rows of $P_e(z)$ and $\tilde{P}_e(z)$ are the given matrices $P(z)$ and $\tilde{P}(z)$, respectively.
- (D2) Biorthogonality property: $P_e(z)\tilde{P}_e^*(z) = I_s$ for all $z \in \mathbb{C} \setminus \{0\}$; that is, $P_e(z)$ and $\tilde{P}_e(z)$ are *biorthogonal*.
- (D3) Compatible symmetry: when $P(z)$ and $\tilde{P}(z)$ have *compatible symmetry*, i.e., SP and $S\tilde{P}$ have certain symmetry pattern, the extension matrices $P_e(z)$ and $\tilde{P}_e(z)$ also have compatible symmetry.
- (D4) Support control: the lengths of coefficient supports of $P_e(z)$ and $\tilde{P}_e(z)$ are controlled by those of $P(z)$ and $\tilde{P}(z)$ in some way.

When the given pair $(P(z), \tilde{P}(z))$ satisfies $P(z) = \tilde{P}(z)$, the above biorthogonal matrix extension problem becomes the paraunitary matrix extension problem, which requires the extension pair $(P_e(z), \tilde{P}_e(z))$ satisfies $P_e(z) = \tilde{P}_e(z)$. In Section 2, we shall give a precise definition of *compatible symmetry* using the symmetry operator S . The connection of the biorthogonal matrix extension problem to biorthogonal multiwavelets is discussed in Section 3.

1.2. Related work

The matrix extension problem plays a fundamental role in many areas such as electronic engineering, system sciences, mathematics, etc. We mention only a few references here on this topic, see [1–3,5,7,16,19,22,25,29,31–33,35,36]. For example, matrix extension is an indispensable tool in the design of filter banks in electronic engineering [29,35,36] and in the construction of multiwavelets in wavelet analysis [3,5,7,8,16,19,22,28,30,31,34,32].

For the construction of orthonormal multiwavelets, the biorthogonal matrix extension problem is reduced to the paraunitary matrix extension problem, i.e., $P(z) = \tilde{P}(z)$ and the extension matrices $P_e(z) = \tilde{P}_e(z)$. The paraunitary matrix extension problem is thus a special case of the biorthogonal matrix extension problem. It has been studied in [35] in electronic engineering and in [31] in wavelet analysis without consider any symmetry issue. In [32,22], this problem is solved with symmetry constrain for the special case $r = 1$. For general case $r > 1$ with symmetry constrain, this problem has been completely solved by Han and Zhuang in [26,27].

For $r = 1$, that is, the extension of a pair of vectors of Laurent polynomials to a pair of matrices of Laurent polynomials, Goh and Yap in [18] studied the biorthogonal matrix extension problem and presented a step-by-step algorithm for deriving the extension matrices. Yet neither did they concern about the lengths of the coefficient supports of the extension matrices, nor they considered any symmetry constrain on the extension matrices. In [2], Chui, Han, and Zhuang proposed a dual-chain approach for this problem, which first constructs a top-down dual-chain that essentially reduces the lengths of the coefficient supports of the given pair of vectors to zero and then derives a bottom-up dual-chain that produces the desired pair of extension matrices. Using this approach, symmetry can be easily adapted in the top-down and bottom-up dual-chain so that the extension matrices do have desired symmetry pattern.

Without symmetry constrain, Goh and Yap's algorithm works for a more general situation $r > 1$, i.e., the extension of a pair of submatrices to a pair of square matrices, while at this point, the dual-chain approach in [2] does not apply to this situation even without symmetry constrain. When consider symmetry for the general case $r > 1$, there are only a few results in the literature (e.g. [1,6]) and most of them concern only about some very special cases, for example $r = 2$ or $d = 2$.

In higher dimensions, the biorthogonal matrix extension problem is closely related to the famous Serre conjecture saying that a unimodular line of algebraic polynomials can be extended to a unimodular matrix. The Serre conjecture was solved by Quillen and Suslin independently, see [36]. Though the problem is solved theoretically, implementable algorithms are not known in general. Using results on syzygy modules over a multivariate polynomial ring, the biorthogonal matrix extension problem can be partially solved for some special cases, for example, see [14,17,30,34]. But in high dimensions, when comes to symmetry, it is not even clear what appropriate symmetry patterns should one consider for the given pair of vectors or submatrices. No general algorithm is available for this problem as far as we concern.

1.3. Our contributions

Due to the flexibility of the biorthogonality relation $P(z)\tilde{P}^*(z) = I_r$ for all $z \in \mathbb{C} \setminus \{0\}$, the biorthogonal matrix extension problem is more complicated than that for the paraunitary matrix extension problem considered in [26], which shows that an $r \times s$ paraunitary matrix $P(z)$ (i.e., $P(z)P^*(z) = I_r$ for all $z \in \mathbb{C} \setminus \{0\}$) with compatible symmetry *if and only if* the extension matrix $P_e(z)$ has the following properties.

- (P1) Extension: $[I_r, \mathbf{0}]P_e(z) = P(z)$, the submatrix of the first r rows of $P_e(z)$ is $P(z)$.
- (P2) Paraunitary property: $P_e(z)P_e^*(z) = I_s$ for all $z \in \mathbb{C} \setminus \{0\}$.
- (P3) Compatible symmetry: $P_e(z)$ has compatible symmetry.

(P4) Support control: $|\text{csupp}([P_e]_{j,k})| \leq \max_{1 \leq n \leq r} |\text{csupp}([P]_{n,k})|$ for $1 \leq j \leq s$, i.e., the length of the coefficient support of $P_e(z)$ never exceeds that of $P(z)$.

For the biorthogonal matrix extension problem, we certainly expect that the pair of the extension matrices should have properties similar to above properties (P1)–(P4); that is, (D1)–(D4) as we stated in the *biorthogonal matrix extension problem*. Moreover, when designing the algorithm for this biorthogonal matrix extension problem, it should be able to recover (P1)–(P4) when the algorithm is applied to a pair of matrices $(P(z), \tilde{P}(z))$ s.t. $P(z) = \tilde{P}(z)$. In other words, the biorthogonal results should be able to generalize the results of paraunitary matrix extension. To this end, we need to face two difficulties. One is to provide suitable characterizations for the symmetry pattern of the given pair $(P(z), \tilde{P}(z))$. Another is to design an algorithm that gives appropriate support control of the extension matrices.

For the symmetry pattern of the given pair, as in [26], compatible symmetry (see Section 2.1 for the definition) characterized by the symmetry operator S plays an important role in our study of the biorthogonal matrix extension problem. Compatible symmetry characterizes the condition on how and when multiplication of two matrices of Laurent polynomials results in a matrix of Laurent polynomials having compatible symmetry. For the support control, one might expect the coefficient support lengths of the extension matrices $P_e(z)$ and $\tilde{P}_e(z)$ being controlled by the maximal length of coefficient supports of $P(z)$ and $\tilde{P}(z)$ similar to (P4) for the paraunitary case. It turns out that this is *not* true. There are counter-examples (see Example 1 in Section 2) showing that we can no longer expect such a nice result as property (P4) for the biorthogonal situation. That is, in this case, the length of the coefficient supports of the extension matrices might not be controlled by one of the given matrices. Instead, our results show that it is possible to control the lengths of the coefficient supports of the extension matrices $P_e(z)$ and $\tilde{P}_e(z)$ by the sum of those of two submatrices $P(z)$ and $\tilde{P}(z)$. In summary, our main result (also see Theorem 1 in Section 2.1) reads:

Solution (to the *biorthogonal matrix extension problem*). Let $(P(z), \tilde{P}(z))$ be a pair of $r \times s$ matrices of Laurent polynomials with coefficients in a subfield \mathbb{F} of \mathbb{C} . Then, $P(z)\tilde{P}^*(z) = I_r$ for all $z \in \mathbb{C} \setminus \{0\}$ and $P(z), \tilde{P}(z)$ have compatible symmetry if and only if there exists a pair $(P_e(z), \tilde{P}_e(z))$ of extension matrices of Laurent polynomials with coefficients in \mathbb{F} as well such that desiderata (D1)–(D4) hold. In particular, when $P(z) = \tilde{P}(z)$, properties (P1)–(P4) hold.

The contributions of this paper lie in the following aspects. First, we satisfactorily solve the biorthogonal matrix extension problem with symmetry for any integers r and s such that $1 \leq r \leq s$. More importantly, we obtain a complete representation of any pair of $r \times s$ matrices $(P(z), \tilde{P}(z))$ having compatible symmetry with $1 \leq r \leq s$. This representation leads to a step-by-step algorithm for deriving a pair of desired extension matrices $(P_e(z), \tilde{P}_e(z))$ from a given pair of submatrices $(P(z), \tilde{P}(z))$. Second, our characterizations of matrices of Laurent polynomials with symmetry in terms of compatibility symmetry plays a critical role in the study of the biorthogonal matrix extension problem. Third, we provide a result on controlling the coefficient support of the desired matrices, which is *optimal* in view of the counter-example in Example 1. Support control is of importance in both theory and application, since short support of a filter or a multiwavelet is a highly desirable property and short support usually means a fast algorithm and simple implementation in practice. Fourth, we provide a complete analysis and a systematic construction algorithm for symmetric biorthogonal multiwavelets. Fifth, our algorithm extends the results in [26] in the sense that when setting $P(z) = \tilde{P}(z)$, our results recover results in [26]. Finally, most of the literature on the matrix extension problem considers only Laurent polynomials with coefficients in the special field \mathbb{C} or \mathbb{R} . In this paper, our setting is under a general field \mathbb{F} , which can be any subfield of \mathbb{C} .

1.4. Contents

Here is the structure of this paper. In Section 2, we shall introduce our main theorem on biorthogonal matrix extension with symmetry, prove it based on a key lemma, and provide a step-by-step algorithm for the construction of the extension matrices. In Section 3, we shall discuss the applications of our results to the construction of symmetric biorthogonal multiwavelets in wavelet analysis. Examples will be provided to illustrate our results and algorithms in Section 4. Proof of the key lemma is postponed to Section 5. Remarks shall be given in the last section.

2. Main theorem and algorithm

In this section, we first introduce the notion of compatible symmetry, then present our main theorem of this paper, and finally provide a step-by-step algorithm for constructing a pair of extension matrices $(P_e(z), \tilde{P}_e(z))$ having compatible symmetry from a given pair of submatrices $(P(z), \tilde{P}(z))$.

2.1. Compatible symmetry and our main theorem

Recall that the symmetry operator S is given by $Sp(z) := \frac{p(z)}{p(z^{-1})}$ for $z \in \mathbb{C} \setminus \{0\}$ and for an $r \times s$ matrix $P(z) = \sum_{k \in \mathbb{Z}} P_k z^k$, SP is an $r \times s$ matrix such that $[SP]_{j,k} := S([P]_{j,k})$. For two matrices $P(z)$ and $Q(z)$ of Laurent polynomials with symmetry, even though all the entries in $P(z)$ and $Q(z)$ have symmetry, their sum $P(z) + Q(z)$, difference $P(z) - Q(z)$, or product

$P(z)Q(z)$, if well defined, generally may not have symmetry any more. This is one of the difficulties for matrix extension with symmetry. In order for $P(z) \pm Q(z)$ or $P(z)Q(z)$ to possess some symmetry, the symmetry patterns of $P(z)$ and $Q(z)$ should be compatible. For example, if $SP = SQ$, that is, both $P(z)$ and $Q(z)$ have the same symmetry pattern, then indeed $P(z) \pm Q(z)$ has symmetry and $S(P \pm Q) = SP = SQ$. But how to make sure the multiplication of two matrices $P(z)$ and $Q(z)$ possesses certain symmetry pattern is not that obvious since not only do we need to guarantee that the product of each row of $P(z)$ and each column of $Q(z)$ has symmetry, but also the product matrix itself has certain symmetry pattern as well. For this, we need the concept of compatible symmetry. We say that *the symmetry of $P(z)$ is compatible* or *$P(z)$ has compatible symmetry*, if

$$SP(z) = (S\theta_1)^*(z)S\theta_2(z), \tag{2.1}$$

for some $1 \times r$ and $1 \times s$ row vectors $\theta_1(z)$ and $\theta_2(z)$ of Laurent polynomials with symmetry. Using the concept of compatible symmetry, we can define mutually compatible symmetry for two matrices having compatible symmetry so that their product matrix has compatible symmetry as well. For an $r \times s$ matrix $P(z)$ and an $s \times t$ matrix $Q(z)$ of Laurent polynomials, we say that $(P(z), Q(z))$ has mutually compatible symmetry if

$$SP(z) = (S\theta_1)^*(z)S\theta(z) \quad \text{and} \quad SQ(z) = (S\theta)^*(z)S\theta_2(z) \tag{2.2}$$

for some $1 \times r$, $1 \times s$, $1 \times t$ row vectors $\theta_1(z)$, $\theta(z)$, $\theta_2(z)$ of Laurent polynomials with symmetry. If $(P(z), Q(z))$ has mutually compatible symmetry as in (2.2), then it is easy to check that their product $P(z)Q(z)$ has compatible symmetry. In fact, noting that $[PQ]_{j,k} = \sum_{\ell=1}^s [P]_{j,\ell}[Q]_{\ell,k}$ and $S([P]_{j,\ell}[Q]_{\ell,k}) = [S\theta_1]_j^*[S\theta]_\ell[S\theta_2]_k = [S\theta_1]_j^*[S\theta_2]_k$, we have $S(PQ) = (S\theta_1)^*S\theta_2$.

Thanks to the compatible symmetry, we can state our main theorem on biorthogonal matrix extension with symmetry as follows, which shows that for a pair $(P(z), \tilde{P}(z))$ of $r \times s$ biorthogonal matrices having compatible symmetry, we can extend them to a pair $(P_e(z), \tilde{P}_e(z))$ of $s \times s$ biorthogonal matrices having compatible symmetry as well.

Theorem 1. *Let \mathbb{F} be a subfield of \mathbb{C} . Let $(P(z), \tilde{P}(z))$ be a pair of $r \times s$ matrices of Laurent polynomials with coefficients in \mathbb{F} . Then $P(z)\tilde{P}^*(z) = I_r$ for all $z \in \mathbb{C} \setminus \{0\}$ and $SP = S\tilde{P} = (S\theta_1)^*S\theta_2$ for some $1 \times r$, $1 \times s$ vectors $\theta_1(z)$, $\theta_2(z)$ of Laurent polynomials with symmetry, i.e., $P(z)$ and $\tilde{P}(z)$ are biorthogonal and both have the same compatible symmetry, if and only if, there exists a pair $(P_e(z), \tilde{P}_e(z))$ of $s \times s$ square matrices of Laurent polynomials with coefficients in \mathbb{F} such that*

- (i) $[I_r, \mathbf{0}]P_e(z) = P(z)$ and $[I_r, \mathbf{0}]\tilde{P}_e(z) = \tilde{P}(z)$; that is, the submatrices of the first r rows of $P_e(z)$ and $\tilde{P}_e(z)$ are $P(z)$ and $\tilde{P}(z)$, respectively.
- (ii) $P_e(z)$ and $\tilde{P}_e(z)$ are biorthogonal: $P_e(z)\tilde{P}_e^*(z) = I_s$ for all $z \in \mathbb{C} \setminus \{0\}$.
- (iii) Both $P_e(z)$ and $\tilde{P}_e(z)$ have compatible symmetry: $SP_e = S\tilde{P}_e = (S\theta)^*S\theta_2$ for some $1 \times s$ vector $\theta(z)$ of Laurent polynomials with symmetry.
- (iv) $P_e(z)$ and $\tilde{P}_e(z)$ can be represented as products of matrices:

$$P_e(z) = P_J(z) \cdots P_1(z), \quad \tilde{P}_e(z) = \tilde{P}_J(z) \cdots \tilde{P}_1(z), \tag{2.3}$$

where $(P_j(z), \tilde{P}_j(z))$, $j = 1, \dots, J$ are pairs of biorthogonal matrices of Laurent polynomials with symmetry. Moreover, each pair of $(P_{j+1}(z), P_j(z))$ and $(\tilde{P}_{j+1}(z), \tilde{P}_j(z))$ has mutually compatible symmetry for all $j = 1, \dots, J - 1$.

- (v) The coefficient supports of $P_e(z)$ and $\tilde{P}_e(z)$ are controlled by those of $P(z)$ and $\tilde{P}(z)$ in the following sense:

$$\max\{|\text{csupp}(P_e)|, |\text{csupp}(\tilde{P}_e)|\} \leq r \cdot (|\text{csupp}(P)| + |\text{csupp}(\tilde{P})|). \tag{2.4}$$

In particular, when $P(z) = \tilde{P}(z)$, items (i)–(iv) hold with $P_e(z) = \tilde{P}_e(z)$ and item (v) is replaced by

$$|\text{csupp}([P_e]_{j,k})| \leq \max_{1 \leq n \leq r} |\text{csupp}([P]_{n,k})|, \quad 1 \leq j \leq s; \tag{2.5}$$

that is, the length of the coefficient support of $P_e(z)$ never exceeds that of $P(z)$.

The representation in (2.3) (without symmetry) is often called the cascade structure in engineering literature, see [35]. Comparing (2.4) and (2.5), one may wonder why the support control for biorthogonal matrix extension with symmetry is worse than and not as subtle as that for the paraunitary matrix extension problem. We would like to point out that (2.4) is the worst case support control estimate. In practice, the situation is not as bad as (2.4) (see Examples 2–4). But in general, as shown by the following example for $r = 1$, (2.4) is optimal in the sense that one cannot expect support control better than (2.4).

Example 1. Consider two 1×3 vectors of Laurent polynomials $p(z) = [1, 0, a(z)]$ and $\tilde{p}(z) = [1, \tilde{a}(z), 0]$ with $a(z)$ and $\tilde{a}(z)$ being two Laurent polynomials with symmetry satisfying $|\text{csupp}(a(z))| > 0$ and $|\text{csupp}(\tilde{a}(z))| > 0$, for instance, $a(z) = z + 1 + z^{-1}$ and $\tilde{a}(z) = z - z^{-1}$. It is obvious that $p(z)\tilde{p}^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$ and the coefficient supports of $p(z)$ and $\tilde{p}(z)$ are completely determined by those of $a(z)$ and $\tilde{a}(z)$.

Let $P_e(z)$ and $\tilde{P}_e(z)$ be their extension matrices of Laurent polynomials with symmetry such that $P_e(z)\tilde{P}_e^*(z) = I_3$ for all $z \in \mathbb{C} \setminus \{0\}$ (the existence of such extension matrices is guaranteed by the Quillen–Suslin theorem). Then $P_e(z)$ and $\tilde{P}_e(z)$ must take the following form:

$$P_e(z) = \begin{bmatrix} 1 & 0 & a(z) \\ -b_1(z)\tilde{a}^*(z) & b_1(z) & c_1(z) \\ -b_2(z)\tilde{a}^*(z) & b_2(z) & c_2(z) \end{bmatrix}, \quad \tilde{P}_e(z) = \begin{bmatrix} 1 & \tilde{a}(z) & 0 \\ -\tilde{c}_1(z)a^*(z) & \tilde{b}_1(z) & \tilde{c}_1(z) \\ -\tilde{c}_2(z)a^*(z) & \tilde{b}_2(z) & \tilde{c}_2(z) \end{bmatrix},$$

where $b_1(z), \tilde{b}_1(z), b_2(z), \tilde{b}_2(z), c_1(z), \tilde{c}_1(z), c_2(z), \tilde{c}_2(z)$ are Laurent polynomials with symmetry. Both $P_e(z)$ and $\tilde{P}_e(z)$ are unimodular and hence the determinant $\det(P_e)$ of $P_e(z)$ must be a monomial. Note that $\det(P_e) = b_1(z)c_2(z) - b_2(z)c_1(z)$. Without loss of generality, we can assume $b_1(z)c_2(z) - b_2(z)c_1(z) = 1$. Using cofactors of $P_e(z)$, it is easy to show that $\tilde{P}_e(z) = (P_e(z)^{-1})^*$ must take the following form:

$$\tilde{P}_e(z) = \begin{bmatrix} 1 & \tilde{a}(z) & 0 \\ b_2^*(z)a^*(z) & c_2^*(z) + \tilde{a}(z)a^*(z)b_2^*(z) & -b_2^*(z) \\ -b_1^*(z)a^*(z) & -c_1^*(z) - \tilde{a}(z)a^*(z)b_1^*(z) & b_1^*(z) \end{bmatrix}.$$

On one hand, if $|\text{csupp}(b_1(z))| > 0$ or $|\text{csupp}(b_2(z))| > 0$, then from the first columns of $P_e(z)$ and $\tilde{P}_e(z)$, we see that one of the extension matrices $P_e(z)$ and $\tilde{P}_e(z)$ already has support length exceeding the maximal length of coefficient supports of the given columns $p(z)$ and $\tilde{p}(z)$. On the other hand, if both $|\text{csupp}(b_1(z))| = 0$ and $|\text{csupp}(b_2(z))| = 0$ (in this case, both $b_1(z)$ and $b_2(z)$ are monomials), then since $\tilde{b}_1(z) = c_2^*(z) + \tilde{a}(z)a^*(z)b_2^*(z)$ and $\tilde{b}_2(z) = -c_1^*(z) - \tilde{a}(z)a^*(z)b_1^*(z)$, either we choose $c_1(z), c_2(z)$ with coefficient supports comparable to $\tilde{a}(z)a^*(z)$ to make the lengths of coefficient supports of $\tilde{b}_1(z)$ and $\tilde{b}_2(z)$ small, in which case will result in long coefficient support of $P_e(z)$; or we choose $c_1(z), c_2(z)$ with coefficient supports comparable to $a(z)$ or $\tilde{a}(z)$ to make the coefficient support of $P_e(z)$ short, in which case will result in long coefficient support of $\tilde{P}_e(z)$.

In any case, the coefficient supports of $P_e(z)$ and $\tilde{P}_e(z)$ cannot be controlled by one of $P(z)$ and $\tilde{P}(z)$. For example, if $a(z) = z + 1 + z^{-1}$ and $\tilde{a}(z) = z - z^{-1}$, then $\tilde{a}(z)a^*(z) = z^2 + z - z^{-1} - z^{-2}$. No matter how we choose $b_1(z), b_2(z), c_1(z), c_2(z)$ in $P_e(z)$, one of $P_e(z)$ and $\tilde{P}_e(z)$ will have support length no less than 4, which is the sum of $|\text{csupp}(p)|$ and $|\text{csupp}(\tilde{p})|$. □

It is possible to control the coefficient supports of $P_e(z)$ and $\tilde{P}_e(z)$ by both $p(z)$ and $\tilde{p}(z)$. In Example 1, by properly choosing $b_1(z), b_2(z), c_1(z), c_2(z)$, we indeed have $\max\{|\text{csupp}(P_e)|, |\text{csupp}(\tilde{P}_e)|\} \leq |\text{csupp}(p)| + |\text{csupp}(\tilde{p})|$.

2.2. Proof of Theorem 1

In this subsection, we shall prove our main result, Theorem 1, and based on the proof, we shall provide a step-by-step extension algorithm for deriving the desired pair of extension matrices in the next subsection.

Before we continue to the proof, let us lay out the main idea: given a pair of $1 \times s$ vectors $(p(z), \tilde{p}(z))$ satisfying $p(z)\tilde{p}^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$ (note that with the \star notation, we are actually working on $|z| = 1, z \in \mathbb{C}$), we first find a pair $(B_1(z), \tilde{B}_1(z))$ of $s \times s$ biorthogonal matrices, which have some simple structure and reduce the length of the coefficient support of $p(z)$ or $\tilde{p}(z)$ (or both of them). Moreover, these two matrices preserve the compatible symmetry of both $p(z)$ and $\tilde{p}(z)$. That is, $p_1(z) := p(z)B_1(z)$ and $\tilde{p}_1(z) := \tilde{p}(z)\tilde{B}_1(z)$ are both vectors of Laurent polynomials with symmetry and $|\text{csupp}(p_1)|$ or $|\text{csupp}(\tilde{p}_1)|$ is reduced. Next, replace $(p(z), \tilde{p}(z))$ by the new pair $(p_1(z), \tilde{p}_1(z))$ which again satisfies $p_1(z)\tilde{p}_1^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$. Continuing this procedure, we can find a sequence of pairs of biorthogonal matrices $(B_1(z), \tilde{B}_1(z)), \dots, (B_K(z), \tilde{B}_K(z))$ that eventually reduce the lengths of coefficient supports of both $p(z)$ and $\tilde{p}(z)$ to 0. Then the product matrices $A(z) := B_1(z) \cdots B_K(z)$ and $\tilde{A}(z) := \tilde{B}_1(z) \cdots \tilde{B}_K(z)$ give us the desired extension matrices. This is the idea for the case of extending a pair $(p(z), \tilde{p}(z))$ of $1 \times s$ vectors. For extending a pair of $r \times s$ biorthogonal matrices $(P(z), \tilde{P}(z))$, we can apply the above procedure to each pair of rows of $P(z)$ and $\tilde{P}(z)$ and obtain the desired extension matrices.

Using the symmetry operator S , as mentioned, up to center shifting, there are at most four basic symmetry types for a Laurent polynomial with symmetry: $1, -1, z^{-1}, -z^{-1}$. For a vector of Laurent polynomials with symmetry, we can normalize the symmetry of this vector to be a vector of symmetry types involving only these four basic symmetry types too. Throughout this paper, $\mathbf{1}_n$ denotes the $1 \times n$ row vector $[1, \dots, 1]$ and $\mathbf{e}_j = [0, \dots, 0, 1, 0, \dots, 0]$ is the j th standard unit row vector. $[A]_{j,\cdot}$ denotes the j th row of a matrix A and $[A]_{j,k;\ell}$ is the row vector $[[A]_{j,k}, [A]_{j,k+1}, \dots, [A]_{j,\ell}]$.

Let $\theta(z)$ be a $1 \times n$ row vector of Laurent polynomials with symmetry such that $S\theta(z) = [\varepsilon_1 z^{c_1}, \dots, \varepsilon_n z^{c_n}]$ for some $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and $c_1, \dots, c_n \in \mathbb{Z}$. Then, the symmetry of any entry in the vector $\theta(z) \text{diag}(z^{-\lceil c_1/2 \rceil}, \dots, z^{-\lceil c_n/2 \rceil})$ belongs to $\{\pm 1, \pm z^{-1}\}$. Here $\lceil \cdot \rceil$ is the ceiling operation: $\lceil x \rceil$ denotes the smallest integer no less than x for $x \in \mathbb{R}$. Thus, there is a permutation matrix E_θ to regroup these four types of symmetries together so that

$$S(\theta U_{S\theta}) = [\mathbf{1}_{n_1}, -\mathbf{1}_{n_2}, z^{-1}\mathbf{1}_{n_3}, -z^{-1}\mathbf{1}_{n_4}], \tag{2.6}$$

where $U_{S\theta}(z) := \text{diag}(z^{-\lceil c_1/2 \rceil}, \dots, z^{-\lceil c_n/2 \rceil})E_\theta$ and n_1, \dots, n_4 are nonnegative integers uniquely determined by $S\theta$ so that $n_1 + \dots + n_4 = n$. Note that $U_{S\theta}$ is paraunitary.

Given a pair $(p(z), \tilde{p}(z))$ of vectors of Laurent polynomials with symmetry satisfying $p(z)\tilde{p}^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$, using the paraunitary matrix $U_{S\theta}$, we can assume that $p(z)$ and $\tilde{p}(z)$ are both vectors containing Laurent polynomials with at most four symmetry types. We have the following key lemma on the existence and properties of a pair $(B(z), \tilde{B}(z))$ of biorthogonal matrices that reduces the length of the coefficient support of the pair $(p(z), \tilde{p}(z))$.

Lemma 1. *Let $(p(z), \tilde{p}(z))$ be a pair of $1 \times s$ vectors of Laurent polynomials with symmetry and with coefficients in a subfield \mathbb{F} of \mathbb{C} such that $p(z)\tilde{p}^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$ and $S_p = S_{\tilde{p}} =: S\theta$ with $S\theta(z) = \varepsilon z^c [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some nonnegative integers s_1, \dots, s_4 satisfying $s_1 + \dots + s_4 = s$ and some $\varepsilon \in \{1, -1\}, c \in \{0, 1\}$. Suppose $|\text{csupp}(p)| > 0$. Then there exists a pair $(B(z), \tilde{B}(z))$ of $s \times s$ matrices of Laurent polynomials with symmetry and with coefficients in \mathbb{F} as well such that*

- (a) $B(z)$ and $\tilde{B}(z)$ are biorthogonal: $B(z)\tilde{B}^*(z) = I_s$ for all $z \in \mathbb{C} \setminus \{0\}$.
- (b) $SB = S\tilde{B} = (S\theta)^*S\theta_1$ with $S\theta_1 = \varepsilon z^c [\mathbf{1}_{s'_1}, -\mathbf{1}_{s'_2}, z^{-1}\mathbf{1}_{s'_3}, -z^{-1}\mathbf{1}_{s'_4}]$ for some nonnegative integers s'_1, \dots, s'_4 such that $s'_1 + \dots + s'_4 = s$.
- (c) The length of the coefficient support of $p(z)$ is reduced by that of $B(z)$, and $\tilde{B}(z)$ does not increase the length of the coefficient support of $\tilde{p}(z)$. That is,

$$|\text{csupp}(pB)| \leq |\text{csupp}(p)| - |\text{csupp}(B)| \quad \text{and} \quad |\text{csupp}(\tilde{p}\tilde{B})| \leq |\text{csupp}(\tilde{p})|.$$

In particular, when $p(z) = \tilde{p}(z)$, items (a)–(c) hold with $B(z) = \tilde{B}(z)$ and the coefficient support of $B(z)$ satisfies $\text{csupp}(B) \subseteq [-1, 1]$.

The above lemma shows that we can always find a pair of biorthogonal matrices with some simple structure and with compatible symmetry to reduce the length of the coefficient support of the given pair $(p(z), \tilde{p}(z))$. Proof of this lemma is postponed to Section 5 for the purpose of clear presentation. Now, we can prove Theorem 1 using Lemma 1.

Proof of Theorem 1. The proof for the sufficiency part of the theorem is straightforward. Let us prove the necessity part of the theorem.

For an $r \times s$ matrix $P(z)$ of Laurent polynomials with compatible symmetry as in (2.1), it is easy to see that $Q(z) := U_{S\theta_1}^*(z)P(z)U_{S\theta_2}(z)$ (given θ , $U_{S\theta}$ is obtained by (2.6)) has the symmetry pattern as follows.

$$SQ = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}], \tag{2.7}$$

where r_1, \dots, r_4 and s_1, \dots, s_4 are nonnegative integers such that $r_1 + \dots + r_4 = r$ and $s_1 + \dots + s_4 = s$. Note that $U_{S\theta_1}(z)$ and $U_{S\theta_2}(z)$ do not increase the length of the coefficient support of $P(z)$.

First, we normalize the symmetry patterns of $P(z)$ and $\tilde{P}(z)$ to the standard form as in (2.7). Let $Q(z) := U_{S\theta_1}^*(z)P(z)U_{S\theta_2}(z)$ and $\tilde{Q}(z) := U_{S\theta_1}^*(z)\tilde{P}(z)U_{S\theta_2}(z)$. Then the symmetry of each row of $Q(z)$ or $\tilde{Q}(z)$ is of the form $\varepsilon z^c [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some $\varepsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$.

Next, let $p(z) := [Q(z)]_{1,:}$ and $\tilde{p}(z) := [\tilde{Q}(z)]_{1,:}$, the first rows of $Q(z)$ and $\tilde{Q}(z)$, respectively. Applying Lemma 1 recursively and in view of item (c) in Lemma 1, we can find pairs of biorthogonal matrices of Laurent polynomials $(B_1(z), \tilde{B}_1(z)), \dots, (B_K(z), \tilde{B}_K(z))$ such that $p(z)B_1(z) \cdots B_K(z) = [1, 0, \dots, 0]$ and $\tilde{p}(z)\tilde{B}_1(z) \cdots \tilde{B}_K(z) = [1, q(z)]$ for some $1 \times (s - 1)$ vector $q(z)$ of Laurent polynomials with symmetry. Note that by item (b) of Lemma 1, all pairs $(B_j(z), B_{j+1}(z))$ and $(\tilde{B}_j(z), \tilde{B}_{j+1}(z))$ for $j = 1, \dots, K - 1$ have mutually compatible symmetry. Now construct $B_{K+1}(z), \tilde{B}_{K+1}(z)$ as follows:

$$B_{K+1}(z) = \begin{bmatrix} 1 & \mathbf{0} \\ q^*(z) & I_{s-1} \end{bmatrix}, \quad \tilde{B}_{K+1}(z) = \begin{bmatrix} 1 & -q(z) \\ \mathbf{0} & I_{s-1} \end{bmatrix}.$$

$B_{K+1}(z)$ and $\tilde{B}_{K+1}(z)$ are biorthogonal and have compatible symmetry. Let $A_1(z) := B_1(z) \cdots B_K(z)B_{K+1}(z)$ and $\tilde{A}_1(z) := \tilde{B}_1(z) \cdots \tilde{B}_K(z)\tilde{B}_{K+1}(z)$. Then $A_1(z)$ and $\tilde{A}_1(z)$ are biorthogonal and we have $p(z)A_1(z) = \tilde{p}(z)\tilde{A}_1(z) = \mathbf{e}_1 = [1, 0, \dots, 0]$.

Now, we show that $Q(z)A_1(z)$ and $\tilde{Q}(z)\tilde{A}_1(z)$ are of the forms:

$$Q(z)A_1(z) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_1(z) \end{bmatrix}, \quad \tilde{Q}(z)\tilde{A}_1(z) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{Q}_1(z) \end{bmatrix}$$

for some $(r - 1) \times (s - 1)$ matrices $Q_1(z)$ and $\tilde{Q}_1(z)$ of Laurent polynomials with symmetry. In fact, by the biorthogonality relation, we have $(Q(z)A_1(z))(\tilde{A}_1^*(z)\tilde{Q}^*(z)) = I_r$ for all $z \in \mathbb{C} \setminus \{0\}$. Since the first row of $(Q(z)A_1(z))$ is $[1, 0, \dots, 0]$, the first row of $(\tilde{A}_1^*(z)\tilde{Q}^*(z))$ must be $[1, 0, \dots, 0]$. Consequently, the first column of $\tilde{Q}(z)\tilde{A}_1(z)$ is $[1, 0, \dots, 0]^T$. Similarly, the first column of $Q(z)A_1(z)$ must be $[1, 0, \dots, 0]^T$. Therefore, $Q(z)A_1(z)$ and $\tilde{Q}(z)\tilde{A}_1(z)$ are of forms as above. Moreover, in view of item (b) of Lemma 1, $Q_1(z)$ and $\tilde{Q}_1(z)$ have compatible symmetry and satisfy $SQ_1 = S\tilde{Q}_1$.

Employing the standard procedure of induction, we can find a sequence of pairs of biorthogonal matrices $(A_1(z), \tilde{A}_1(z)), \dots, (A_J(z), \tilde{A}_J(z))$ having compatible symmetry for some integer J such that

$$Q(z)A_1(z) \cdots A_J(z) = \tilde{Q}(z)\tilde{A}_1(z) \cdots \tilde{A}_J(z) = I_r.$$

Consequently, the pair of biorthogonal extension matrices $(P_e(z), \tilde{P}_e(z))$ is given by

$$P_e(z) = \text{diag}(U_{S\theta_1}(z), I_{s-r}) \tilde{A}_J^*(z) \cdots \tilde{A}_1^*(z) U_{S\theta_2}^*(z),$$

$$\tilde{P}_e(z) = \text{diag}(U_{S\theta_1}(z), I_{s-r}) A_J^*(z) \cdots A_1^*(z) U_{S\theta_2}^*(z).$$

Now, items (i) and (ii) of Theorem 1 follow from the above construction and item (a) of Lemma 1, items (iii) and (iv) are the result of item (b) of Lemma 1, and finally item (v) is due to (c) of Lemma 1. In particular, when $P(z) = \tilde{P}(z)$, we have $P_e(z) = \tilde{P}_e(z)$ and (2.4) is replaced by (2.5). \square

2.3. Algorithm

According to the proof of Theorem 1, we have an extension algorithm for Theorem 1, see Algorithm 1. Our algorithm has three main steps: initialization, support reduction, and finalization. The step of initialization reduces the symmetry pattern of $(P(z), \tilde{P}(z))$ to a standard form. The step of support reduction is the main body of the algorithm, producing a sequence of pairs of biorthogonal matrices $(A_1(z), \tilde{A}_1(z)), \dots, (A_J(z), \tilde{A}_J(z))$ that reduce the lengths of the coefficient supports of both $P(z)$ and $\tilde{P}(z)$ to 0. The step of finalization generates the desired pair of extension matrices $(P_e(z), \tilde{P}_e(z))$ as in Theorem 1.

Algorithm 1 Biorthogonal matrix extension with symmetry.

- (a) **Input:** A pair $(P(z), \tilde{P}(z))$ as in Theorem 1 with $SP = S\tilde{P} = (S\theta_1)^* S\theta_2$ for two $1 \times r, 1 \times s$ row vectors $\theta_1(z), \theta_2(z)$ of Laurent polynomials with symmetry.
- (b) **Output:** A desired pair of matrices $(P_e(z), \tilde{P}_e(z))$ satisfying all the properties in Theorem 1.
- (c) **Initialization:** Let $Q(z) := U_{S\theta_1}^*(z)P(z)U_{S\theta_2}(z)$ and $\tilde{Q}(z) := U_{S\theta_1}^*(z)\tilde{P}(z)U_{S\theta_2}(z)$. Then both $Q(z)$ and $\tilde{Q}(z)$ have the same symmetry pattern as in (2.7).
- (d) **Support reduction:**
 - 1: Let $U_0(z) := U_{S\theta_2}^*(z)$ and $A(z) = \tilde{A}(z) := I_s$.
 - 2: **for** $k = 1$ to r **do**
 - 3: Let $p(z) := [Q(z)]_{k,k:s}$ and $\tilde{p}(z) := [\tilde{Q}(z)]_{k,k:s}$.
 - 4: **while** $|\text{csupp}(p)| > 0$ and $|\text{csupp}(\tilde{p})| > 0$ **do**
 - 5: Construct a pair of biorthogonal matrices $(B(z), \tilde{B}(z))$ with respect to the pair $(p(z), \tilde{p}(z))$ by Lemma 1 such that

$$|\text{csupp}(pB)| + |\text{csupp}(\tilde{p}\tilde{B})| < |\text{csupp}(p)| + |\text{csupp}(\tilde{p})|.$$

- 6: Replace $(p(z), \tilde{p}(z))$ by $(p(z)B(z), \tilde{p}(z)\tilde{B}(z))$.
 - 7: Set $A(z) := A(z) \text{diag}(I_{k-1}, B(z))$ and $\tilde{A}(z) := \tilde{A}(z) \text{diag}(I_{k-1}, \tilde{B}(z))$.
 - 8: **end while**
 - 9: The pair $(p(z), \tilde{p}(z))$ is of the form: $([1, 0, \dots, 0], [1, q(z)])$ for some $1 \times (s - k)$ vector of Laurent polynomials $q(z)$. Construct $B(z), \tilde{B}(z)$ as follows:

$$B(z) = \begin{bmatrix} 1 & \mathbf{0} \\ q^*(z) & I_{s-k} \end{bmatrix}, \quad \tilde{B}(z) = \begin{bmatrix} 1 & -q(z) \\ \mathbf{0} & I_{s-k} \end{bmatrix}.$$
 - 10: Set $A(z) := A(z) \text{diag}(I_{k-1}, B(z))$ and $\tilde{A}(z) := \tilde{A}(z) \text{diag}(I_{k-1}, \tilde{B}(z))$.
 - 11: Set $Q(z) := Q(z)A(z)$ and $\tilde{Q}(z) := \tilde{Q}(z)\tilde{A}(z)$.
 - 12: **end for**
 - (e) **Finalization:** Let $U_1(z) := \text{diag}(U_{S\theta_1}(z), I_{s-r})$. Set $P_e := U_1(z)\tilde{A}^*(z)U_0(z)$ and $\tilde{P}_e(z) := U_1(z)A^*(z)U_0(z)$.
-

3. Application to biorthogonal multiwavelets with symmetry

In this section, we shall discuss the connection between biorthogonal matrix extension and the construction of biorthogonal multiwavelets. We shall also discuss the application of our results obtained in the previous section to the construction of biorthogonal multiwavelets with symmetry. Examples will be provided in the next section.

We say that d is a *dilation factor* if d is an integer with $|d| > 1$. Throughout this section, d denotes a dilation factor. For simplicity of presentation, we further assume that d is positive, while multiwavelets and filter banks with a negative dilation factor can be handled similarly by a slight modification of the statements in this paper.

Let \mathbb{F} be a subfield of \mathbb{C} . A low-pass filter $a_0 : \mathbb{Z} \mapsto \mathbb{F}^{r \times r}$ with multiplicity r is a finitely supported sequence of $r \times r$ matrices on \mathbb{Z} ; that is $a_0 = \{a_0(k) \in \mathbb{F}^{r \times r} : k \in \mathbb{Z}\}$. The *symbol* of the filter a_0 is defined to be $a_0(z) := \sum_{k \in \mathbb{Z}} a_0(k)z^k$, which is a matrix of Laurent polynomials with coefficients in \mathbb{F} . Let d be a dilation factor and d, \tilde{d} be two fixed numbers in \mathbb{F} such that $d = d \cdot \tilde{d}$ (for instance, we can set $d = 1, \tilde{d} = 2$ for $\mathbb{F} = \mathbb{Q}$; or $d = \tilde{d} = \sqrt{2}$ for $\mathbb{F} = \mathbb{R}$ when $d = 2$). Let (a_0, \tilde{a}_0) be a pair of low-pass filters with multiplicity r . We say that (a_0, \tilde{a}_0) is a pair of *d-dual filters* if

$$\sum_{\gamma=0}^{d-1} a_{0;\gamma}(z) \tilde{a}_{0;\gamma}^*(z) = I_r, \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.1}$$

where $a_{0;\gamma}(z)$ and $\tilde{a}_{0;\gamma}(z)$ are *d-subsymbols* (polyphase components, cosets) of $a_0(z)$ and $\tilde{a}_0(z)$ defined to be

$$a_{0;\gamma}(z) := d \sum_{k \in \mathbb{Z}} a_0(\gamma + dk)z^k, \quad \tilde{a}_{0;\gamma}(z) := \tilde{d} \sum_{k \in \mathbb{Z}} \tilde{a}_0(\gamma + dk)z^k, \quad \gamma \in \mathbb{Z}. \quad (3.2)$$

Quite often, a low-pass filter a_0 is obtained beforehand. To construction a pair of d -dual low-pass filters (a_0, \tilde{a}_0) , i.e., (3.1) holds, one can use the CBC (*coset-by-coset*) algorithm proposed in [20] to derive \tilde{a}_0 from a_0 . There are two key ingredients in the CBC algorithm. One is that the CBC algorithm reduces the nonlinear system in the definition of sum rules for \tilde{a}_0 to a system of linear equations. Another is that the CBC algorithm reduces the big system of linear equation of biorthogonality relation for the pair (a_0, \tilde{a}_0) to a small system of linear equations in (3.1). Moreover, the CBC algorithm guarantees that for any given positive integers \tilde{k} , there always exists a finitely supported filter \tilde{a}_0 that satisfies the sum rules of order \tilde{k} . For more details on the CBC algorithm, one may refer to [20,24]. In our first two examples presented in the next section, the pairs of d -dual low-pass filters are obtained in this way using the CBC algorithm (see examples in [24]).

To construct biorthogonal multiwavelets in $L_2(\mathbb{R})$, we need to design high-pass filters $a_1, \dots, a_{d-1} : \mathbb{Z} \rightarrow \mathbb{F}^{r \times r}$ and $\tilde{a}_1, \dots, \tilde{a}_{d-1} : \mathbb{Z} \rightarrow \mathbb{F}^{r \times r}$ such that the *polyphase matrices* with respect to the filter bank pair $(\{a_0; a_1, \dots, a_{d-1}\}, \{\tilde{a}_0; \tilde{a}_1, \dots, \tilde{a}_{d-1}\})$

$$\mathbf{P}(z) = \begin{bmatrix} a_{0;0}(z) & \cdots & a_{0;d-1}(z) \\ a_{1;0}(z) & \cdots & a_{1;d-1}(z) \\ \vdots & \vdots & \vdots \\ a_{d-1;0}(z) & \cdots & a_{d-1;d-1}(z) \end{bmatrix}, \quad \tilde{\mathbf{P}}(z) = \begin{bmatrix} \tilde{a}_{0;0}(z) & \cdots & \tilde{a}_{0;d-1}(z) \\ \tilde{a}_{1;0}(z) & \cdots & \tilde{a}_{1;d-1}(z) \\ \vdots & \vdots & \vdots \\ \tilde{a}_{d-1;0}(z) & \cdots & \tilde{a}_{d-1;d-1}(z) \end{bmatrix} \quad (3.3)$$

are biorthogonal; that is, $\mathbf{P}(z)\tilde{\mathbf{P}}^*(z) = I_{dr}$ for $z \in \mathbb{C} \setminus \{0\}$, where $a_{m;\gamma}(z)$ and $\tilde{a}_{m;\gamma}(z)$ are d -subsymbols of $a_m(z)$ and $\tilde{a}_m(z)$ defined similar to (3.2) for $m, \gamma = 0, \dots, d-1$, respectively. The pair of filter banks $(\{a_0; a_1, \dots, a_{d-1}\}, \{\tilde{a}_0; \tilde{a}_1, \dots, \tilde{a}_{d-1}\})$ whose corresponding polyphase matrices $\mathbf{P}(z)$ and $\tilde{\mathbf{P}}(z)$ satisfy $\mathbf{P}(z)\tilde{\mathbf{P}}^*(z) = I_{dr}$ for $z \in \mathbb{C} \setminus \{0\}$ is called a *pair of d -dual filter banks (with the perfect reconstruction property)*.

For $f \in L_1(\mathbb{R})$, the Fourier transform is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi} dx$ and can be naturally extended to $L_2(\mathbb{R})$ functions. Here $i = \sqrt{-1}$ is the imaginary unit. For a pair (a_0, \tilde{a}_0) of d -dual low-pass filter, we assume that there exists a *pair of biorthogonal d -refinable function vectors* $(\phi, \tilde{\phi})$ associated with (a_0, \tilde{a}_0) . That is,

$$\hat{\phi}(d\xi) = a_0(e^{-i\xi})\hat{\phi}(\xi), \quad \hat{\tilde{\phi}}(d\xi) = \tilde{a}_0(e^{-i\xi})\hat{\tilde{\phi}}(\xi), \quad \xi \in \mathbb{R}, \quad (3.4)$$

and

$$(\phi(\cdot - k), \tilde{\phi}) := \int_{\mathbb{R}} \phi(x - k)\overline{\tilde{\phi}(x)}^T dx = \delta(k)I_r, \quad k \in \mathbb{Z}, \quad (3.5)$$

where δ denotes the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$, and $\phi = [\phi_1, \dots, \phi_r]^T$, $\tilde{\phi} = [\tilde{\phi}_1, \dots, \tilde{\phi}_r]^T$ are vectors of functions in $L_2(\mathbb{R})$.

Symmetry of the filters in a filter bank is a very much desirable property in many applications. We say that the low-pass filter a_0 with multiplicity r has *symmetry* if its symbol $a_0(z)$ satisfies

$$a_0(z) = \text{diag}(\varepsilon_1 z^{dc_1}, \dots, \varepsilon_r z^{dc_r}) a_0(z^{-1}) \text{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}) \quad (3.6)$$

for some $\varepsilon_1, \dots, \varepsilon_r \in \{-1, 1\}$ and some $c_1, \dots, c_r \in \mathbb{R}$ such that $dc_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$. Let (a_0, \tilde{a}_0) be a pair of d -dual low-pass filters. The symmetry pattern in (3.6) comes from the following fact: If a d -refinable function vector ϕ associated with a low-pass filter a_0 satisfies $\phi_1(c_1 - \cdot) = \varepsilon_1 \phi_1, \dots, \phi_r(c_r - \cdot) = \varepsilon_r \phi_r$, and the shifts of ϕ are stable, then the symbol a_0 of a_0 must satisfy (3.6). Conversely, if both a_0 and \tilde{a}_0 have symmetry as in (3.6) and 1 is a simple eigenvalue of both $a_0(1)$ and $\tilde{a}_0(1)$, then the pair $(\phi, \tilde{\phi})$ of biorthogonal d -refinable function vectors associated with (a_0, \tilde{a}_0) as in (3.4) has the following symmetry:

$$\begin{aligned} \phi_1(c_1 - \cdot) &= \varepsilon_1 \phi_1, & \phi_2(c_2 - \cdot) &= \varepsilon_2 \phi_2, & \dots, & & \phi_r(c_r - \cdot) &= \varepsilon_r \phi_r, \\ \tilde{\phi}_1(c_1 - \cdot) &= \varepsilon_1 \tilde{\phi}_1, & \tilde{\phi}_2(c_2 - \cdot) &= \varepsilon_2 \tilde{\phi}_2, & \dots, & & \tilde{\phi}_r(c_r - \cdot) &= \varepsilon_r \tilde{\phi}_r. \end{aligned} \quad (3.7)$$

As shown in the previous section, compatible symmetry plays an important role in our biorthogonal matrix extension with symmetry. Though a low-pass filter a_0 might have certain symmetry, its polyphase submatrix $\mathbf{P}_{a_0}(z) := [a_{0;0}(z), \dots, a_{0;d-1}(z)]$ – the first ‘row’ of $\mathbf{P}(z)$ in (3.3) – not necessarily has compatible symmetry. However, under the symmetry condition in (3.6), the next lemma shows that there always exists a suitable *unimodular symmetrization matrix* $\mathbf{U}(z)$ of Laurent polynomials with symmetry, which acts on $\mathbf{P}_{a_0}(z) := [a_{0;0}(z), \dots, a_{0;d-1}(z)]$ producing a new submatrix $\mathbf{P}(z) := \mathbf{P}_{a_0}(z)\mathbf{U}(z)$ having compatible symmetry.

Lemma 2. *Let $\mathbf{P}_{a_0}(z) := [a_{0;0}(z), \dots, a_{0;d-1}(z)]$, where $a_{0;0}(z), \dots, a_{0;d-1}(z)$ are d -subsymbols of $a_0(z)$ for a low-pass filter a_0 satisfying (3.6). Then there exists a $dr \times dr$ unimodular matrix $\mathbf{U}(z)$ of Laurent polynomials with symmetry such that $\mathbf{P}(z) := \mathbf{P}_{a_0}(z)\mathbf{U}(z)$ has compatible symmetry.*

Proof. From (3.6), we deduce that

$$[\mathbf{a}_{0;\gamma}(z)]_{\ell,j} = \varepsilon_\ell \varepsilon_j z^{R_{\ell,j}^\gamma} [\mathbf{a}_{0;Q_{\ell,j}^\gamma}(z^{-1})]_{\ell,j}, \quad \gamma = 0, \dots, d-1; \ell, j = 1, \dots, r, \tag{3.8}$$

where $\gamma, Q_{\ell,j}^\gamma \in \Gamma := \{0, \dots, d-1\}$ and $R_{\ell,j}^\gamma, Q_{\ell,j}^\gamma$ are uniquely determined by

$$dc_\ell - c_j - \gamma = dR_{\ell,j}^\gamma + Q_{\ell,j}^\gamma \quad \text{with } R_{\ell,j}^\gamma \in \mathbb{Z}, Q_{\ell,j}^\gamma \in \Gamma. \tag{3.9}$$

Since $dc_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$, we have $c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$ and therefore, $Q_{\ell,j}^\gamma$ is independent of ℓ . Consequently, by (3.8), for every $1 \leq j \leq r$, the j th column of the matrix $\mathbf{a}_{0;\gamma}(z)$ is a flipped version of the j th column of the matrix $\mathbf{a}_{0;Q_{\ell,j}^\gamma}(z)$. Let $\kappa_{j,\gamma} \in \mathbb{Z}$ be an integer such that $\lfloor \text{csupp}([\mathbf{a}_{0;\gamma}]_{:,j} + z^{\kappa_{j,\gamma}} [\mathbf{a}_{0;Q_{\ell,j}^\gamma}]_{:,j}) \rfloor$ is the smallest possible integer. Define $\mathbf{P}(z) := [\mathbf{b}_{0;0}(z), \dots, \mathbf{b}_{0;d-1}(z)]$ as follows:

$$[\mathbf{b}_{0;\gamma}(z)]_{:,j} := \begin{cases} [\mathbf{a}_{0;\gamma}(z)]_{:,j}, & \gamma = Q_{\ell,j}^\gamma; \\ \frac{1}{2}([\mathbf{a}_{0;\gamma}(z)]_{:,j} + z^{\kappa_{j,\gamma}} [\mathbf{a}_{0;Q_{\ell,j}^\gamma}(z)]_{:,j}), & \gamma < Q_{\ell,j}^\gamma; \\ \frac{1}{2}([\mathbf{a}_{0;\gamma}(z)]_{:,j} - z^{\kappa_{j,\gamma}} [\mathbf{a}_{0;Q_{\ell,j}^\gamma}(z)]_{:,j}), & \gamma > Q_{\ell,j}^\gamma. \end{cases} \tag{3.10}$$

Let $\mathbf{U}(z)$ denote the unique transform matrix corresponding to (3.10) such that

$$\mathbf{P}(z) := [\mathbf{b}_{0;0}(z), \dots, \mathbf{b}_{0;d-1}(z)] = [\mathbf{a}_{0;0}(z), \dots, \mathbf{a}_{0;d-1}(z)]\mathbf{U}(z).$$

It is evident that $\mathbf{U}(z)$ is unimodular and $\mathbf{P}(z) = \mathbf{P}_{\mathbf{a}_0}(z)\mathbf{U}(z)$. Also note that $\mathbf{U}(z)$ and $\tilde{\mathbf{U}}(z) := (\mathbf{U}^*(z))^{-1}$ have the same structure in view of (3.10). In fact, $\mathbf{U}(z)\mathbf{U}^*(z)$ is a diagonal constant matrix. We now show that $\mathbf{P}(z)$ has compatible symmetry. Indeed, by (3.8) and (3.10),

$$[\mathbf{Sb}_{0;\gamma}]_{\ell,j} = \text{sgn}(Q_{\ell,j}^\gamma - \gamma) \varepsilon_\ell \varepsilon_j z^{R_{\ell,j}^\gamma + \kappa_{j,\gamma}}, \tag{3.11}$$

where $\text{sgn}(x) = 1$ for $x \geq 0$ and $\text{sgn}(x) = -1$ for $x < 0$. By (3.9) and noting that $Q_{\ell,j}^\gamma$ is independent of ℓ , we have

$$\frac{[\mathbf{Sb}_{0;\gamma}]_{\ell,j}}{[\mathbf{Sb}_{0;\gamma}]_{n,j}} = \varepsilon_\ell \varepsilon_n z^{R_{\ell,j}^\gamma - R_{n,j}^\gamma} = \varepsilon_\ell \varepsilon_n z^{c_\ell - c_n}, \quad 1 \leq \ell, n \leq r,$$

which is equivalent to saying that $\mathbf{P}(z)$ has compatible symmetry. \square

Given a pair (a_0, \tilde{a}_0) of d -dual low-pass filters with multiplicity r , if both a_0 and \tilde{a}_0 have symmetry, it is quite natural to require a_0 and \tilde{a}_0 having the same symmetry as (3.6). Then, by the above lemma, we can always find a pair $(\mathbf{U}(z), \tilde{\mathbf{U}}(z))$ of symmetrization matrices such that the pair $(\mathbf{P}(z), \tilde{\mathbf{P}}(z))$ given by

$$\begin{aligned} \mathbf{P}(z) &:= \mathbf{P}_{\mathbf{a}_0}(z)\mathbf{U}(z) = [\mathbf{a}_{0;0}(z), \dots, \mathbf{a}_{0;d-1}(z)]\mathbf{U}(z), \\ \tilde{\mathbf{P}}(z) &:= \tilde{\mathbf{P}}_{\tilde{\mathbf{a}}_0}(z)\tilde{\mathbf{U}}(z) = [\tilde{\mathbf{a}}_{0;0}(z), \dots, \tilde{\mathbf{a}}_{0;d-1}(z)]\tilde{\mathbf{U}}(z), \end{aligned}$$

is biorthogonal and both $\mathbf{P}(z)$ and $\tilde{\mathbf{P}}(z)$ have the same compatible symmetry. Applying Algorithm 1 to the pair $(\mathbf{P}(z), \tilde{\mathbf{P}}(z))$, we obtain the pair $(\mathbf{P}_e(z), \tilde{\mathbf{P}}_e(z))$ of extension matrices, from which we derive a pair $(\mathbf{P}(z), \tilde{\mathbf{P}}(z))$ of polyphase matrices and construct high-pass filters a_1, \dots, a_{d-1} and $\tilde{a}_1, \dots, \tilde{a}_{d-1}$. This is summarized in Algorithm 2 below.

Proof of Algorithm 2. Rewrite $\mathbf{P}_e(z) = (\mathbf{b}_{m;\gamma}(z))_{0 \leq m, \gamma \leq d-1}$ as a $d \times d$ block matrix with $r \times r$ blocks $\mathbf{b}_{m;\gamma}(z)$. Since $\mathbf{P}_e(z)$ has compatible symmetry as in (3.18), we have $[\mathbf{Sb}_{m;\gamma}(z)]_{\ell,:} = \varepsilon_\ell^m \varepsilon_\ell z^{k_\ell^m - k_\ell} [\mathbf{Sb}_{0;\gamma}(z)]_{\ell,:}$ for $\ell = 1, \dots, r$ and $m = 1, \dots, d-1$. By (3.11), we have

$$[\mathbf{Sb}_{m;\gamma}]_{\ell,j} = \text{sgn}(Q_{\ell,j}^\gamma - \gamma) \varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^\gamma + \kappa_{j,\gamma} + k_\ell^m - k_\ell}, \quad \ell, j = 1, \dots, r. \tag{3.12}$$

By (3.12) and the definition of $\mathbf{U}^*(z)$ in (3.10), we deduce that

$$[\mathbf{a}_{m;\gamma}(z)]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^\gamma + k_\ell^m - k_\ell} [\mathbf{a}_{m;Q_{\ell,j}^\gamma}(z^{-1})]_{\ell,j}. \tag{3.13}$$

This implies that $[\mathbf{Sa}_m]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{d(k_\ell^m - k_\ell + c_\ell) - c_j}$, which is equivalent to (3.17) with $c_\ell^m := k_\ell^m - k_\ell + c_\ell$ for $m = 1, \dots, d-1$ and $\ell = 1, \dots, r$. The proof for the symmetry of $\tilde{a}_1, \dots, \tilde{a}_{d-1}$ as in (3.17) is similar. We are done. \square

Let $(\phi, \tilde{\phi})$ be a pair of biorthogonal d -refinable function vectors in $L_2(\mathbb{R})$ associated with a pair (a_0, \tilde{a}_0) of d -dual low-pass filters and $\phi = [\phi_1, \dots, \phi_r]^T, \tilde{\phi} = [\tilde{\phi}_1, \dots, \tilde{\phi}_r]^T$. Let (a_1, \dots, a_{d-1}) and $(\tilde{a}_1, \dots, \tilde{a}_{d-1})$ be high-pass filters derived from

(a_0, \tilde{a}_0) using Algorithm 2. Define multiwavelet function vectors ψ^m and $\tilde{\psi}^m$ associated with the high-pass filters a_m and \tilde{a}_m with symbols $a_m(z)$ and $\tilde{a}_m(z)$ for $m = 1, \dots, d - 1$ by the following relations:

$$\widehat{\psi^m}(d\xi) := a_m(e^{-i\xi})\hat{\phi}(\xi), \quad \widehat{\tilde{\psi}^m}(d\xi) := \tilde{a}_m(e^{-i\xi})\hat{\phi}(\xi), \quad \xi \in \mathbb{R}. \tag{3.14}$$

It is well known that $(\{\phi; \psi^1, \dots, \psi^{d-1}\}, \{\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^{d-1}\})$ generates a pair of biorthogonal d -multiwavelet bases in $L_2(\mathbb{R})$. Moreover, since the high-pass filters a_1, \dots, a_{d-1} and $\tilde{a}_1, \dots, \tilde{a}_{d-1}$ satisfy (3.17), it is easy to verify that $\psi^m = [\psi_1^m, \dots, \psi_r^m]^T$ and $\tilde{\psi}^m = [\tilde{\psi}_1^m, \dots, \tilde{\psi}_r^m]^T$ defined in (3.14) for $m = 1, \dots, d - 1$ also have the following symmetry:

$$\begin{aligned} \psi_1^m(c_1^m - \cdot) &= \varepsilon_1^m \psi_1^m, & \psi_2^m(c_2^m - \cdot) &= \varepsilon_2^m \psi_2^m, & \dots, & & \psi_r^m(c_r^m - \cdot) &= \varepsilon_r^m \psi_r^m, \\ \tilde{\psi}_1^m(c_1^m - \cdot) &= \varepsilon_1^m \tilde{\psi}_1^m, & \tilde{\psi}_2^m(c_2^m - \cdot) &= \varepsilon_2^m \tilde{\psi}_2^m, & \dots, & & \tilde{\psi}_r^m(c_r^m - \cdot) &= \varepsilon_r^m \tilde{\psi}_r^m, \end{aligned} \tag{3.15}$$

for $m = 1, \dots, d - 1$. Summarizing, we have the following theorem.

Theorem 2. Let \mathbb{F} be any subfield of \mathbb{C} and d be a dilation factor. Let $(\phi, \tilde{\phi})$ be a pair of biorthogonal d -refinable function vectors in $L_2(\mathbb{R})$ associated with a pair (a_0, \tilde{a}_0) of d -dual low-pass filters with multiplicity r , and the symbols $a_0(z), \tilde{a}_0(z)$ of a_0, \tilde{a}_0 are $r \times r$ matrices of Laurent polynomials with coefficients in \mathbb{F} . Let $(\{a_1, \dots, a_{d-1}\}, \{\tilde{a}_1, \dots, \tilde{a}_{d-1}\})$ be high-pass filters derived from the pair (a_0, \tilde{a}_0) via Algorithm 2. Suppose both a_0 and \tilde{a}_0 have symmetry as in (3.6) and 1 is a simple eigenvalue of both $a_0(1)$ and $\tilde{a}_0(1)$. Then $(\phi, \tilde{\phi})$ has symmetry as in (3.7) and the following statements hold.

- (i) The high-pass filters a_1, \dots, a_{d-1} and $\tilde{a}_1, \dots, \tilde{a}_{d-1}$ have symmetry as in (3.17), and their symbols $a_1(z), \dots, a_{d-1}(z)$ and $\tilde{a}_1(z), \dots, \tilde{a}_{d-1}(z)$ are $r \times r$ matrices of Laurent polynomials with coefficients in \mathbb{F} .
- (ii) The pair $(\{a_0; a_1, \dots, a_{d-1}\}, \{\tilde{a}_0; \tilde{a}_1, \dots, \tilde{a}_{d-1}\})$ is a pair of d -dual filter banks with the perfect reconstruction property.
- (iii) The pair $(\{\phi; \psi^1, \dots, \psi^{d-1}\}, \{\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^{d-1}\})$ defined in (3.14) has symmetry satisfying (3.15) and generates a pair of biorthogonal d -multiwavelet bases in $L_2(\mathbb{R})$.

Algorithm 2 Construction of biorthogonal multiwavelets with symmetry.

- (a) **Input:** A pair (a_0, \tilde{a}_0) of d -dual low-pass filters with multiplicity r and with the same symmetry as in (3.6).
- (b) **Output:** A pair $(\{a_0; a_1, \dots, a_{d-1}\}, \{\tilde{a}_0; \tilde{a}_1, \dots, \tilde{a}_{d-1}\})$ of d -dual filter banks with symmetry and with the perfect reconstruction property.
- (c) **Initialization:** Construct a pair of biorthogonal matrices $(U(z), \tilde{U}(z))$ in \mathbb{F} by Lemma 2 such that both $P(z) := P_{a_0}(z)U(z)$ and $\tilde{P}(z) := \tilde{P}_{\tilde{a}_0}(z)\tilde{U}(z)$ ($\tilde{U}(z) = (U^*(z))^{-1}$) are matrices of Laurent polynomials with coefficient in \mathbb{F} having compatible symmetry: $SP = S\tilde{P} := [\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}]^T S\theta$ for some $k_1, \dots, k_r \in \mathbb{Z}$ and some $1 \times dr$ row vector $\theta(z)$ of Laurent polynomials with symmetry.
- (d) **Extension:** Derive a desired pair of extension matrices $(P_e(z), \tilde{P}_e(z))$ with all the properties as in Theorem 1 from the pair $(P(z), \tilde{P}(z))$ by Algorithm 1.
- (e) **High-pass filters:** Let $\mathbf{P}(z) := P_e(z)\tilde{U}^*(z) = (a_{m;\gamma}(z))_{0 \leq m, \gamma \leq d-1}$ and $\tilde{\mathbf{P}}(z) := \tilde{P}_e(z)U^*(z) = (\tilde{a}_{m;\gamma}(z))_{0 \leq m, \gamma \leq d-1}$ as in (3.3). For $m = 1, \dots, d - 1$, define high-pass filters a_1, \dots, a_{d-1} and $\tilde{a}_1, \dots, \tilde{a}_{d-1}$ through their symbols

$$a_m(z) := \frac{1}{d_1} \sum_{\gamma=0}^{d-1} a_{m;\gamma}(z^d)z^\gamma, \quad \tilde{a}_m(z) := \frac{1}{d_2} \sum_{\gamma=0}^{d-1} \tilde{a}_{m;\gamma}(z^d)z^\gamma. \tag{3.16}$$

$\mathbf{P}(z)$ and $\tilde{\mathbf{P}}(z)$ as in (3.3) are biorthogonal and all filters $a_m, \tilde{a}_m, m = 1, \dots, d - 1$, have symmetry:

$$\begin{aligned} a_m(z) &= \text{diag}(\varepsilon_1^m z^{d c_1^m}, \dots, \varepsilon_r^m z^{d c_r^m}) a_m(z^{-1}) \text{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}), \\ \tilde{a}_m(z) &= \text{diag}(\varepsilon_1^m z^{d c_1^m}, \dots, \varepsilon_r^m z^{d c_r^m}) \tilde{a}_m(z^{-1}) \text{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}), \end{aligned} \tag{3.17}$$

where $c_\ell^m := (k_\ell^m - k_\ell) + c_\ell \in \mathbb{R}$ and all $\varepsilon_\ell^m \in \{-1, 1\}, k_\ell^m \in \mathbb{Z}$, for $\ell = 1, \dots, r$ and $m = 1, \dots, d - 1$, are determined by the symmetry pattern of $(P_e(z), \tilde{P}_e(z))$:

$$[\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}, \varepsilon_1^1 z^{k_1^1}, \dots, \varepsilon_r^1 z^{k_r^1}, \dots, z^{k_1^{d-1}}, \dots, \varepsilon_r^{d-1} z^{k_r^{d-1}}]^T S\theta := SP_e = S\tilde{P}_e. \tag{3.18}$$

4. Illustrative examples

In this section, we will illustrate our algorithms and results stated in Sections 2 and 3 on the construction of biorthogonal multiwavelets with symmetry by three examples. For each example, a pair (a_0, \tilde{a}_0) of d -dual low-pass filters with symmetry is obtained beforehand and we apply Algorithm 2 to constructing the corresponding high-pass filters a_1, \dots, a_{d-1} and $\tilde{a}_1, \dots, \tilde{a}_{d-1}$ so that $(\{a_0; a_1, \dots, a_{d-1}\}, \{\tilde{a}_0; \tilde{a}_1, \dots, \tilde{a}_{d-1}\})$ forms a pair of d -dual filter banks with the perfect reconstruction property and with symmetry. The first two examples are biorthogonal multiwavelets with coefficients in the rational number field $\mathbb{F} = \mathbb{Q}$ and with $r = 2$, while the third example are biorthogonal wavelets with coefficients in the field $\mathbb{F} = \mathbb{Q}(\sqrt{5}i) = \{c_1 + c_2\sqrt{5}i; c_1, c_2 \in \mathbb{Q}\}$.

Example 2. Consider $d = r = 2$. A pair (a_0, \tilde{a}_0) of 2-dual low-pass filters with symbols $a_0(z), \tilde{a}_0(z)$ (cf. [24]) is given by

$$a_0(z) = \frac{1}{16} \begin{bmatrix} 8 & 6z^{-1} + 6 \\ 8z & -z^{-1} + 3 + 3z - z^2 \end{bmatrix},$$

$$\tilde{a}_0(z) = \frac{1}{384} \begin{bmatrix} -28z^{-1} + 216 & 112z^{-1} + 112 \\ 21z^{-1} - 18 + 330z - 18z^2 + 21z^3 & -36z^{-1} + 60 + 60z - 36z^2 \end{bmatrix}.$$

Both $a_0(z)$ and $\tilde{a}_0(z)$ have the same symmetry pattern and satisfy (3.6) with $\varepsilon_1 = \varepsilon_2 = 1$, and $c_1 = 0, c_2 = 1$.

Let $d = d\tilde{d}$ with $d = 1$ and $\tilde{d} = 2$. Then, following Algorithm 2, we first construct $P_{a_0}(z) := [a_{0;0}(z), a_{0;1}(z)]$ and $\tilde{P}_{\tilde{a}_0}(z) := [\tilde{a}_{0;0}(z), \tilde{a}_{0;1}(z)]$ as follows:

$$P_{a_0}(z) = \frac{1}{16} \begin{bmatrix} 8 & 6 & 0 & 6z^{-1} \\ 0 & 3 - z & 8 & -z^{-1} + 3 \end{bmatrix},$$

$$\tilde{P}_{\tilde{a}_0}(z) = \frac{1}{192} \begin{bmatrix} 216 & 112 & -28(z^{-1} + 1) & 112z^{-1} \\ -18(1 + z) & 12(5 - 3z) & 3(7z^{-1} + 110 + 7z) & 12(5 - 3z^{-1}) \end{bmatrix}.$$

The unimodular symmetrization matrices $U(z)$ and $\tilde{U}(z)$ as stated in Lemma 2 are given by

$$U(z) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & -z \end{bmatrix}, \quad \tilde{U}(z) := \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & z & 0 & -z \end{bmatrix}.$$

It is obvious that $U(z)\tilde{U}^*(z) = I_4$. Let $P(z) := P_{a_0}(z)U(z)$ and $\tilde{P}(z) := \tilde{P}_{\tilde{a}_0}(z)\tilde{U}(z)$. Then we have $SP = S\tilde{P} = [1, z]^T [1, 1, z^{-1}, -1]$; that is, both $P(z)$ and $\tilde{P}(z)$ have compatible symmetry. $(P(z), \tilde{P}(z))$ is a pair of biorthogonal matrices of Laurent polynomials with coefficients in \mathbb{Q} given as follows:

$$P(z) = \frac{1}{8} \begin{bmatrix} 4 & 6 & 0 & 0 \\ 0 & 1(1 + z) & 4 & 2(1 - z) \end{bmatrix},$$

$$\tilde{P}(z) = \frac{1}{192} \begin{bmatrix} 216 & 112 & -28(1 + z^{-1}) & 0 \\ -18(1 + z) & 12(1 + z) & 3(7z^{-1} + 110 + 7z) & 48(1 - z) \end{bmatrix}.$$

Now applying Algorithm 1, we obtain a pair of extension matrices $(P_e(z), \tilde{P}_e(z))$ as follows:

$$P_e(z) = \frac{1}{192} \begin{bmatrix} 96 & 144 & 0 & 0 \\ 0 & 24(1 + z) & 96 & 48(1 - z) \\ -112 & -3(z^{-1} - 70 + z) & -12(1 + z^{-1}) & -6(z^{-1} - z) \\ 0 & -6(z - z^{-1}) & -24(1 - z^{-1}) & 12(z + 14 + z^{-1}) \end{bmatrix},$$

$$\tilde{P}_e(z) = \frac{1}{192} \begin{bmatrix} 216 & 112 & -28(1 + z^{-1}) & 0 \\ -18(1 + z) & 12(1 + z) & 3(7z^{-1} + 110 + 7z) & 48(1 - z) \\ -144 & 96 & -24(1 + z^{-1}) & 0 \\ 0 & 0 & -96(1 - z^{-1}) & 192 \end{bmatrix}.$$

Note that $SP_e = S\tilde{P}_e = [1, z, 1, -1]^T [1, 1, z^{-1}, -1]$. Moreover, for the support control, we have $\max\{|\text{csupp}(P_e)|, |\text{csupp}(\tilde{P}_e)|\} \leq \max\{|\text{csupp}(P)|, |\text{csupp}(\tilde{P})|\}$.

Finally, as in the last part of Algorithm 2, from the polyphase matrices $\mathbf{P}(z) := P_e(z)\tilde{U}^*(z) =: (a_{m;\gamma}(z))_{0 \leq m, \gamma \leq 1}$ and $\tilde{\mathbf{P}}(z) := \tilde{P}_e(z)U^*(z) =: (\tilde{a}_{m;\gamma}(z))_{0 \leq m, \gamma \leq 1}$, we derive two high-pass filters a_1, \tilde{a}_1 as follows:

$$a_1(z) = \frac{1}{384} \begin{bmatrix} -8(3z + 28 + 3z^{-1}) & 3(z^2 - 3z + 70 + 70z^{-1} - 3z^{-2} + z^{-3}) \\ -48(z - z^{-1}) & 6(z^2 - 3z + 28 - 28z^{-1} + 3z^{-2} - z^{-3}) \end{bmatrix},$$

$$\tilde{a}_1(z) = \frac{1}{16} \begin{bmatrix} -(z + 6 + z^{-1}) & 4(1 + z^{-1}) \\ -4(z - z^{-1}) & 8(1 - z^{-1}) \end{bmatrix}.$$

Moreover, $a_1(z), \tilde{a}_1(z)$ satisfy (3.17) with $\varepsilon_1^1 = 1, \varepsilon_2^1 = -1$, and $c_1^1 = c_2^1 = 0$.

By the biorthogonality relation of the polyphase matrices $\mathbf{P}(z)$ and $\tilde{\mathbf{P}}(z)$, the pair $(\{a_0; a_1\}, \{\tilde{a}_0; \tilde{a}_1\})$ is a pair of d -dual filter banks. Let $\phi = [\phi_1, \phi_2]^T, \tilde{\phi} = [\tilde{\phi}_1, \tilde{\phi}_2]^T$, and $\psi = [\psi_1, \psi_2]^T, \tilde{\psi} = [\tilde{\psi}_1, \tilde{\psi}_2]^T$ be d -refinable function vectors and multiwavelet generators associated with a_0, \tilde{a}_0 , and a_1, \tilde{a}_1 , respectively. Then the pair $(\{\phi; \psi\}, \{\tilde{\phi}; \tilde{\psi}\})$ generates a pair of biorthogonal d -multiwavelet bases in $L_2(\mathbb{R})$ and satisfies the following symmetry:

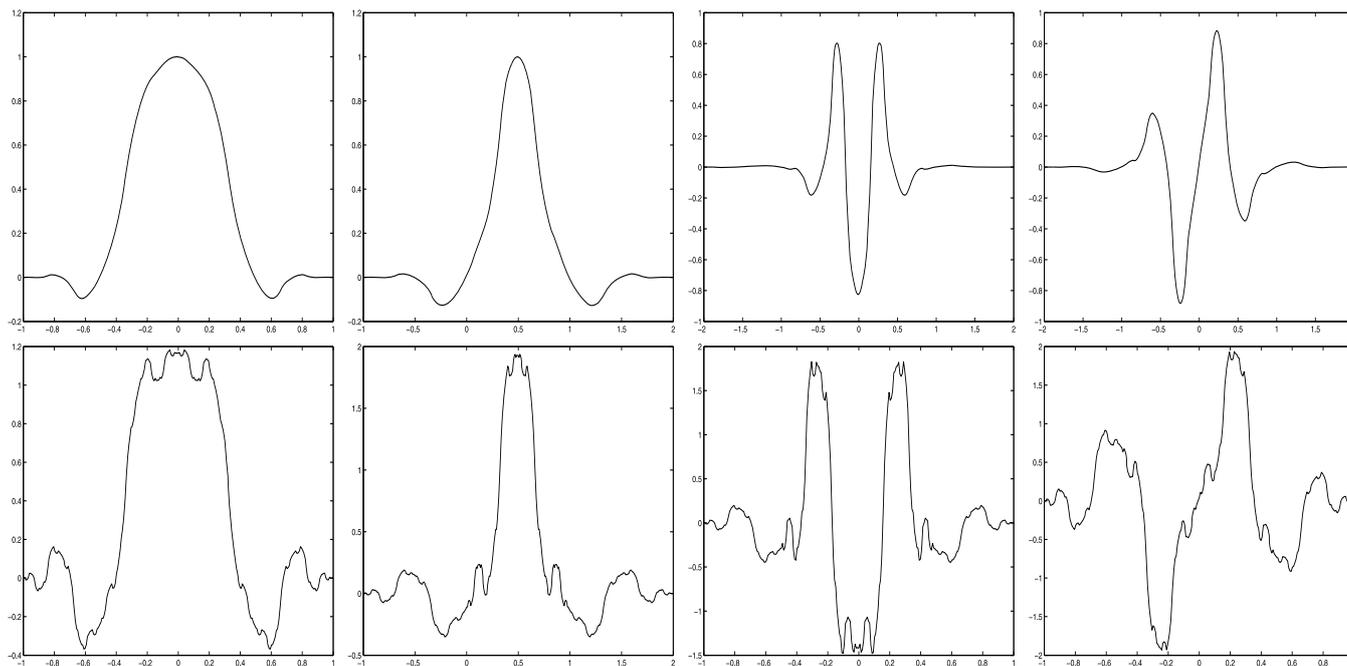


Fig. 1. The graphs of $\phi = [\phi_1, \phi_2]^T$, $\psi = [\psi_1, \psi_2]^T$ (top row, left to right), and $\tilde{\phi} = [\tilde{\phi}_1, \tilde{\phi}_2]^T$, $\tilde{\psi} = [\tilde{\psi}_1, \tilde{\psi}_2]^T$ (bottom row, left to right) in Example 2.

$$\begin{aligned} \phi_1(-\cdot) &= \phi_1, & \phi_2(1-\cdot) &= \phi_2, & \psi_1(-\cdot) &= \psi_1, & \psi_2(-\cdot) &= -\psi_2, \\ \tilde{\phi}_1(-\cdot) &= \tilde{\phi}_1, & \tilde{\phi}_2(1-\cdot) &= \tilde{\phi}_2, & \tilde{\psi}_1(-\cdot) &= \tilde{\psi}_1, & \tilde{\psi}_2(-\cdot) &= -\tilde{\psi}_2. \end{aligned}$$

See Fig. 1 for the graphs of ϕ , $\tilde{\phi}$ and ψ , $\tilde{\psi}$. \square

Example 3. Consider $d = 3$ and $r = 2$. A pair (a_0, \tilde{a}_0) of 3-dual low-pass filters with symbols $a_0(z), \tilde{a}_0(z)$ (cf. [24]) are given by

$$a_0(z) = \frac{1}{243} \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}, \quad \tilde{a}_0(z) = \frac{1}{34884} \begin{bmatrix} \tilde{a}_{11}(z) & \tilde{a}_{12}(z) \\ \tilde{a}_{21}(z) & \tilde{a}_{22}(z) \end{bmatrix}.$$

where

$$\begin{aligned} a_{11}(z) &= -21z^{-2} + 30z^{-1} + 81 + 14z - 5z^2, & a_{12}(z) &= 60z^{-1} + 84 - 4z^2 + 4z^3, \\ a_{21}(z) &= 4z^{-2} - 4z^{-1} + 84z + 60z^2, & a_{22}(z) &= -5z^{-1} + 14 + 81z + 30z^2 - 21z^3, \end{aligned}$$

and

$$\begin{aligned} \tilde{a}_{11}(z) &= 1292z^{-2} + 2844z^{-1} + 17496 + 2590z - 1284z^2 + 1866z^3, \\ \tilde{a}_{12}(z) &= -4773z^{-2} + 9682z^{-1} + 8715 - 2961z + 386z^2 - 969z^3, \\ \tilde{a}_{21}(z) &= -969z^{-2} + 386z^{-1} - 2961 + 8715z + 9682z^2 - 4773z^3, \\ \tilde{a}_{22}(z) &= 1866z^{-2} - 1284z^{-1} + 2590 + 17496z + 2844z^2 + 1292z^3. \end{aligned}$$

The low-pass filters a_0 and \tilde{a}_0 do not satisfy (3.6). However, we can employ a very simple transform matrix $E := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ to a_0 and \tilde{a}_0 so that the symmetry in (3.6) holds. That is, the new pair (b_0, \tilde{b}_0) with symbols $b_0(z) := Ea_0(z)E^{-1}$ and $\tilde{b}_0(z) := E^{-1}\tilde{a}_0(z)E$ is a pair of 3-dual low-pass filters satisfying the symmetry condition in (3.6) with $c_1 = c_2 = 1/2$ and $\varepsilon_1 = 1, \varepsilon_2 = -1$.

First, following Algorithm 2, let $d = \tilde{d}$ with $d = 1$ and $\tilde{d} = 3$. Construct polyphase submatrices $P_{b_0}(z) := [b_{0;0}(z), b_{0;1}(z), b_{0;2}(z)]$ and $\tilde{P}_{\tilde{b}_0}(z) := [\tilde{b}_{0;0}(z), \tilde{b}_{0;1}(z), \tilde{b}_{0;2}(z)]$ from $b_0(z)$ and $\tilde{b}_0(z)$. The unimodular symmetrization matrices is $U(z) = \text{diag}(U_0(z), I_2)$ with $U_0(z)$ being given by

$$U_0(z) = \begin{bmatrix} z^{-1} & 0 & z^{-1} & 0 \\ 0 & z^{-1} & 0 & z^{-1} \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

and $\tilde{U}(z) := (U^*(z))^{-1}$. Let $P(z) := P_{b_0}(z)U(z)$ and $\tilde{P}(z) := \tilde{P}_{b_0}(z)\tilde{U}(z)$. Then we have $SP = S\tilde{P} = [z^{-1}, -z^{-1}]^T [1, -1, -1, 1, 1, -1]$ and $(P(z), \tilde{P}(z))$ is a pair of biorthogonal matrices with

$$P(z) = c \begin{bmatrix} t_{11}(1 + \frac{1}{z}) & t_{12}(1 - \frac{1}{z}) & t_{13}(1 - \frac{1}{z}) & t_{14} & t_{15}(1 + \frac{1}{z}) & t_{16}(1 - \frac{1}{z}) \\ t_{21}(1 - \frac{1}{z}) & t_{22}(1 + \frac{1}{z}) & t_{23}(1 + \frac{1}{z}) & t_{24}(1 - \frac{1}{z}) & t_{25}(1 - \frac{1}{z}) & t_{26}(1 + \frac{1}{z}) \end{bmatrix},$$

$$\tilde{P}(z) = \tilde{c} \begin{bmatrix} \tilde{t}_{11}(1 + \frac{1}{z}) & \tilde{t}_{12}(1 - \frac{1}{z}) & \tilde{t}_{13}(1 - \frac{1}{z}) & \tilde{t}_{14} & \tilde{t}_{15}(1 + \frac{1}{z}) & \tilde{t}_{16}(1 - \frac{1}{z}) \\ \tilde{t}_{21}(1 - \frac{1}{z}) & \tilde{t}_{22}(1 + \frac{1}{z}) & \tilde{t}_{23}(1 + \frac{1}{z}) & \tilde{t}_{24}(1 - \frac{1}{z}) & \tilde{t}_{25}(1 - \frac{1}{z}) & \tilde{t}_{26}(1 + \frac{1}{z}) \end{bmatrix},$$

where $c = \frac{1}{486}$, $\tilde{c} = \frac{3}{34884}$, and t_{jk} 's, \tilde{t}_{jk} 's are constants defined to be:

$$\begin{matrix} t_{11} = 162, & t_{12} = 34, & t_{13} = -196, & t_{14} = 0, & t_{15} = 81, & t_{16} = 29, \\ t_{21} = -126, & t_{22} = -14, & t_{23} = 176, & t_{24} = -36, & t_{25} = -99, & t_{26} = -31, \\ \tilde{t}_{11} = 5814, & \tilde{t}_{12} = -1615, & \tilde{t}_{13} = -7160, & \tilde{t}_{14} = 0, & \tilde{t}_{15} = 5814, & \tilde{t}_{16} = 2584, \\ \tilde{t}_{21} = -5551, & \tilde{t}_{22} = 5808, & \tilde{t}_{23} = 7740, & \tilde{t}_{24} = -1358, & \tilde{t}_{25} = -6712, & \tilde{t}_{26} = -4254. \end{matrix}$$

Next, applying Algorithm 1, we obtain a pair of extension matrices $(P_e(z), \tilde{P}_e(z))$ as follows:

$$P_e(z) = c \begin{bmatrix} t_{11}(1 + \frac{1}{z}) & t_{12}(1 - \frac{1}{z}) & t_{13}(1 - \frac{1}{z}) & t_{14} & t_{15}(1 + \frac{1}{z}) & t_{16}(1 - \frac{1}{z}) \\ t_{21}(1 - \frac{1}{z}) & t_{22}(1 + \frac{1}{z}) & t_{23}(1 + \frac{1}{z}) & t_{24}(1 - \frac{1}{z}) & t_{25}(1 - \frac{1}{z}) & t_{26}(1 + \frac{1}{z}) \\ \hline t_{31}(1 + \frac{1}{z}) & t_{32}(1 - \frac{1}{z}) & t_{33}(1 - \frac{1}{z}) & t_{34}(1 + \frac{1}{z}) & t_{35}(1 + \frac{1}{z}) & t_{36}(1 - \frac{1}{z}) \\ t_{41} & 0 & 0 & t_{44} & t_{45} & 0 \\ \hline 0 & t_{52} & t_{53} & 0 & 0 & t_{56} \\ \hline t_{61}(1 - \frac{1}{z}) & t_{62}(1 + \frac{1}{z}) & t_{63}(1 + \frac{1}{z}) & t_{64}(1 - \frac{1}{z}) & t_{65}(1 - \frac{1}{z}) & t_{66}(1 + \frac{1}{z}) \end{bmatrix},$$

where all t_{jk} 's are constants given by:

$$\begin{matrix} t_{31} = 24, & t_{32} = \frac{472}{27}, & t_{33} = -\frac{148}{27}, & t_{34} = -36, & t_{35} = -24, \\ t_{36} = -\frac{112}{27}, & t_{41} = \frac{109998}{533}, & t_{44} = \frac{94041}{533}, & t_{45} = -\frac{109989}{533}, & t_{52} = 406c_0, \\ t_{53} = 323c_0, & t_{56} = 1142c_0, & t_{61} = 24210c_1, & t_{62} = 14318c_1, & t_{63} = -11807c_1, \\ t_{64} = -26721c_1, & t_{65} = -14616c_1, & t_{66} = -1934c_1, & c_1 = \frac{200}{26163}, & c_0 = \frac{1609537}{13122}, \end{matrix}$$

and

$$\tilde{P}_e(z) = \tilde{c} \begin{bmatrix} \tilde{t}_{11}(1 + \frac{1}{z}) & \tilde{t}_{12}(1 - \frac{1}{z}) & \tilde{t}_{13}(1 - \frac{1}{z}) & \tilde{t}_{14} & \tilde{t}_{15}(1 + \frac{1}{z}) & \tilde{t}_{16}(1 - \frac{1}{z}) \\ \tilde{t}_{21}(1 - \frac{1}{z}) & \tilde{t}_{22}(1 + \frac{1}{z}) & \tilde{t}_{23}(1 + \frac{1}{z}) & \tilde{t}_{24}(1 - \frac{1}{z}) & \tilde{t}_{25}(1 - \frac{1}{z}) & \tilde{t}_{26}(1 + \frac{1}{z}) \\ \hline \tilde{t}_{31}(1 + \frac{1}{z}) & \tilde{t}_{32}(1 - \frac{1}{z}) & \tilde{t}_{33}(1 - \frac{1}{z}) & \tilde{t}_{34}(1 + \frac{1}{z}) & \tilde{t}_{35}(1 + \frac{1}{z}) & \tilde{t}_{36}(1 - \frac{1}{z}) \\ \tilde{t}_{41} & 0 & 0 & \tilde{t}_{44} & \tilde{t}_{45} & 0 \\ \hline 0 & \tilde{t}_{52} & \tilde{t}_{53} & 0 & 0 & \tilde{t}_{56} \\ \hline \tilde{t}_{61}(1 - \frac{1}{z}) & \tilde{t}_{62}(1 + \frac{1}{z}) & \tilde{t}_{63}(1 + \frac{1}{z}) & \tilde{t}_{64}(1 - \frac{1}{z}) & \tilde{t}_{65}(1 - \frac{1}{z}) & \tilde{t}_{66}(1 + \frac{1}{z}) \end{bmatrix},$$

where all \tilde{t}_{jk} 's are constants given by:

$$\begin{matrix} \tilde{t}_{31} = 3483\tilde{c}_0, & \tilde{t}_{32} = 37427\tilde{c}_0, & \tilde{t}_{33} = 4342\tilde{c}_0, & \tilde{t}_{34} = -12222\tilde{c}_0, & \tilde{t}_{35} = -3483\tilde{c}_0, \\ \tilde{t}_{36} = -7267, & \tilde{c}_0 = \frac{8721}{4264}, & \tilde{t}_{41} = 5814, & \tilde{t}_{44} = 11628, & \tilde{t}_{45} = -11628, \\ \tilde{t}_{52} = 3\tilde{c}_1, & \tilde{t}_{53} = 2\tilde{c}_1, & \tilde{t}_{56} = 10\tilde{c}_1, & \tilde{c}_1 = \frac{12680011}{243}, & \\ \tilde{t}_{61} = 18203\tilde{c}_2, & \tilde{t}_{62} = 101595\tilde{c}_2, & \tilde{t}_{63} = 1638\tilde{c}_2, & \tilde{t}_{64} = -33950\tilde{c}_2, & \\ \tilde{t}_{65} = -10822\tilde{c}_2, & \tilde{t}_{66} = -36582\tilde{c}_2, & \tilde{c}_2 = \frac{26163}{213200}. & & \end{matrix}$$

Note that $SP_e = S\tilde{P}_e = [z^{-1}, -z^{-1}, z^{-1}, 1, -1, -z^{-1}]^T [1, -1, -1, 1, 1, -1]$. Moreover, for the support control, we have $|\text{csupp}(P_e)| = |\text{csupp}(\tilde{P}_e)| = |\text{csupp}(P)| = |\text{csupp}(\tilde{P})|$.

Finally, as in the last part of Algorithm 2, from the polyphase matrices $\mathbf{P}(z) := \mathbf{P}_e(z)\tilde{\mathbf{U}}^*(z)$ and $\tilde{\mathbf{P}}(z) := \tilde{\mathbf{P}}_e(z)\mathbf{U}^*(z)$, we derive high-pass filters b_1, b_2 and \tilde{b}_1, \tilde{b}_2 as follows:

$$b_1(z) = \begin{bmatrix} b_{11}^1(z) & b_{12}^1(z) \\ b_{21}^1(z) & b_{22}^1(z) \end{bmatrix}, \quad b_2(z) = \begin{bmatrix} b_{11}^2(z) & b_{12}^2(z) \\ b_{21}^2(z) & b_{22}^2(z) \end{bmatrix},$$

where

$$\begin{aligned} b_{11}^1(z) &= \frac{1}{6561}(199 + 125z^3 - 324z^2 + 199z - 324z^{-1} + 125z^{-2}), \\ b_{12}^1(z) &= \frac{1}{6561}(-361 - 125z^3 - 56z^2 + 361z + 56z^{-1} + 125z^{-2}), \\ b_{21}^1(z) &= \frac{679}{3198}(z^3 + z - 2z^2), \quad b_{22}^1(z) = \frac{387}{2132}(z^3 - z), \\ b_{11}^2(z) &= \frac{8721}{3219074}(z^3 - z), \quad b_{12}^2(z) = \frac{17442}{3219074}(203z^3 + 1142z^2 + 203z), \\ b_{21}^2(z) &= c_4(-36017 + 12403z^3 - 29232z^2 + 36017z + 29232z^{-1} - 12403z^{-2}), \\ b_{22}^2(z) &= c_4(41039 - 12403z^3 - 3868z^2 + 41039z - 3868z^{-1} - 12403z^{-2}) \end{aligned}$$

with $c_4 = \frac{50}{6357609}$, and

$$\tilde{b}_1(z) = \begin{bmatrix} \tilde{b}_{11}^1(z) & \tilde{b}_{12}^1(z) \\ \tilde{b}_{21}^1(z) & \tilde{b}_{22}^1(z) \end{bmatrix}, \quad \tilde{b}_2(z) = \begin{bmatrix} \tilde{b}_{11}^2(z) & \tilde{b}_{12}^2(z) \\ \tilde{b}_{21}^2(z) & \tilde{b}_{22}^2(z) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{b}_{11}^1(z) &= \frac{1}{17056}(-859 + 7852z^3 - 6966z^2 - 859z - 6966z^{-1} + 7825z^{-2}), \\ \tilde{b}_{12}^1(z) &= \frac{1}{17056}(-49649 + 25205z^3 - 14534z^2 + 49649z + 14534z^{-1} - 25205z^{-2}), \\ \tilde{b}_{21}^1(z) &= \frac{1}{6}(z^3 + z - 2z^2), \quad \tilde{b}_{22}^1(z) = \frac{1}{3}(z^3 - z), \\ \tilde{b}_{11}^2(z) &= 2\tilde{c}_3(z^3 - z), \quad \tilde{b}_{12}^2(z) = \tilde{c}_3(3z^3 + 10z^2 + 3z), \quad \tilde{c}_3 = \frac{39257}{26244}, \\ \tilde{b}_{21}^2(z) &= \frac{1}{852800}(49696(z - 1) + 59523(z^3 - z^{-2}) + 32466(z^{-1} - z^2)), \\ \tilde{b}_{22}^2(z) &= \frac{1}{170560}(81327(1 + z) + 40587(z^{-2} + z^3)) - \frac{4221}{32800}(z^{-1} + z^2). \end{aligned}$$

The high-pass filters b_1, b_2 and \tilde{b}_1, \tilde{b}_2 satisfy (3.17) with $c_1^1 = c_2^1 = 1/2$, $\varepsilon_1^1 = 1$, $\varepsilon_2^1 = 1$ and $c_1^2 = c_2^2 = 3/2$, $\varepsilon_1^2 = -1$, $\varepsilon_2^2 = -1$, respectively.

Let a_1, a_2 and \tilde{a}_1, \tilde{a}_2 be high-pass filters constructed from b_1, b_2 and \tilde{b}_1, \tilde{b}_2 through their symbols

$$a_1(z) := E^{-1}b_1(z)E, \quad a_2 := E^{-1}b_2E; \quad \tilde{a}_1(z) := E\tilde{b}_1(z)E^{-1}, \quad \tilde{a}_2 := E\tilde{b}_2E^{-1}.$$

Then both pairs $(\{a_0; a_1, a_2\}, \{\tilde{a}_0; \tilde{a}_1, \tilde{a}_2\})$ and $(\{b_0; b_1, b_2\}, \{\tilde{b}_0; \tilde{b}_1, \tilde{b}_2\})$ are pairs of 3-dual filter banks with the perfect reconstruction property. Let $(\{\phi; \psi^1, \psi^2\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2\})$ and $(\{\eta; \zeta^1, \zeta^2\}, \{\tilde{\eta}; \tilde{\zeta}^1, \tilde{\zeta}^2\})$ be pairs of biorthogonal 3-refinable function vectors and multiwavelet generators associated with the filter banks $(\{a_0; a_1, a_2\}, \{\tilde{a}_0; \tilde{a}_1, \tilde{a}_2\})$ and $(\{b_0; b_1, b_2\}, \{\tilde{b}_0; \tilde{b}_1, \tilde{b}_2\})$, respectively. Then, we have

$$\begin{aligned} \eta_1\left(\frac{1}{2} - \cdot\right) &= \eta_1, & \eta_2\left(\frac{1}{2} - \cdot\right) &= -\eta_2, & \tilde{\eta}_1\left(\frac{1}{2} - \cdot\right) &= \tilde{\eta}_1, & \tilde{\eta}_2\left(\frac{1}{2} - \cdot\right) &= -\tilde{\eta}_2, \\ \zeta_1^1\left(\frac{1}{2} - \cdot\right) &= \zeta_1^1, & \zeta_2^1\left(\frac{1}{2} - \cdot\right) &= \zeta_2^1, & \tilde{\zeta}_1^1\left(\frac{1}{2} - \cdot\right) &= \tilde{\zeta}_1^1, & \tilde{\zeta}_2^1\left(\frac{1}{2} - \cdot\right) &= \tilde{\zeta}_2^1, \\ \zeta_1^2\left(\frac{3}{2} - \cdot\right) &= -\zeta_1^2, & \zeta_2^2\left(\frac{3}{2} - \cdot\right) &= -\zeta_2^2, & \tilde{\zeta}_1^2\left(\frac{3}{2} - \cdot\right) &= -\tilde{\zeta}_1^2, & \tilde{\zeta}_2^2\left(\frac{3}{2} - \cdot\right) &= -\tilde{\zeta}_2^2. \end{aligned}$$

See Fig. 2 for the graphs of the 3-refinable function vectors $\eta, \tilde{\eta}$ associated with the low-pass filters b_0, \tilde{b}_0 , respectively, and the biorthogonal multiwavelet function vectors ζ^1, ζ^2 and $\tilde{\zeta}^1, \tilde{\zeta}^2$ associated with the high-pass filters b_1, b_2 and \tilde{b}_1, \tilde{b}_2 , respectively. Also see Fig. 2 for the graphs of the 3-refinable function vectors $\phi, \tilde{\phi}$ associated with the low-pass filters a_0, \tilde{a}_0 , respectively, and the biorthogonal multiwavelet function vectors ψ^1, ψ^2 and $\tilde{\psi}^1, \tilde{\psi}^2$ associated with the high-pass filters a_1, a_2 and \tilde{a}_1, \tilde{a}_2 , respectively. \square

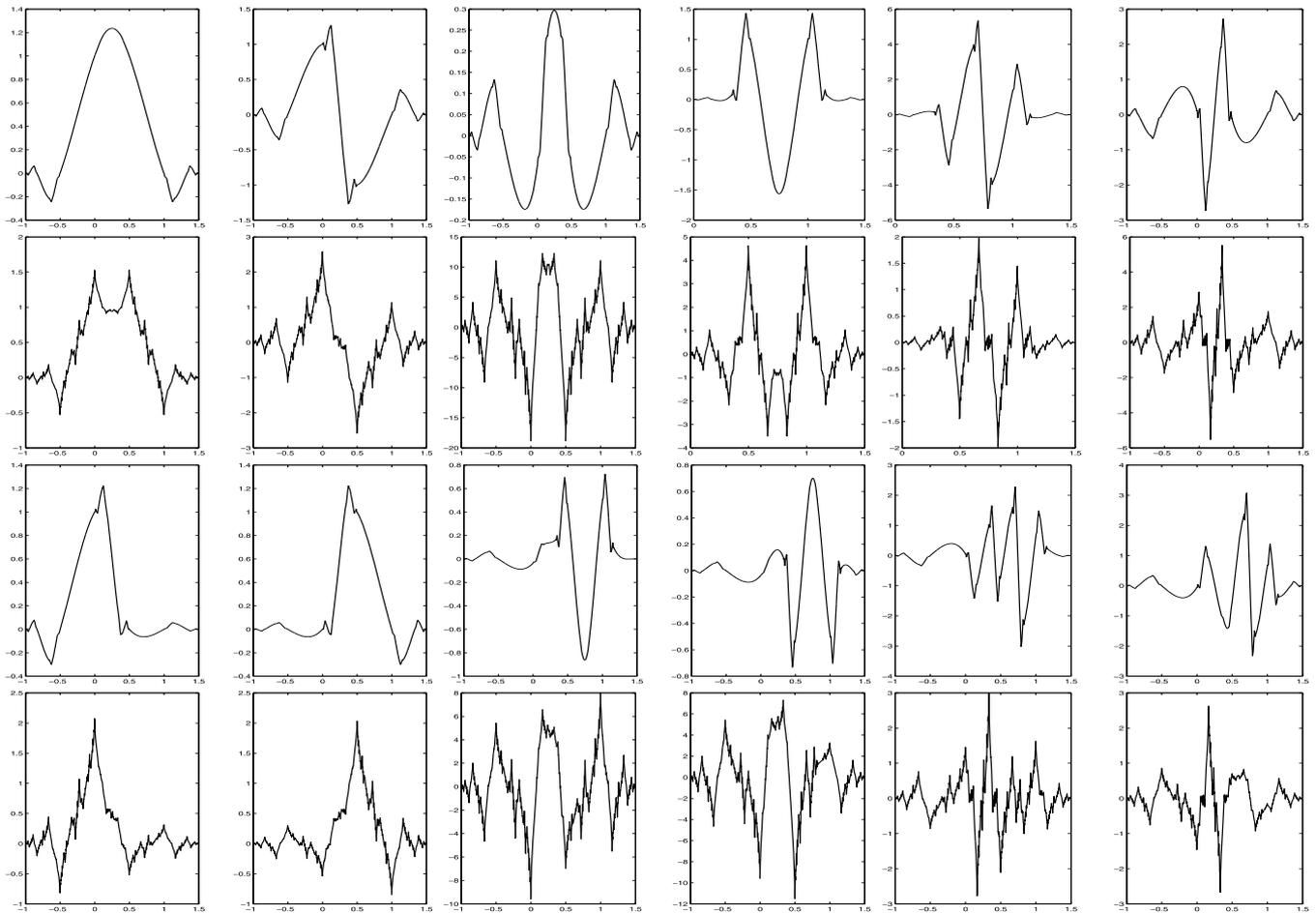


Fig. 2. The graphs of $\eta = [\eta_1, \eta_2]^T$, $\zeta^1 = [\zeta_1^1, \zeta_2^1]^T$, and $\zeta^2 = [\zeta_1^2, \zeta_2^2]^T$ (top row, left to right), and $\tilde{\eta} = [\tilde{\eta}_1, \tilde{\eta}_2]^T$, $\tilde{\zeta}^1 = [\tilde{\zeta}_1^1, \tilde{\zeta}_2^1]^T$, and $\tilde{\zeta}^2 = [\tilde{\zeta}_1^2, \tilde{\zeta}_2^2]^T$ (second row, left to right), and the graphs of $\phi = [\phi_1, \phi_2]^T$, $\psi^1 = [\psi_1^1, \psi_2^1]^T$, and $\psi^2 = [\psi_1^2, \psi_2^2]^T$ (third row, left to right), and $\tilde{\phi} = [\tilde{\phi}_1, \tilde{\phi}_2]^T$, $\tilde{\psi}^1 = [\tilde{\psi}_1^1, \tilde{\psi}_2^1]^T$, and $\tilde{\psi}^2 = [\tilde{\psi}_1^2, \tilde{\psi}_2^2]^T$ (bottom row, left to right) in Example 3.

Example 4. Consider dilation factor $d = 3$ and $r = 1$. Then we have a pair (a_0, \tilde{a}_0) 3-dual low-pass filters having filter taps in the coefficient field $\mathbb{F} = \mathbb{Q}(\sqrt{5}i)$ as follows:

$$a_0(z) = \left(\frac{1+z+1+z}{3}\right)^4 \left[\left(-\frac{4}{3} - \frac{2\sqrt{5}i}{3}\right) \frac{1}{z} + \left(\frac{11}{3} + \frac{4\sqrt{5}i}{3}\right) + \left(-\frac{4}{3} - \frac{2\sqrt{5}i}{3}\right) z \right],$$

$$\tilde{a}_0(z) = \left(\frac{1+z+1+z}{3}\right)^8 (b(z) + b(z^{-1})),$$

where

$$b(z) = \frac{329387}{2754} + \frac{209689}{1377} \sqrt{5}i - \left(\frac{102661}{816} + \frac{5464379}{22032} \sqrt{5}i\right) z - \left(\frac{177727}{2754} - \frac{551620}{4131} \sqrt{5}i\right) z^2$$

$$+ \left(\frac{2967467}{22032} - \frac{1034833}{22032} \sqrt{5}i\right) z^3 + \left(-\frac{375253}{4131} + \frac{158555}{15147} \sqrt{5}i\right) z^4 + \left(\frac{24620753}{727056} - \frac{29059}{22032} \sqrt{5}i\right) z^5$$

$$- \frac{24103}{3366} z^6 + \left(\frac{11}{16} + \frac{21391}{727056} \sqrt{5}i\right) z^7.$$

Note that both $a_0(z)$ and $\tilde{a}_0(z)$ have symmetry: $Sa_0 = S\tilde{a}_0 = 1$.

First, following Algorithm 2, let $d = d\tilde{d}$ with $d = 1$ and $\tilde{d} = 3$. We construct polyphase vectors $p_{a_0}(z) = [a_{0;0}(z), a_{0;1}(z), a_{0;2}(z)]$ and $\tilde{p}_{\tilde{a}_0}(z) = [\tilde{a}_{0;0}(z), \tilde{a}_{0;1}(z), \tilde{a}_{0;2}(z)]$. Note that $a_{0;2}(z) = z^{-1}a_{0;3}(z^{-1})$ and $\tilde{a}_{0;2}(z) = z^{-1}\tilde{a}_{0;3}(z^{-1})$. By Lemma 2, the unimodular symmetrization matrices $U(z)$ and $\tilde{U}(z)$ are given by $U(z) = \text{diag}(1, U_0)$ and $\tilde{U}(z) = \text{diag}(1, \frac{1}{2}U_0)$ with $U_0 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $Sp = S\tilde{p} = [1, z^{-1}, -z^{-1}]$ and $(p(z), \tilde{p}(z))$ is a pair of biorthogonal vectors; that is $p(z)\tilde{p}^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$. Applying Algorithm 1, we obtain a pair $(P_e(z), \tilde{P}_e(z))$ of 3×3 extension matrices. We omit the exact forms of $(p(z), \tilde{p}(z))$ and $(P_e(z), \tilde{P}_e(z))$ here. Instead, we summarize the result as follows. $(P_e(z), \tilde{P}_e(z))$ is a pair

of 3×3 biorthogonal matrices extending the pair $(\rho(z), \tilde{\rho}(z))$, $P_e(z)$ and $\tilde{P}_e(z)$ both have compatible symmetry satisfying $SP_e = S\tilde{P}_e = [1, 1, -1]^T [1, z^{-1}, -z^{-1}]$, and for the support control in terms of the input pair $(\rho(z), \tilde{\rho}(z))$, we have $\max\{|\text{csupp}(P_e)|, |\text{csupp}(\tilde{P}_e)|\} = \max\{|\text{csupp}(\rho)|, |\text{csupp}(\tilde{\rho})|\} + 1 < |\text{csupp}(\rho)| + |\text{csupp}(\tilde{\rho})|$.

Finally, as in the last part of Algorithm 2, from $\mathbf{P}(z) = P_e(z)\tilde{U}^*(z)$ and $\tilde{\mathbf{P}}(z) = \tilde{P}_e(z)U^*(z)$, we can derive high-pass filters a_1, a_2 and \tilde{a}_1, \tilde{a}_2 via (3.16) as follows:

$$\begin{aligned} a_1(z) &= b_1(z) + b_1(z^{-1}), & a_2(z) &= b_2(z) - b_2(z^{-1}), \\ \tilde{a}_1(z) &= \tilde{b}_1(z) + \tilde{b}_1(z^{-1}), & \tilde{a}_2(z) &= \tilde{b}_2(z) - \tilde{b}_2(z^{-1}), \end{aligned}$$

where

$$\tilde{b}_1(z) = \frac{(c_0 + d_0\sqrt{5}i) + (c_1 + d_1\sqrt{5}i)z + (c_2 + d_2\sqrt{5}i)z^2 + (c_3 + d_3\sqrt{5}i)z^3}{1715154023316894153611}, \quad \tilde{b}_2(z) = \frac{2\sqrt{5}i}{3}(z^3 - 4z^2 + 5z)$$

with

$$\begin{aligned} c_0 &= 114117839473968000, & d_0 &= -28945947289152960, \\ c_1 &= -126469325847476700, & d_1 &= 41505677372992140, \\ c_2 &= 99644865829391520, & d_2 &= -40357731236549760, \\ c_3 &= -30234459718898820, & d_3 &= 13325027508134100, \end{aligned}$$

and $b_1(z), b_2(z)$ are also Laurent polynomials with coefficients in the field $\mathbb{Q}(\sqrt{5}i)$. Since their exact coefficients are too long to be presented here, we only show their numerical coefficients as follows (note that, once $(a_0(z), \tilde{a}_0(z))$ and $(\tilde{a}_1(z), \tilde{a}_2(z))$ are given, $(a_1(z), a_2(z))$ are exactly determined by the biorthogonality relation in (3.3)):

$$\begin{aligned} b_1(z) &\approx (4247.95 - 488.11\sqrt{5}i) + (-1823.52 + 269.05\sqrt{5}i)z + (-1881.57 + 199.50\sqrt{5}i)z^2 \\ &\quad + (1559.41 - 51.16\sqrt{5}i)z^3 + (400.30 - 249.56\sqrt{5}i)z^4 + (-83.46 - 131.24\sqrt{5}i)z^5 \\ &\quad + (-201.19 + 156.04\sqrt{5}i)z^6 + (-140.85 + 75.68\sqrt{5}i)z^7 + (-49.48 + 18.56\sqrt{5}i)z^8 \\ &\quad + (47.18 - 13.26\sqrt{5}i)z^9 + (42.03 - 22.25\sqrt{5}i)z^{10} + (21.62 - 13.90\sqrt{5}i)z^{11} \\ &\quad + (-8.85 + 1.80\sqrt{5}i)z^{12} + (-3.36 + 2.75\sqrt{5}i)z^{13} + (-1.14 + 1.87\sqrt{5}i)z^{14} \\ &\quad + (-0.54 - 0.0017\sqrt{5}i)z^{15} + (-0.36 + 0.089\sqrt{5}i)z^{16} + (-0.18 + 0.071\sqrt{5}i)z^{17}, \\ b_2(z) &\approx \frac{1}{10^4}((-43.74 + 231.31\sqrt{5}i)z + (-30.02 + 131.54\sqrt{5}i)z^2 + (98.64 - 130.36\sqrt{5}i)z^3 \\ &\quad + (-26.84 - 76.66\sqrt{5}i)z^4 + (-37.61 - 21.27\sqrt{5}i)z^5 + (8.62 + 30.73\sqrt{5}i)z^6 \\ &\quad + (4.58 + 22.17\sqrt{5}i)z^7 + (2.06 + 9.10\sqrt{5}i)z^8 + (3.25 - 4.49\sqrt{5}i)z^9 \\ &\quad + (-0.92 - 4.74\sqrt{5}i)z^{10} + (-1.42 - 2.68\sqrt{5}i)z^{11} + (-0.51 + 0.50\sqrt{5}i)z^{12} \\ &\quad + (0.14 + 0.37\sqrt{5}i)z^{13} + (0.21 + 0.21\sqrt{5}i)z^{14} + (-0.025 + 0.027\sqrt{5}i)z^{15} \\ &\quad + (0.0058 + 0.022\sqrt{5}i)z^{16} + (0.0096 + 0.012\sqrt{5}i)z^{17}). \end{aligned}$$

We have $Sa_1 = S\tilde{a}_1 = 1$ and $Sa_2 = S\tilde{a}_2 = -1$.

The pair $(\{a_0; a_1, a_2\}, \{\tilde{a}_0; \tilde{a}_1, \tilde{a}_2\})$ is a pair of 3-dual filter banks with the perfect reconstruction property. Let $(\{\phi; \psi^1, \psi^2\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2\})$ be a pair of biorthogonal 3-refinable functions and wavelet generators associated with the pair $(\{a_0; a_1, a_2\}, \{\tilde{a}_0; \tilde{a}_1, \tilde{a}_2\})$. Then, we have

$$\phi(-\cdot) = \phi, \quad \psi^1(-\cdot) = \psi^1, \quad \psi^2(-\cdot) = -\psi^2; \quad \tilde{\phi}(-\cdot) = \tilde{\phi}, \quad \tilde{\psi}^1(-\cdot) = \tilde{\psi}^1, \quad \tilde{\psi}^2(-\cdot) = -\tilde{\psi}^2.$$

See Fig. 3 for the graphs of ϕ, ψ^1, ψ^2 and $\tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2$. \square

5. Proof of Lemma 1

In this section, we shall prove our key lemma: Lemma 1. Our proof is constructive, which produces a pair $(B(z), \tilde{B}(z))$ of biorthogonal matrices having compatible symmetry and simple structure that reduces the length of coefficient support of a given pair $(\rho(z), \tilde{\rho}(z))$. The construction of the pair $(B(z), \tilde{B}(z))$ only relates to a few coefficient vectors of the pair $(\rho(z), \tilde{\rho}(z))$. The main idea is to normalize those coefficient vectors in a way that only a few nonzero coefficients are involved in the construction of the pair $(B(z), \tilde{B}(z))$. To this end, let us introduce the following lemma – which generalizes [18, Lemma 2.1] – that normalizes a given pair of constant vectors in a field \mathbb{F} to a pair of unit coordinate vectors in \mathbb{F} .

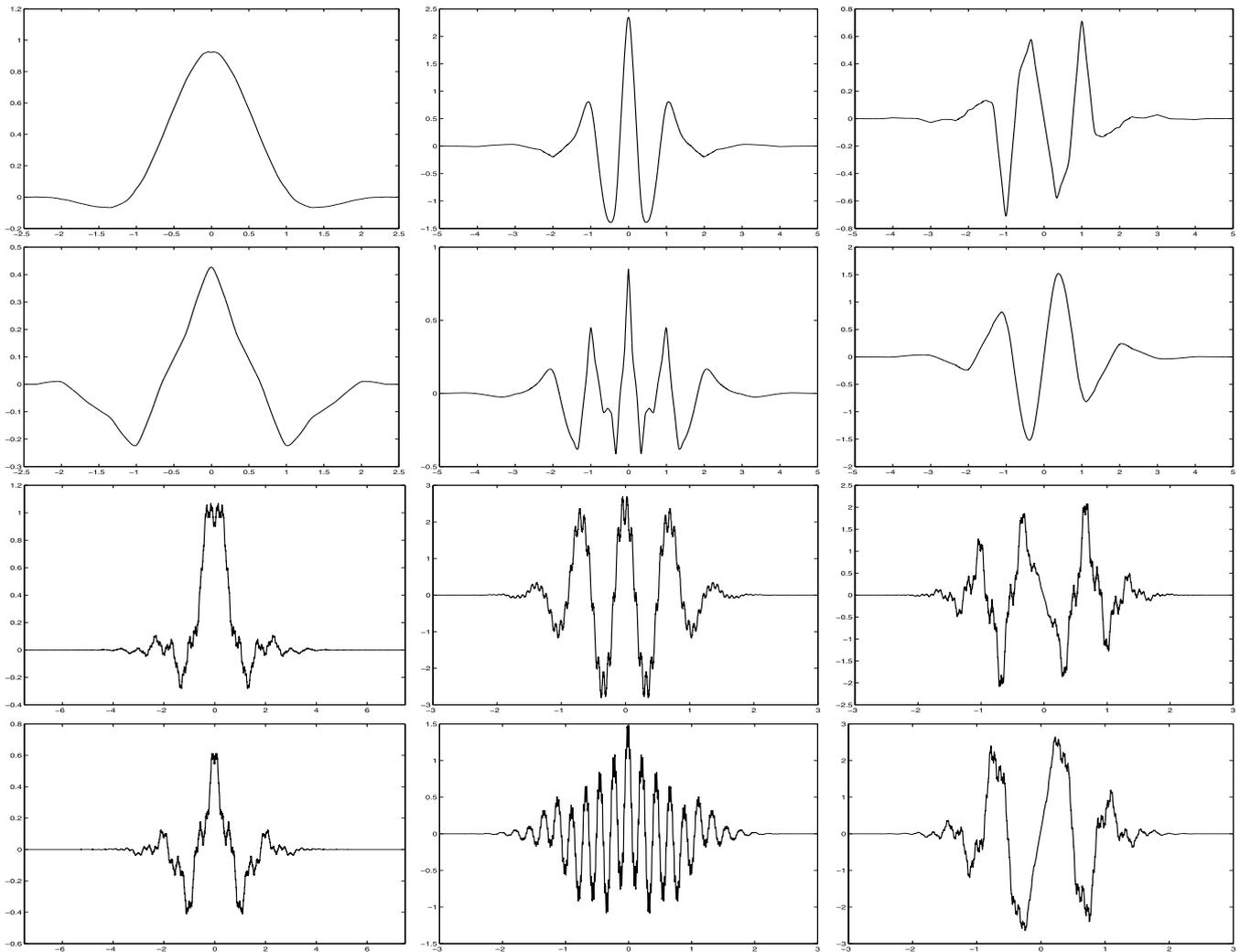


Fig. 3. The graph of ϕ, ψ^1, ψ^2 (left to right) in Example 4. Real part: top row. Imaginary part: second row. The graph of $\tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2$ (left to right) in Example 4. Real part: third row. Imaginary part: bottom row.

More precisely, given a pair $(\mathfrak{f}, \tilde{\mathfrak{f}})$ of constant vectors in \mathbb{F} , we are going to construct a pair (U, \tilde{U}) of constant matrices such that U and \tilde{U} are biorthogonal to each other ($U\tilde{U}^* = I$), and up to a constant multiplication, they normalize $\mathfrak{f}, \tilde{\mathfrak{f}}$ to two standard unit coordinate vectors.

In what follows, we shall use $\mathfrak{f}, \mathfrak{g}$ to denote constant vectors with entries in a subfield \mathbb{F} of \mathbb{C} and U, V, E, F, G to denote constant matrices with entries in \mathbb{F} . Also, recall that \star is the operator of transpose of complex conjugate, i.e., $A^* = \bar{A}^T$ for a constant matrix, and $[\mathfrak{f}]_k$ is the k th entry of a vector \mathfrak{f} . The norm of a vector \mathfrak{f} is defined to be $\|\mathfrak{f}\| = \sqrt{\mathfrak{f}\mathfrak{f}^*}$.

Lemma 3. Let $(\mathfrak{f}, \tilde{\mathfrak{f}})$ be a pair of nonzero $1 \times n$ constant vectors in \mathbb{F}^n . Then the following statements hold.

- (1) If $\mathfrak{f}\tilde{\mathfrak{f}}^* \neq 0$, then there exists a pair of $n \times n$ matrices (U, \tilde{U}) in $\mathbb{F}^{n \times n}$ such that $U\tilde{U}^* = I_n$ and

$$U = \left[\left(\frac{\tilde{\mathfrak{f}}}{\tilde{c}} \right)^*, F \right], \quad \tilde{U} = \left[\left(\frac{\mathfrak{f}}{c} \right)^*, \tilde{F} \right],$$

where F and \tilde{F} are two $n \times (n-1)$ constant matrices in $\mathbb{F}^{n \times (n-1)}$ and c, \tilde{c} are any two nonzero numbers in \mathbb{F} such that $\mathfrak{f}\tilde{\mathfrak{f}}^* = c\tilde{c}$. In this case, $\mathfrak{f}U = c\mathbf{e}_1$ and $\tilde{\mathfrak{f}}\tilde{U} = \tilde{c}\mathbf{e}_1$.

- (2) If $\mathfrak{f}\tilde{\mathfrak{f}}^* = 0$, then there exists a pair of $n \times n$ matrices (U, \tilde{U}) in $\mathbb{F}^{n \times n}$ such that $U\tilde{U}^* = I_n$ and

$$U = \left[\left(\frac{\mathfrak{f}}{c_1} \right)^*, \left(\frac{\tilde{\mathfrak{f}}}{c_2} \right)^*, F \right], \quad \tilde{U} = \left[\left(\frac{\mathfrak{f}}{c_1} \right)^*, \left(\frac{\tilde{\mathfrak{f}}}{\tilde{c}_2} \right)^*, \tilde{F} \right],$$

where F and \tilde{F} are two $n \times (n-2)$ constant matrices in $\mathbb{F}^{n \times (n-2)}$ and $c_1, c_2, \tilde{c}_1, \tilde{c}_2$ are nonzero numbers in \mathbb{F} such that $\|\mathfrak{f}\|^2 = c_1\tilde{c}_1$ and $\|\tilde{\mathfrak{f}}\|^2 = c_2\tilde{c}_2$. In this case, $\mathfrak{f}U = c_1\mathbf{e}_1$ and $\tilde{\mathfrak{f}}\tilde{U} = c_2\mathbf{e}_2$.

Proof. If $\tilde{f}\tilde{f}^* \neq 0$, then there exist $n - 1$ constant vectors f_2, \dots, f_n of size $1 \times n$ in \mathbb{F}^n such that $\{f_2, \dots, f_n\}$ is a basis of the orthogonal compliment of the linear span of $\{f\}$ in \mathbb{F}^n . It is easy to obtain such a basis $\{f, f_2, \dots, f_n\}$ using Gram–Schmidt orthogonalization process. In fact, without loss of generality, we can assume the first component $[f]_1$ of f is nonzero. Then, $\{f, e_2, \dots, e_n\}$ is a set of linear independent vectors in \mathbb{F}^n and applying the Gram–Schmidt orthogonalization process to $\{f, e_2, \dots, e_n\}$ gives the desired basis $\{f, f_2, \dots, f_n\}$. Next, let $F := [f_2^*, \dots, f_n^*]$, which is an $n \times (n - 1)$ matrix, and define $U := [(\frac{\tilde{f}}{c_1})^*, F]$. Then U is invertible and hence we could define $\tilde{U} := (U^*)^{-1}$. It is easy to show that (U, \tilde{U}) is the desired pair of matrices.

If $\tilde{f}\tilde{f}^* = 0$, similarly, let $\{f_3, \dots, f_n\}$ be a basis of the orthogonal compliment of the linear span of $\{f, \tilde{f}\}$ in \mathbb{F}^n and define $U = [(\frac{f}{c_1})^*, (\frac{\tilde{f}}{c_2})^*, F]$ with $F := [f_3^*, \dots, f_n^*]$. Then (U, \tilde{U}) with $\tilde{U} = (U^*)^{-1}$ is the desired pair of matrices. \square

Thanks to Lemma 3, we are now ready to prove Lemma 1.

Proof of Lemma 1. Suppose that $S\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ (proofs for other symmetry patterns are similar). By their symmetry patterns, $p(z)$ and $\tilde{p}(z)$ must take the form as follows with $\ell > 0$ and $\text{coeff}(p, -\ell) \neq 0$:

$$\begin{aligned}
 p(z) &= [f_1, -f_2, g_1, -g_2]z^{-\ell} + [f_3, -f_4, g_3, -g_4]z^{-\ell+1} + \sum_{k=-\ell+2}^{\ell-2} \text{coeff}(p, k)z^k \\
 &\quad + [f_3, f_4, g_1, g_2]z^{\ell-1} + [f_1, f_2, \mathbf{0}, \mathbf{0}]z^\ell, \\
 \tilde{p}(z) &= [\tilde{f}_1, -\tilde{f}_2, \tilde{g}_1, -\tilde{g}_2]z^{-\tilde{\ell}} + [\tilde{f}_3, -\tilde{f}_4, \tilde{g}_3, -\tilde{g}_4]z^{-\tilde{\ell}+1} + \sum_{k=-\tilde{\ell}+2}^{\tilde{\ell}-2} \text{coeff}(\tilde{p}, k)z^k \\
 &\quad + [\tilde{f}_3, \tilde{f}_4, \tilde{g}_1, \tilde{g}_2]z^{\tilde{\ell}-1} + [\tilde{f}_1, \tilde{f}_2, \mathbf{0}, \mathbf{0}]z^{\tilde{\ell}}.
 \end{aligned} \tag{5.1}$$

Then, either $\|f_1\| + \|f_2\| \neq 0$ or $\|g_1\| + \|g_2\| \neq 0$. Without loss of generality, we assume $\|f_1\| + \|f_2\| \neq 0$ (for the case $\|f_1\| + \|f_2\| = 0$, simply swap the roles of f and g), due to $p(z)\tilde{p}^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$ and $|\text{csupp}(p)| > 0$, we have $f_1\tilde{f}_1^* - f_2\tilde{f}_2^* = 0$. Hence, $f_1\tilde{f}_1^* = f_2\tilde{f}_2^* =: c$. Then there are at most three cases: (a) $c \neq 0$; (b) $c = 0$ but both f_1, f_2 are nonzero vectors; (c) $c = 0$ and one of f_1, f_2 is $\mathbf{0}$. For case (a), we will show that we can find a pair $(B(z), \tilde{B}(z))$ of biorthogonal matrices reduces the lengths of coefficient support of both $p(z)$ and $\tilde{p}(z)$ simultaneously. For cases (b) and (c), the idea is to construct $B(z)$ and $\tilde{B}(z)$ so that $B(z)$ reduces the length of $\text{csupp}(p)$ while $\tilde{B}(z)$ does not increase the length of $\text{csupp}(\tilde{p})$.

Case (a): In this case, we have $f_1\tilde{f}_1^* \neq 0$ and $f_2\tilde{f}_2^* \neq 0$. By Lemma 3, we can construct two pairs (U_1, \tilde{U}_1) and (U_2, \tilde{U}_2) of constant matrices with respect to the pairs (f_1, \tilde{f}_1) and (f_2, \tilde{f}_2) such that

$$\begin{aligned}
 U_1 &= \left[\begin{pmatrix} \tilde{f}_1 \\ \tilde{c}_1 \end{pmatrix}^*, F_1 \right], & \tilde{U}_1 &= \left[\begin{pmatrix} f_1 \\ c_1 \end{pmatrix}^*, \tilde{F}_1 \right], & f_1 U_1 &= c_1 \mathbf{e}_1, & \tilde{f}_1 \tilde{U}_1 &= \tilde{c}_1 \mathbf{e}_1, \\
 U_2 &= \left[\begin{pmatrix} \tilde{f}_2 \\ \tilde{c}_1 \end{pmatrix}^*, F_2 \right], & \tilde{U}_2 &= \left[\begin{pmatrix} f_2 \\ c_1 \end{pmatrix}^*, \tilde{F}_2 \right], & f_2 U_2 &= c_1 \mathbf{e}_1, & \tilde{f}_2 \tilde{U}_2 &= \tilde{c}_1 \mathbf{e}_1,
 \end{aligned}$$

where c_1, \tilde{c}_1 are constants in \mathbb{F} such that $c = c_1 \tilde{c}_1$. Define $B_0(z), \tilde{B}_0(z)$ as follows:

$$\begin{aligned}
 B_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\tilde{f}_1}{\tilde{c}_1})^* & F_1 & -\frac{1-z^{-1}}{2}(\frac{\tilde{f}_1}{\tilde{c}_1})^* & \mathbf{0} & \mathbf{0} \\ -\frac{1-z^{-1}}{2}(\frac{\tilde{f}_2}{\tilde{c}_1})^* & \mathbf{0} & \frac{1+z^{-1}}{2}(\frac{\tilde{f}_2}{\tilde{c}_1})^* & F_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array} \right], \\
 \tilde{B}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{f_1}{c_1})^* & \tilde{F}_1 & -\frac{1-z^{-1}}{2}(\frac{f_1}{c_1})^* & \mathbf{0} & \mathbf{0} \\ -\frac{1-z^{-1}}{2}(\frac{f_2}{c_1})^* & \mathbf{0} & \frac{1+z^{-1}}{2}(\frac{f_2}{c_1})^* & \tilde{F}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array} \right].
 \end{aligned} \tag{5.2}$$

It is easy to show that $B_0(z)\tilde{B}_0(z)^* = I_s$ for all $z \in \mathbb{C} \setminus \{0\}$ in view of properties of (U_1, \tilde{U}_1) and (U_2, \tilde{U}_2) . Moreover, the symmetry patterns of $p(z)B_0(z)$ and $\tilde{p}(z)\tilde{B}_0(z)$ satisfy

$$S(pB_0) = S(\tilde{p}\tilde{B}_0) = [z^{-1}, \mathbf{1}_{s_1-1}, -z^{-1}, -\mathbf{1}_{s_2-1}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}].$$

Let E be a permutation matrix such that

$$S(pB_0)E = S(\tilde{p}\tilde{B}_0)E = [\mathbf{1}_{s_1-1}, -\mathbf{1}_{s_2-1}, z^{-1}\mathbf{1}_{s_3+1}, -z^{-1}\mathbf{1}_{s_4+1}] =: S\theta_1.$$

Define $B(z) = B_0(z)E$ and $\tilde{B}(z) = \tilde{B}_0(z)E$. Then $B(z)$ and $\tilde{B}(z)$ are the desired matrices. It is easy to show that $B(z)$ and $\tilde{B}(z)$ reduce the lengths of the coefficient support of both $p(z)$ and $\tilde{p}(z)$ by 1, respectively. In fact, in view of the above symmetry patterns and the structures of $B_0(z)$ and $\tilde{B}_0(z)$, we only need to show that $\text{coeff}([pB_0]_j, \ell) = \text{coeff}([\tilde{p}\tilde{B}_0]_j, \tilde{\ell}) = 0$ for $j = 1, s_1 + 1$. Indeed, for $j = 1$,

$$\text{coeff}([pB_0]_1, \ell) = \text{coeff}(p, \ell) \text{coeff}([B_0]_{:,1}, 0) = \frac{1}{2\tilde{c}_1} (\tilde{f}_1 \tilde{f}_1^* - \tilde{f}_2 \tilde{f}_2^*) = 0.$$

Similar computations apply for other terms. In this case, $|\text{csupp}(pB)| = |\text{csupp}(p)| - 1$ and $|\text{csupp}(\tilde{p}\tilde{B})| = |\text{csupp}(\tilde{p})| - 1$. When $p(z) = \tilde{p}(z)$, case (a) always holds and as shown in [26, (3.5)]. The matrix $B(z) = \tilde{B}(z)$ are given by

$$B^*(z) := \frac{1}{c_f} \begin{bmatrix} \left[\begin{array}{cc|cc} \tilde{f}_1(z + \frac{c_0}{c_{\tilde{f}_1}} + \frac{1}{z}) & \tilde{f}_2(z - \frac{1}{z}) & g_1(1 + \frac{1}{z}) & g_2(1 - \frac{1}{z}) \\ c_f F_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \\ \left[\begin{array}{cc|cc} -\tilde{f}_1(z - \frac{1}{z}) & -\tilde{f}_2(z - \frac{c_0}{c_{\tilde{f}_1}} + \frac{1}{z}) & -g_1(1 - \frac{1}{z}) & -g_2(1 + \frac{1}{z}) \\ \mathbf{0} & c_f F_2 & \mathbf{0} & \mathbf{0} \end{array} \right] \\ \left[\begin{array}{cc|cc} \frac{c_{g_1}}{c_{\tilde{f}_1}} \tilde{f}_1(1+z) & -\frac{c_{g_1}}{c_{\tilde{f}_1}} \tilde{f}_2(1-z) & c_{g'_1} g'_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_f G_1 & \mathbf{0} \end{array} \right] \\ \left[\begin{array}{cc|cc} \frac{c_{g_2}}{c_{\tilde{f}_1}} \tilde{f}_1(1-z) & -\frac{c_{g_2}}{c_{\tilde{f}_1}} \tilde{f}_2(1+z) & \mathbf{0} & c_{g'_2} g'_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & c_f G_2 \end{array} \right] \end{bmatrix}. \tag{5.3}$$

In such circumstance, $B(z)$ reduces the length of the coefficient support of $p(z)$ by 2. See [26, (3.5)] for more details on the construction of $B(z)$ in (5.3).

Case (b): In this case, $\tilde{f}_1 \tilde{f}_1^* = \tilde{f}_2 \tilde{f}_2^* = 0$ and both \tilde{f}_1, \tilde{f}_2 are nonzero vectors. We have $\tilde{f}_1 \tilde{f}_1^* \neq 0$ and $\tilde{f}_2 \tilde{f}_2^* \neq 0$. Again, by Lemma 3, we can construct two pairs (U_1, \tilde{U}_1) and (U_2, \tilde{U}_2) of matrices with respect to the pairs $(\tilde{f}_1, \tilde{f}_1)$ and $(\tilde{f}_2, \tilde{f}_2)$ such that

$$\begin{aligned} U_1 &= \left[\left(\begin{array}{c} \tilde{f}_1 \\ \tilde{c}_1 \end{array} \right)^*, F_1 \right], & \tilde{U}_1 &= \left[\left(\begin{array}{c} \tilde{f}_1 \\ c_0 \end{array} \right)^*, F_1 \right], & \tilde{f}_1 U_1 &= c_0 \mathbf{e}_1, \\ U_2 &= \left[\left(\begin{array}{c} \tilde{f}_2 \\ \tilde{c}_2 \end{array} \right)^*, F_2 \right], & \tilde{U}_2 &= \left[\left(\begin{array}{c} \tilde{f}_2 \\ c_0 \end{array} \right)^*, F_2 \right], & \tilde{f}_2 U_2 &= c_0 \mathbf{e}_1, \end{aligned}$$

where $c_0, \tilde{c}_1, \tilde{c}_2$ are constants in \mathbb{F} such that $\tilde{f}_1 \tilde{f}_1^* = c_0 \tilde{c}_1$ and $\tilde{f}_2 \tilde{f}_2^* = c_0 \tilde{c}_2$. Let $B_0(z)$ and $\tilde{B}_0(z)$ be defined as follows:

$$\begin{aligned} B_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2} \left(\frac{\tilde{f}_1}{\tilde{c}_1} \right)^* & F_1 & -\frac{1-z^{-1}}{2} \left(\frac{\tilde{f}_1}{\tilde{c}_1} \right)^* & \mathbf{0} & \mathbf{0} \\ -\frac{1-z^{-1}}{2} \left(\frac{\tilde{f}_2}{\tilde{c}_2} \right)^* & \mathbf{0} & \frac{1+z^{-1}}{2} \left(\frac{\tilde{f}_2}{\tilde{c}_2} \right)^* & F_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array} \right], \\ \tilde{B}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2} \left(\frac{\tilde{f}_1}{c_0} \right)^* & F_1 & -\frac{1-z^{-1}}{2} \left(\frac{\tilde{f}_1}{c_0} \right)^* & \mathbf{0} & \mathbf{0} \\ -\frac{1-z^{-1}}{2} \left(\frac{\tilde{f}_2}{c_0} \right)^* & \mathbf{0} & \frac{1+z^{-1}}{2} \left(\frac{\tilde{f}_2}{c_0} \right)^* & F_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{s_3+s_4} \end{array} \right]. \end{aligned} \tag{5.4}$$

Then, $B_0(z)$ reduces the length of the coefficient support of $p(z)$ by 1 while $\tilde{B}_0(z)$ does not increase the support length of $\tilde{p}(z)$. Moreover,

$$S(pB_0) = S(\tilde{p}\tilde{B}_0) = [z^{-1}, \mathbf{1}_{s_1-1}, -z^{-1}, -\mathbf{1}_{s_2-1}, z^{-1} \mathbf{1}_{s_3}, -z^{-1} \mathbf{1}_{s_4}],$$

and similar to case (a), we can find a permutation matrix E such that

$$S(pB_0)E = S(\tilde{p}\tilde{B}_0)E = [\mathbf{1}_{s_1-1}, -\mathbf{1}_{s_2-1}, z^{-1} \mathbf{1}_{s_3+1}, -z^{-1} \mathbf{1}_{s_4+1}] =: S\theta_1.$$

Define $B(z) = B_0(z)E$ and $\tilde{B}(z) = \tilde{B}_0(z)E$. Then $B(z)$ and $\tilde{B}(z)$ are the desired matrices. In this case, we have $|\text{csupp}(pB)| \leq |\text{csupp}(p)| - 1$ and $|\text{csupp}(\tilde{p}\tilde{B})| \leq |\text{csupp}(\tilde{p})|$.

Case (c): In this case, $\tilde{f}_1 \tilde{f}_1^* = \tilde{f}_2 \tilde{f}_2^* = 0$ and one of \tilde{f}_1 and \tilde{f}_2 is nonzero. Without loss of generality, we assume that $\tilde{f}_1 \neq \mathbf{0}$ and $\tilde{f}_2 = \mathbf{0}$. Construct a pair (U_1, \tilde{U}_1) of matrices with respect to $(\tilde{f}_1, \tilde{f}_1)$ by Lemma 3 such that $\tilde{f}_1 U_1 = c_1 \mathbf{e}_1$ and $\tilde{f}_1 \tilde{U}_1 = c_2 \mathbf{e}_2$ (when $\tilde{f}_1 = \mathbf{0}$, construct the pair (U_1, \tilde{U}_1) with respect to $(\tilde{f}_1, \tilde{f}_1)$). Extend this pair to a pair (U, \tilde{U}) of $s \times s$ matrices by $U := \text{diag}(U_1, I_{s_2+s_3+s_4})$ and $\tilde{U} := \text{diag}(\tilde{U}_1, I_{s_2+s_3+s_4})$. Then $p(z)U$ and $\tilde{p}(z)\tilde{U}$ must be of the form:

$$\begin{aligned}
 q(z) &:= p(z)U = [c_1, 0, \dots, 0, -\bar{f}_2, g_1, -g_2]z^{-\ell} + [\bar{f}_3, -\bar{f}_4, g_3, -g_4]z^{-\ell+1} \\
 &\quad + \sum_{k=-\ell+2}^{\ell-2} \text{coeff}(q, k)z^k + [\bar{f}_3, \bar{f}_4, g_1, g_2]z^{\ell-1} + [c_1, 0, \dots, 0, \bar{f}_2, \mathbf{0}, \mathbf{0}]z^\ell, \\
 \tilde{q}(z) &:= \tilde{p}(z)\tilde{U} = [0, c_2, \dots, 0, -\tilde{\bar{f}}_2, \tilde{g}_1, -\tilde{g}_2]z^{-\tilde{\ell}} + [\tilde{\bar{f}}_3, -\tilde{\bar{f}}_4, \tilde{g}_3, -\tilde{g}_4]z^{-\tilde{\ell}+1} \\
 &\quad + \sum_{k=-\tilde{\ell}+2}^{\tilde{\ell}-2} \text{coeff}(\tilde{q}, k)z^k + [\tilde{\bar{f}}_3, \tilde{\bar{f}}_4, \tilde{g}_1, \tilde{g}_2]z^{\tilde{\ell}-1} + [0, c_2, \dots, 0, \tilde{\bar{f}}_2, \mathbf{0}, \mathbf{0}]z^{\tilde{\ell}}.
 \end{aligned}$$

Here, in the above, we abuse the notation and still use $\bar{f}, g, \tilde{\bar{f}}, \tilde{g}$, and so on. We next construct $B_0(z), \tilde{B}_0(z)$ so that $|\text{csupp}(qB_0)| \leq |\text{csupp}(q)| - 1$ and $|\text{csupp}(\tilde{q}\tilde{B}_0)| \leq |\text{csupp}(\tilde{q})|$. If $[\tilde{q}(z)]_1 \equiv 0$, we choose k such that

$$k = \arg \min_{\ell \neq 1} \{ |\text{csupp}([q]_1)| - |\text{csupp}([q]_\ell)| \},$$

i.e., k is such that the length $|\text{csupp}([q]_1)| - |\text{csupp}([q]_k)|$ is minimal among those of all $|\text{csupp}([q]_1)| - |\text{csupp}([q]_\ell)|$, $\ell = 2, \dots, s$; otherwise, due to $q(z)\tilde{q}^*(z) = 1$ for all $z \in \mathbb{C} \setminus \{0\}$, there must exist $k \neq 1$ such that

$$|\text{csupp}([q]_1)| - |\text{csupp}([q]_k)| \leq \max_{2 \leq j \leq s} |\text{csupp}([\tilde{q}]_j)| - |\text{csupp}([\tilde{q}]_1)|$$

(k might not be unique, we can choose one of such k so that $|\text{csupp}([q]_1)| - |\text{csupp}([q]_k)|$ is minimal among all $|\text{csupp}([q]_1)| - |\text{csupp}([q]_\ell)|$, $\ell = 2, \dots, s$). For such k (in the case of either $[\tilde{q}(z)]_1 = 0$ or $[\tilde{q}(z)]_1 \neq 0$), define two matrices $B_0(z), \tilde{B}_0(z)$ as follows:

$$B_0(z) = \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & 1 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ -a(z) & 0 & \dots & 1 & \\ \hline & & & & I_{s-k} \end{array} \right], \quad \tilde{B}_0(z) = \left[\begin{array}{cccc|c} 1 & 0 & \dots & a^*(z) & \\ 0 & 1 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & \\ \hline & & & & I_{s-k} \end{array} \right],$$

where $a(z)$ in $B_0(z)$ and $\tilde{B}_0(z)$ is a Laurent polynomial with symmetry such that

$$\begin{aligned}
 Sa(z) &= \frac{S([q]_1)}{S([q]_k)}, \\
 |\text{csupp}([q]_1 - a(z)[q]_k)| &< |\text{csupp}([q]_k)|, \quad \text{and} \\
 |\text{csupp}([\tilde{q}]_k - a^*(z)[\tilde{q}]_1)| &\leq \max_{1 \leq \ell \leq s} |\text{csupp}([\tilde{q}]_\ell)|.
 \end{aligned}$$

Such a Laurent polynomial $a(z)$ can be easily obtained by applying long division to $([q(z)]_1, [q(z)]_k)$. It is straightforward to show that $B_0(z)\tilde{B}_0^*(z) = I_s$ for all $z \in \mathbb{C} \setminus \{0\}$. $B_0(z)$ reduces the length of the coefficient support of $q(z)$ by that of $a(z)$ due to $|\text{csupp}([q]_1 - a(z)[q]_k)| < |\text{csupp}([q]_k)|$. Moreover, by our choice of k , $\tilde{B}_0(z)$ does not increase the length of the coefficient support of $\tilde{q}(z)$. Define $B(z) := UB_0(z)$ and $\tilde{B}(z) := \tilde{U}\tilde{B}_0(z)$. Then $B(z)$ and $\tilde{B}(z)$ are the desired matrices. Note in this case, the symmetry of both $p(z)$ and $\tilde{p}(z)$ are preserved, i.e., $S(pB) = Sp$ and $S(\tilde{p}\tilde{B}) = S\tilde{p}$.

In summary, for all cases (a), (b), and (c), we can always find a pair $(B(z), \tilde{B}(z))$ of biorthogonal matrices of Laurent polynomials having compatible symmetry such that $B(z)$ reduces the length of the coefficient support of $p(z)$ while $\tilde{B}(z)$ does not increase the length of the coefficient support of $\tilde{p}(z)$. \square

We remark that for the case $\|\bar{f}_1\| + \|\bar{f}_2\| = 0$, i.e., $\|g_1\| + \|g_2\| \neq 0$. The discussion for this case is similar to above. We can find two matrices $B(z), \tilde{B}(z)$ such that all items in the lemma hold. For instance, in the case that $g_1\tilde{g}_1^* = g_2\tilde{g}_2^* = c_1\bar{c}_1 \neq 0$, the pair $(B_0(z), \tilde{B}_0(z))$ similar to (5.2) is of the form:

$$\begin{aligned}
 B_0(z) &= \left[\begin{array}{cc|cc|cc} I_{s_1+s_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1+z}{2} \left(\frac{\tilde{g}_1}{c_1}\right)^* & G_1 & -\frac{1-z}{2} \left(\frac{\tilde{g}_1}{c_1}\right)^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{1-z}{2} \left(\frac{\tilde{g}_2}{c_1}\right)^* & \mathbf{0} & \frac{1+z}{2} \left(\frac{\tilde{g}_2}{c_1}\right)^* & G_2 & \mathbf{0} \end{array} \right], \\
 \tilde{B}_0(z) &= \left[\begin{array}{cc|cc|cc} I_{s_1+s_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1+z}{2} \left(\frac{g_1}{c_1}\right)^* & \tilde{G}_1 & -\frac{1-z}{2} \left(\frac{g_1}{c_1}\right)^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{1-z}{2} \left(\frac{g_2}{c_1}\right)^* & \mathbf{0} & \frac{1+z}{2} \left(\frac{g_2}{c_1}\right)^* & \tilde{G}_2 & \mathbf{0} \end{array} \right]. \tag{5.5}
 \end{aligned}$$

The pairs $(B(z), \tilde{B}(z))$ for other cases can be obtained in a similar way.

6. Final remarks

- (1) For the construction of filter banks for multirate systems (see for example, [35]), we do not consider the existence of their corresponding d -refinable function vectors and multiwavelet generators. In a recent paper [23], a one–one correspondence has been established between filter banks for multirate systems and frequency-based wavelets in the distribution space. For the existence of such biorthogonal multiwavelets in Sobolev spaces associated with a pair of d -dual filter banks, see [20,21]. Further studies about multivariate bi-frames, see [9–15] and references therein.
- (2) In Example 2, since the last entry of the first row of $P(z)$ is 0, one can apply an alternative approach as in [34] (see [34, Algorithm 2]) to construct a paraunitary matrix $P_e(z)$, and then $\tilde{P}_e(z)$ is given by $(P_e^*(z))^{-1}$. We would like to point out that this approach might result in longer lengths of coefficient supports of the extension matrices.
- (3) For Example 4, since $r = 1$, i.e., the filters are filters with scalar filter taps, one can also apply the dual-chain approach introduced in [2] for the construction of the corresponding high-pass filters.
- (4) In higher dimensions, symmetry is more complicated than that in one dimension. For one dimension, there are only two main symmetry patterns (symmetric and antisymmetric about some points) while in higher dimensions, symmetry is related to some symmetry groups, which makes it difficult to analyze the corresponding matrix extension problem with symmetry. A systematic approach for matrix extension problem with symmetry in higher dimension remains open to our best knowledge.

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