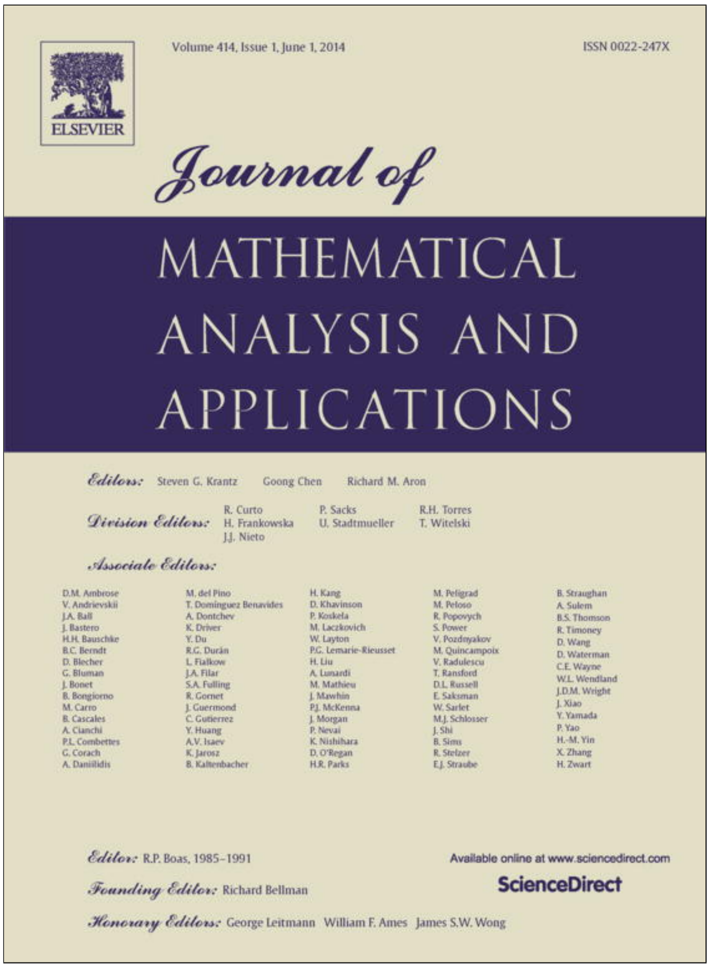
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Note

The common Hardy space and BMO space for singular integral operators associated with isotropic and anisotropic homogeneity

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If T_1 and T_2 are two singular integral operators associated with isotropic and anisotropic homogeneity, respectively, then T_1, T_2 and $T_1 \circ T_2$ are bounded on different Hardy spaces and BMO spaces (see [7,8,12]). In our paper, we show that these operators are actually bounded on a common Hardy space and a common BMO space.

1. Introduction

It is well known that the classical singular integral operators and anisotropic singular integral operators are both bounded on $L^p(\mathbb{R}^m)$ $(1 < p < \infty)$. But for the endpoint spaces, this situation is changed. We have already known that the first one is bounded on the classical isotropic Hardy spaces and isotropic BMO spaces, and the second one is bounded on the anisotropic Hardy spaces and anisotropic BMO spaces, respectively. These Hardy spaces and BMO spaces are essentially different. A natural question is weather there exist a common Hardy space and a common BMO space on which these operators are all bounded. The purpose of this paper is to answer this question. We will show that these operators are all bounded on the product Hardy spaces $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ and the product BMO space $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$.

Our results can be immediately applied to the compositions of operators with different kind of homogeneities which arise naturally in the $\bar{\partial}$-Neumann problem. More precisely, let $e(\xi)$ and $h(\xi)$ be homogeneous functions on $\mathbb{R}^m$ of degree 0 in the classical isotropic sense and the anisotropic sense, respectively, and smooth away from the origin. It is well-known that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\widehat{f}(\xi)$ and T_2

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given by $\widehat{T_2(f)}(\xi) = h(\xi)\hat{f}(\xi)$ are both bounded on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type $(1,1)$. Rivieré in [14] asked the following question: Is the composition $T_1 \circ T_2$ still of weak-type $(1,1)$? Phong and Stein in [11] answered this question affirmatively. Recently, in [8], a new Hardy space was introduced and it was proved that the composition $T_1 \circ T_2$ is bounded on this new Hardy space. In [7], a new BMO_{com} and the Lipschitz spaces CMO^p_{com}, $0 < p \leqslant 1$ are established and it was also shown that the composition $T_1 \circ T_2$ is bounded on them. These results are interesting. However they make the Hardy spaces and the BMO spaces too complicated due to the existence of too many such spaces. It is meaningful if we can find a common Hardy space and a common BMO space on which the operators T_1, T_2 and $T_1 \circ T_2$ are all bounded. Actually, we will show that the common spaces exist and they are the product space $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ and the product space $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$.

To describe our questions and our results more precisely, we begin with considering all functions and operators defined on $\mathbb{R}^m$. For $x \in \mathbb{R}^m$, we write $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}^{m-1}$ and $x_2 \in \mathbb{R}$. We denote by $|x| = (|x_1|^2 + |x_2|^2)^{\frac{1}{2}}$ and $|x|_h = (|x_1|^2 + |x_2|)^{\frac{1}{2}}$. The usual norm $|x|$ is isotropic in the sense that $|tx| = t|x|$ for $t \geqslant 0$ while the norm $|x|_h$ is non-isotropic and it induces the parabolic dilation in the sense that $|\rho_t x|_h = t|x|_h$ with $\rho_t = \mathrm{diag}(t, \ldots, t, t^2)$, $t \geqslant 0$. The parabolic dilation together with rotation operators or shear operators play a crucial role in the recent development of directional multiscale representation systems in wavelet analysis, e.g. [1,2]. These types of systems can be used to capture anisotropic features such as curve singularities in 2D or surface singularities in 3D, etc., which leads to sparse approximation of high-dimensional data that concentrate near low-dimensional structures; see [10] and references therein for more details.

In this paper, the Calderón–Zygmund singular integral operators associated with isotropic homogeneity (we refer readers to [12]) are defined as follows.

Definition 1.1. T_1 is said to be a Calderón–Zygmund singular integral operator associated with isotropic homogeneity, if T_1 is bounded on $L^2(\mathbb{R}^m)$ and $T_1 f(x) = p.v.(\mathcal{K}_1 * f)(x)$ with $\mathcal{K}_1 \in C^2(\mathbb{R}^m \backslash \{0\})$ and $|\partial_x^\alpha \mathcal{K}_1(x)| \leqslant \frac{C}{|x|^{m+|\alpha|}}$ for all $x \in \mathbb{R}^m \backslash \{0\}$, $\alpha \in \mathbb{N}_0^m$ with $|\alpha| \leqslant 1$.

The Calderón–Zygmund singular integral operators associated with anisotropic homogeneity is defined as follows.

Definition 1.2. T_2 is said to be a Calderón–Zygmund singular integral operator associated with anisotropic homogeneity, if T_2 is bounded on $L^2(\mathbb{R}^m)$ and $T_2 f(x) = p.v.(\mathcal{K}_2 * f)(x)$ with $\mathcal{K}_2 \in C^2(\mathbb{R}^m \backslash \{0\})$ and $|\partial_{x_1}^\alpha \partial_{x_2}^\beta \mathcal{K}_2(x_1, x_2)| \leqslant \frac{C}{|x|_h^{m+1+|\alpha|+2\beta}}$ for all $x \in \mathbb{R}^m \backslash \{0\}$, $\alpha \in \mathbb{N}_0^{m-1}$, $\beta \in \mathbb{N}_0$ with $|\alpha|, |\beta| \leqslant 1$.

Note that $\mathcal{K}_1$ is invariant under isotropic dilation, i.e. for all $\delta > 0$, $\delta^m \mathcal{K}_1(\delta x)$ satisfies the same estimates as $\mathcal{K}_1$. Meanwhile, $\mathcal{K}_2$ is invariant under anisotropic dilation, i.e. for all $\delta > 0$, $\delta^{m+1}\mathcal{K}_2(\delta x_1, \delta^2 x_2)$ satisfies the same estimates as $\mathcal{K}_2$. It is well known that T_1 and T_2 are both bounded on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$. But for the endpoint spaces, things become different. It is known that T_1 is bounded on the isotropic BMO space and the classical isotropic Hardy space $H^p(\mathbb{R}^m)$ for $p \leqslant 1$ but p is close to 1. And T_2 is bounded on the anisotropic BMO space and the anisotropic Hardy space $H_h^p(\mathbb{R}^m)$ for $p \leqslant 1$ but p is close to 1 (see [12]).

The purpose of this paper is to show that T_1 and T_2 are bounded on the product Hardy spaces $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ and the product BMO space $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$. Before doing so, we first recall the definitions of the product Hardy space $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ and the product BMO space $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$ (see [3] and [6] for more details).

For $j, k, N \in \mathbb{Z}$, we let $\mathcal{Q}^{j,k} = \{R = I \times J\colon I, J$ are dyadic rectangles on $\mathbb{R}^{m-1}$ and $\mathbb{R}$ with side-lengths $\ell(I) = 2^{-j}$ and $\ell(J) = 2^{-k}$, respectively$\}$ and $\mathcal{Q}_N^{j,k} = \mathcal{Q}^{j+N,k+N}$.

Given $p \leqslant 1$ but p is close to 1 and a function $\psi \in \mathcal{S}(\mathbb{R}^m)$ with the support contained in the unit ball and satisfying $\int_{\mathbb{R}^{m-1}} \psi(x_1, x_2) x_1^\alpha\, dx_1 = \int_{\mathbb{R}} \psi(x_1, x_2) x_2^\beta\, dx_2 = 0$ for all $0 \leqslant |\alpha|, |\beta| \leqslant M_p$ where M_p is a large integer depending on p, and $\sum_{j,k\in\mathbb{Z}} |\hat{\psi}(2^{-j}\xi_1, 2^{-k}\xi_2)|^2 = 1$ for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{m-1} \times \mathbb{R}$ with $\xi_1 \neq 0$ and $\xi_2 \neq 0$. The product Littlewood–Paley square function of f is defined by

$$g_\psi(f)(x) = \left(\sum_{j,k\in\mathbb{Z}} \left|\psi_{j,k} * f(x)\right|^2\right)^{\frac{1}{2}},$$

where $\psi_{j,k}(x) = 2^{j(m-1)+k}\psi(2^j x_1, 2^k x_2)$.

And the discrete product Littlewood–Paley square function is defined by

$$g_\psi^d(f)(x) = \left(\sum_{j,k\in\mathbb{Z}} \sum_{R=I\times J\in\mathcal{Q}^{j,k}} \left|\psi_{j,k} * f(c_R)\right|^2 \chi_R(x)\right)^{\frac{1}{2}},$$

where $\chi_R(x)$ is the characteristic function and $c_R = (c_I, c_J)$ is the center of R.

The product Hardy space $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ and the product BMO space $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$ are defined as follows.

Definition 1.3. Let $f \in \mathcal{S}' \setminus \mathcal{P}$, where $\mathcal{S}' \setminus \mathcal{P}$ denotes the space of temper distributions modulo polynomials.

(a) We say $f \in H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ if $f \in \mathcal{S}' \setminus \mathcal{P}$ with the finite norm:

$$\|f\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})} = \left\|g_\psi^d(f)\right\|_{L^p(\mathbb{R}^m)}.$$

(b) We say $f \in BMO(\mathbb{R}^{m-1} \times \mathbb{R})$ if $f \in \mathcal{S}' \setminus \mathcal{P}$ with the finite norm:

$$\|f\|_{BMO(\mathbb{R}^{m-1}\times\mathbb{R})} = \sup_{\Omega}\left\{\left(\frac{1}{|\Omega|}\sum_{j,k\in\mathbb{Z}} \sum_{\substack{R=I\times J\in\mathcal{Q}^{j,k}\\ R\subset\Omega}} |R|\left|\psi_{j,k} * f(c_R)\right|^2\right)^{\frac{1}{2}} :\ \Omega \subset \mathbb{R}^m \text{ open sets}\right\}.$$

Now we are ready to introduce our main result and the remaining part of this paper is devoted to the proof of this result.

Theorem 1.4. *Suppose that T_1 and T_2 are Calderón–Zygmund singular integral operators associated with isotropic and anisotropic homogeneity, respectively. Then T_1, T_2 and $T_1 \circ T_2$ are all bounded on $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$ and $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$, for $1 - \frac{1}{m} < p \leqslant 1$.*

Throughout the paper, the notation $A \lesssim B$ means $A \leqslant CB$, for some positive constant C, while the notation $A \approx B$ means $C_1 A \leqslant B \leqslant C_2 A$ for some positive constants C_1, C_2. And $j \wedge j'$ means the minimum of j and j'.

Remark 1.5. It has been known that the definitions of product Hardy space $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ and the product BMO space $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$ are the independent choice of ψ, and $\|g_\psi(f)\|_{L^p(\mathbb{R}^m)} \approx \|g_\psi^d(f)\|_{L^p(\mathbb{R}^m)}$. See [9] for more details.

2. Proof of Theorem 1.4

In [13], we have shown that T_1 is bounded on the product Hardy space $H^p(\mathbb{R}^{m-1} \times \mathbb{R})$ and the product BMO space $BMO(\mathbb{R}^{m-1} \times \mathbb{R})$. So we just need to obtain the same result for T_2. The key estimate in the proof of Theorem 1.4 is the following orthogonal estimate.

Lemma 2.1. *Suppose that $\phi(x) \in C_0^\infty(\mathbb{R}^m)$ with $\int_{\mathbb{R}^{m-1}} \phi(x_1, x_2)\, dx_1 = \int_{\mathbb{R}} \phi(x_1, x_2)\, dx_2 = 0$. If $\mathcal{K}_2$ is a Calderón–Zygmund convolution kernel associated with anisotropic homogeneity as given in Definition 1.2, then*

$$\left|\phi_{j,k} * \mathcal{K}_2 * \phi_{j',k'}(x)\right| \leqslant C_\phi 2^{-|j-j'|} 2^{-|k-k'|} \frac{2^{(j\wedge j')(m-1)}}{1 + |2^{j\wedge j'} x_1|^m} \frac{2^{(k\wedge k')}}{1 + |2^{k\wedge k'} x_2|^2}$$

for all $x = (x_1, x_2) \in \mathbb{R}^{m-1} \times \mathbb{R}$, where C_ϕ is a constant depending only on ϕ.

Based on the following two observations: (1) convolution operation is commutative, i.e., $\phi_{j,k} * \mathcal{K}_2 * \phi_{j',k'}(x) = \mathcal{K}_2 * (\phi_{j,k} * \phi_{j',k'})(x)$; (2) $\phi_{j,k} * \phi_{j',k'}$ satisfies the same estimates as $\phi_{j\wedge j', k\wedge k'}$ with the bound $C2^{-|j-j'|}2^{-|k-k'|}$, Lemma 2.1 can be reduced to the following.

Lemma 2.2. *Suppose that $\phi(x) \in C_0^\infty(\mathbb{R}^m)$ with $\int_{\mathbb{R}^{m-1}} \phi(x_1, x_2)\, dx_1 = \int_{\mathbb{R}} \phi(x_1, x_2)\, dx_2 = 0$. If $\mathcal{K}_2$ is a Calderón–Zygmund convolution kernel associated with anisotropic homogeneity as given in Definition 1.2, then*

$$\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| \leqslant C_\phi \frac{2^{j(m-1)}}{1 + |2^j x_1|^m} \frac{2^k}{1 + |2^k x_2|^2}$$

for all $x = (x_1, x_2) \in \mathbb{R}^{m-1} \times \mathbb{R}$, where C_ϕ is a constant depending only on ϕ.

Proof. Without loss of generality, we may assume that $\operatorname{supp}(\phi) \subset \{x\colon |x| \leqslant 1\}$. We prove the required estimate in four cases: (I) $|x_1| \geqslant 2^{-j+1}$, $|x_2| \geqslant 2^{-k+1}$; (II) $|x_1| \geqslant 2^{-j+1}$, $|x_2| < 2^{-k+1}$; (III) $|x_1| < 2^{-j+1}$, $|x_2| \geqslant 2^{-k+1}$; (IV) $|x_1| < 2^{-j+1}$, $|x_2| < 2^{-k+1}$.

For case (I) $|x_1| \geqslant 2^{-j+1}$, $|x_2| \geqslant 2^{-k+1}$, we first point out that:

$$\lim_{\epsilon\to 0} \iint\limits_{|x-y|_h > \epsilon} \mathcal{K}_2(x_1 - y_1, x_2)\phi\big(2^j y_1, 2^k y_2\big)\, dy_1\, dy_2 = 0 \tag{2.1}$$

and

$$\lim_{\epsilon\to 0} \iint\limits_{|x-y|_h > \epsilon} \mathcal{K}_2(x_1, x_2 - y_2)\phi\big(2^j y_1, 2^k y_2\big)\, dy_1\, dy_2 = 0. \tag{2.2}$$

The equality (2.1) can be obtained by the facts that: (1) if $\phi(2^j y_1, 2^k y_2) \neq 0$, then $|x-y|_h^2 \geqslant |x_1 - y_1|^2 \geqslant 2^{-2j}$; (2) $\iint_{\mathbb{R}^m} |\mathcal{K}_2(x_1 - y_1, x_2)\phi(2^j y_1, 2^k y_2)|\, dy_1\, dy_2 < \infty$; (3) $\int_{\mathbb{R}} \phi(2^j y_1, 2^k y_2)\, dy_2 = 0$. The equality (2.2) can be obtained in a similar way.

Now by (2.1) and (2.2), we have

$$\begin{aligned}\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| = 2^{j(m-1)+k}\Bigg|\lim_{\epsilon\to 0} \iint\limits_{|x-y|_h > \epsilon} &\big[\big(\mathcal{K}_2(x_1 - y_1, x_2 - y_2) - \mathcal{K}_2(x_1, x_2 - y_2)\big) \\ &- \big(\mathcal{K}_2(x_1 - y_1, x_2) - \mathcal{K}_2(x_1, x_2)\big)\big]\phi\big(2^j y_1, 2^k y_2\big)\, dy_1\, dy_2\Bigg|.\end{aligned}$$

Note that

$$\big(\mathcal{K}_2(x_1 - y_1, x_2 - y_2) - \mathcal{K}_2(x_1, x_2 - y_2)\big) - \big(\mathcal{K}_2(x_1 - y_1, x_2) - \mathcal{K}_2(x_1, x_2)\big)$$

$$= \int_0^1 \int_0^1 \partial_s^1 \partial_t^1 \left[\mathcal{K}_2(x_1 - sy_1, x_2 - ty_2)\right] ds\, dt$$

$$= \int_0^1 \int_0^1 \sum_{i=1}^{m-1} y_{1i} y_2 \partial_{x_{1i}}^1 \partial_{x_2}^1 \left[\mathcal{K}_2(x_1 - sy_1, x_2 - ty_2)\right] ds\, dt,$$

where $y_1 = (y_{11}, y_{12}, \ldots, y_{1(m-1)})$. Applying the hypothesis on $\mathcal{K}_2$, that is, the second-order difference smoothness condition, yields

$$\begin{aligned}
\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| &\lesssim 2^{j(m-1)+k} \int_{\mathbb{R}} \int_{\mathbb{R}^{m-1}} \frac{|y_1||y_2|}{(|x_1|^2 + |x_2|)^{(m+4)/2}} \left|\phi\left(2^j y_1, 2^k y_2\right)\right| dy_1\, dy_2 \\
&\lesssim \frac{2^{-j}}{(|x_1|^2 + |x_2|)^{m/2}} \frac{2^{-k}}{(|x_1|^2 + |x_2|)^2} \\
&\lesssim \frac{2^{-j}}{|x_1|^m} \frac{2^{-k}}{|x_2|^2} \lesssim \frac{2^{j(m-1)}}{1 + |2^j x_1|^m} \frac{2^k}{1 + |2^k x_2|^2}.
\end{aligned}$$

For case (II) $|x_1| \geqslant 2^{-j+1}$, $|x_2| < 2^{-k+1}$, similar to case (I), we have

$$\lim_{\epsilon \to 0} \iint_{|x-y|_h > \epsilon} \mathcal{K}_2(x_1, x_2 - y_2) \phi\left(2^j y_1, 2^k y_2\right) dy_1\, dy_2 = 0. \tag{2.3}$$

So, we can write

$$\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| = 2^{j(m-1)+k} \left| \lim_{\epsilon \to 0} \iint_{|x-y|_h > \epsilon} \left(\mathcal{K}_2(x_1 - y_1, x_2 - y_2) - \mathcal{K}_2(x_1, x_2 - y_2)\right) \phi\left(2^j y_1, 2^k y_2\right) dy_1\, dy_2 \right|.$$

Applying the mean value theorem and the hypothesis on $\mathcal{K}_2$ implies

$$\begin{aligned}
\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| &\lesssim 2^{j(m-1)+k} \int_{\mathbb{R}} \int_{|y_1| \leqslant 2^{-j}} \frac{|y_1|}{(|x_1|^2 + |x_2 - y_2|)^{(m+2)/2}} dy_1\, dy_2 \\
&\lesssim \frac{2^{-j}}{|x_1|^m} 2^k \lesssim \frac{2^{j(m-1)}}{1 + |2^j x_1|^m} \frac{2^k}{1 + |2^k x_2|^2}.
\end{aligned}$$

For case (III) $|x_1| < 2^{-j+1}$, $|x_2| \geqslant 2^{-k+1}$. Similarly,

$$\lim_{\epsilon \to 0} \iint_{|x-y|_h > \epsilon} \mathcal{K}_2(x_1 - y_1, x_2) \phi\left(2^j y_1, 2^k y_2\right) dy_1\, dy_2 = 0. \tag{2.4}$$

Hence, we can write

$$\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| = 2^{j(m-1)+k} \left| \lim_{\epsilon \to 0} \iint_{|x-y|_h > \epsilon} \left(\mathcal{K}_2(x_1 - y_1, x_2 - y_2) - \mathcal{K}_2(x_1 - y_1, x_2)\right) \phi\left(2^j y_1, 2^k y_2\right) dy_1\, dy_2 \right|.$$

Also apply the mean value theorem and the hypothesis on $\mathcal{K}_2$, we get

$$\begin{aligned}\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| &\lesssim 2^{j(m-1)+k} \int\limits_{|y_2|\leqslant 2^{-k}} \int\limits_{\mathbb{R}^{m-1}} \frac{|y_2|}{(|x_1-y_1|^2+|x_2|)^{(m+3)/2}}\, dy_1\, dy_2 \\ &\lesssim 2^{j(m-1)} \frac{2^{-k}}{|x_2|^2} \lesssim \frac{2^{j(m-1)}}{1+|2^j x_1|^m} \frac{2^k}{1+|2^k x_2|^2}.\end{aligned}$$

For the last case (IV) $|x_1| < 2^{-j+1}$, $|x_2| < 2^{-k+1}$, let $\eta_1 \in C_0^\infty(\mathbb{R}^{m-1})$ with $0 \leqslant \eta_1(x_1) \leqslant 1$ and $\eta_1(x_1) = 1$ when $|x_1| \leqslant 4$, and $\eta_1(x_1) = 0$ when $|x_1| \geqslant 8$. Set $\eta_2(x_2)$ similarly. Then

$$\begin{aligned}\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| &= 2^{j(m-1)+k} \Bigg| \lim_{\epsilon\to 0} \iint\limits_{|y|_h>\epsilon} \mathcal{K}_2(y_1,y_2)\phi\big(2^j(x_1-y_1), 2^k(x_2-y_2)\big) \\ &\quad \times \eta_1\big(2^j(x_1-y_1)\big)\eta_2\big(2^k(x_2-y_2)\big)\, dy_1\, dy_2 \Bigg| \\ &\leqslant 2^{j(m-1)+k} \Bigg| \lim_{\epsilon\to 0} \iint\limits_{|y|_h>\epsilon} \mathcal{K}_2(y_1,y_2)\big(\phi\big(2^j(x_1-y_1), 2^k(x_2-y_2)\big) \\ &\quad - \phi\big(2^j x_1, 2^k x_2\big)\big)\eta_1\big(2^j(x_1-y_1)\big)\eta_2\big(2^k(x_2-y_2)\big)\, dy_1\, dy_2 \Bigg| \\ &\quad + 2^{j(m-1)+k} \Bigg| \lim_{\epsilon\to 0} \iint\limits_{|y|_h>\epsilon} \mathcal{K}_2(y_1,y_2)\eta_1\big(2^j(x_1-y_1)\big)\eta_2\big(2^k(x_2-y_2)\big)\phi\big(2^j x_1, 2^k x_2\big)\, dy_1\, dy_2 \Bigg|.\end{aligned}$$

Using the condition on $\mathcal{K}_2$ and the smoothness condition on ϕ for the above first term, and the fact that $\hat{\mathcal{K}}_2$ is bounded for the above second term, give

$$\begin{aligned}\left|\mathcal{K}_2 * \phi_{j,k}(x)\right| &\lesssim 2^{j(m-1)+k} \int\limits_{|y_2|\leqslant 2^{-k+3}} \int\limits_{|y_1|\leqslant 2^{-j+3}} \frac{1}{(|y_1|^2+|y_2|)^{(m+1)/2}}\big(\big|2^j y_1\big| + \big|2^k y_2\big|\big)\, dy_1\, dy_2 \\ &\quad + \Bigg| \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{m-1}} \hat{\mathcal{K}}_2(\xi_1,\xi_2)\hat{\eta}_1\big(2^{-j}\xi_1\big)\hat{\eta}_2\big(2^{-k}\xi_2\big)\, d\xi_1\, d\xi_2 \Bigg| \\ &\lesssim 2^{j(m-1)+k} \lesssim \frac{2^{j(m-1)}}{1+|2^j x_1|^m} \frac{2^k}{1+|2^k x_2|^2}.\end{aligned}$$

The proof of Lemma 2.2 is complete. □

Thanks to Lemma 2.1, the remaining steps are routine. For the convenience of readers, we complete the proof as follows.

We introduce two lemmas needed for the proof. The first necessary lemma is the so-called discrete Calderón's identity. For its proof, we refer readers to [9].

Lemma 2.3. *Given $0 < p \leqslant 1$. Suppose that $\phi(x) \in C_0^\infty(\mathbb{R}^m)$ with $\operatorname{supp}(\phi) \in \{x\colon |x| \leqslant 1\}$, $\int_{\mathbb{R}^{m-1}} \phi(x_1,x_2)x_1^\alpha\, dx_1 = \int_{\mathbb{R}^n} \phi(x_1,x_2)x_2^\beta\, dx_2 = 0$ for $0 \leqslant |\alpha|, |\beta| \leqslant M_p$, M_p is a fixed large integer depending on p and $\sum_{j,k\in\mathbb{Z}} |\hat{\phi}(2^{-j}\xi_1, 2^{-k}\xi_2)|^2 = 1$ for all $\xi_1 \neq 0$ and $\xi_2 \neq 0$. For a given $f \in L^2(\mathbb{R}^m) \cap H^p(\mathbb{R}^{m-1}\times\mathbb{R})$, there exist a function $h \in L^2(\mathbb{R}^m) \cap H^p(\mathbb{R}^{m-1}\times\mathbb{R})$ and a large integer $N > 0$ such that $f(x_1,x_2) = \sum_{j,k\in\mathbb{Z}} \sum_{R\in Q_N^{j,k}} |R|\phi_{j,k}(x-c_R)(\phi_{j,k}*h)(c_R)$, where the series converges in both $L^2(\mathbb{R}^m)$ and $H^p(\mathbb{R}^{m-1}\times\mathbb{R})$. Moreover, $\|f\|_{L^2(\mathbb{R}^m)} \approx \|h\|_{L^2(\mathbb{R}^m)}$ and $\|f\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})} \approx \|h\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})}$.*

The other necessary lemma is as follows. For its proof, we refer readers to [5].

Lemma 2.4. *Suppose that $\frac{m-1}{m} < \delta \leqslant 1$, $F \in L^2(\mathbb{R}^m)$, $j, k, j', k' \in \mathbb{Z}$ and N is an integer. If $I' \times J' \in \mathcal{Q}^{j',k'}$, then for any $u = (u_1, u_2), v = (v_1, v_2) \in I' \times J'$, we have*

$$\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}} \frac{2^{(j\wedge j')(m-1)}}{(1+2^{j\wedge j'}|u_1-c_I|)^m} \frac{2^{(k\wedge k')}}{(1+2^{k\wedge k'}|u_2-c_J|)^2} |F(c_R)|$$
$$\leqslant C 2^{(m-1)\{(j\wedge j')(1-1/\delta)+j/\delta\}} 2^{(k\wedge k')(1-1/\delta)+k/\delta} \left\{ M_s\left[\left(\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}} \left|F(c_R)\right|^2 \chi_R\right)^{\delta/2}\right]\right\}^{1/\delta}(v),$$

where M_s is the strong maximal function.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. In [13], we have shown that T_1 is bounded on the product Hardy space $H^p(\mathbb{R}^{m-1}\times\mathbb{R})$ and the product BMO space $BMO(\mathbb{R}^{m-1}\times\mathbb{R})$. So we only need to prove boundedness of T_2.

Since $L^2(\mathbb{R}^m) \cap H^p(\mathbb{R}^{m-1}\times\mathbb{R})$ is dense in $H^p(\mathbb{R}^{m-1}\times\mathbb{R})$, we only need to show that

$$\|T_2 f\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})} \leqslant C\|f\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})}$$

for all $f \in L^2(\mathbb{R}^m) \cap H^p(\mathbb{R}^{m-1}\times\mathbb{R})$. By the definition of $H^p(\mathbb{R}^{m-1}\times\mathbb{R})$, we only need to show that for any fixed ψ, we have

$$\left\|g_\psi^d(T_2 f)\right\|_{L^p(\mathbb{R}^m)} \leqslant C\|f\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})}.$$

Note that

$$\left|g_\psi^d(T_2 f)(x)\right|^2 = \sum_{j',k'\in\mathbb{Z}} \sum_{R'=I'\times J'\in\mathcal{Q}^{j',k'}} \left|\psi_{j',k'} * \mathcal{K}_2 * f(c_{R'})\right|^2 \chi_{R'}(x).$$

For any $R' = I' \times J' \in \mathcal{Q}^{j',k'}$ with $x \in R'$, we first apply Lemma 2.3, and then apply Lemma 2.1, and finally apply Lemma 2.4 with $F = \psi_{j,k} * h$ and $\frac{m-1}{m} < \delta < p$. We get

$$\left|\psi_{j',k'} * \mathcal{K}_2 * f(c_{R'})\right| \lesssim \sum_{j,k\in\mathbb{Z}} 2^{-j(m-1)-k} 2^{-|j-j'|} 2^{-|k-k'|} 2^{(m-1)\{(j\wedge j')(1-1/\delta)+j/\delta\}}$$
$$\times 2^{(k\wedge k')(1-1/\delta)+k/\delta} \left\{ M_s\left[\left(\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}} \left|\psi_{j,k} * h(c_R)\right|^2 \chi_R\right)^{\delta/2}\right]\right\}^{1/\delta}(x),$$

where M_s is the strong maximal function.

Denote $c(j,k,j',k') = 2^{-j(m-1)-k} 2^{-|j-j'|} 2^{-|k-k'|} 2^{(m-1)\{(j\wedge j')(1-1/\delta)+j/\delta\}} 2^{(k\wedge k')(1-1/\delta)+k/\delta}$. Then

$$\left|g_\psi^d(T_2 f)(x)\right|^2 \lesssim \sum_{j',k'\in\mathbb{Z}} \left[\sum_{j,k\in\mathbb{Z}} c\big(j,k,j',k'\big) \left\{ M_s\left[\left(\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}} \left|\psi_{j,k} * h(c_R)\right|^2 \chi_R\right)^{\delta/2}\right]\right\}^{1/\delta}(x)\right]^2.$$

Note that $\sum_{j,k\in\mathbb{Z}} c(j,k,j',k') \lesssim 1$ and $\sum_{j',k'\in\mathbb{Z}} c(j,k,j',k') \lesssim 1$. As a consequence, by applying the Cauchy–Schwartz inequality, we get

$$\begin{aligned}
\left|g_\psi^d(T_2f)(x)\right|^2 &\lesssim \sum_{j',k'\in\mathbb{Z}}\left(\sum_{j,k\in\mathbb{Z}} c(j,k,j',k')\left\{M_s\left[\left(\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}}\left|\psi_{j,k}*h(c_R)\right|^2\chi_R\right)^{\delta/2}\right]\right\}^{2/\delta}(x)\right)\\
&\lesssim \sum_{j,k\in\mathbb{Z}}\left\{M_s\left[\left(\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}}\left|\psi_{j,k}*h(c_R)\right|^2\chi_R\right)^{\delta/2}\right]\right\}^{2/\delta}(x).
\end{aligned}$$

Now, applying the Fefferman–Stein vector-valued strong maximal inequality (see [4] and [12] for more details) on $L^{p/\delta}(\ell^{2/\delta})$ yields

$$\begin{aligned}
\left\|T_2(f)\right\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})} &= \left\|g_\psi^d(T_2f)\right\|_{L^p(\mathbb{R}^m)}\\
&\lesssim \left\|\left\{\sum_{j,k\in\mathbb{Z}}\left\{M_s\left[\left(\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}}\left|\psi_{j,k}*h(c_R)\right|^2\chi_R\right)^{\delta/2}\right]\right\}^{2/\delta}\right\}^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^m)}\\
&\lesssim \left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R=I\times J\in\mathcal{Q}_N^{j,k}}\left|\psi_{j,k}*h(c_R)\right|^2\chi_R(x)\right\}^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^m)}\\
&= \|h\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})} \lesssim \|f\|_{H^p(\mathbb{R}^{m-1}\times\mathbb{R})}.
\end{aligned}$$

Finally, by the dual argument, we get that T_2 is also bounded on the product BMO space $BMO(\mathbb{R}^{m-1}\times\mathbb{R})$. Here we omit the details. The proof of Theorem 1.4 is complete. □

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