Appl. Comput. Harmon. Anal. 41 (2016) 266–296

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

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Multirate systems with shortest spline-wavelet filters

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A R T I C L E I N F O

Article history:
Received 15 October 2014
Received in revised form 4 June 2015
Accepted 6 June 2015
Available online 18 June 2015
Communicated by Spec. Issue Guest Editor

MSC:
42C40
41A15
94A12
93C55
93C95
93C30

Keywords:
Multirate system
Filter bank
B-spline
Arbitrarily integer dilation
Shortest wavelet filters
Polyphase matrices
Matrix extension
Wavelet function spaces

A B S T R A C T

Motivated by the need of short FIR filters for perfect-reconstruction multirate systems, the main objective of this paper is to derive the shortest filters for such filter banks with M channels, for any integer $\mathsf{M} \geqslant 2$, based on the M-dilated refinement sequence p_m of the mth order cardinal B-spline. By imposing the additional constraint of ℓth order sum rule on the M-dual low-pass sequence a_m, the smoothness property of the M-dual scaling function, along with its corresponding analysis wavelets, is studied, and consequently yielding the ℓth order vanishing moment for each of the $\mathsf{M}-1$ synthesis (spline) wavelets. Several illustrative examples and tables of the filter systems are also included in this paper.

1. Introduction

Multirate digital signal processing already gained popularity some three decades ago, and has been commonly used for audio/video and adaptive signal processing, as well as in communication systems ever since. The main reason is that the multirate capability allows for the use of older and slower components,

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http://dx.doi.org/10.1016/j.acha.2015.06.002
1063-5203/

cost saving due to lower bitrates, and longer battery life for portable devices, just to name a few. Since the basic operation in multirate signal processing is to decompose a signal into any desirable number of sub-band components, it is implemented as a filter bank. More precisely, for any desired integer $\mathsf{M} \geqslant 2$, a given signal (that is, a bi-infinite bounded sequence of real numbers) $\mathsf{c} = \{c(k) : k \in \mathbb{Z}\}$ (for convenience, $\mathsf{c} = \{c(k)\}$) is processed by applying M FIR (finite-impulse-response) filters

$$\mathsf{a} = \{a(k)\},\ \mathsf{b}_1 = \{b_1(k)\},\ \ldots,\ \mathsf{b}_{\mathsf{M}-1} = \{b_{\mathsf{M}-1}(k)\},$$

followed by down-sampling by M for each of the M components with lower bitrate (or lower frequency), namely:

$$\begin{aligned} c_0(n) &= \sum_k a(n\mathsf{M} - k)c(k), \\ d_1(n) &= \sum_k b_1(n\mathsf{M} - k)c(k), \\ &\vdots \\ d_{\mathsf{M}-1}(n) &= \sum_k b_{\mathsf{M}-1}(n\mathsf{M} - k)c(k). \end{aligned}$$

This process is called *multirate decimation filtering*. To recover some signal $\mathsf{c}^* = \{c^*(k)\}$ with the original bitrate (or frequency), the M signal components

$$\mathsf{c}_0 := \{c_0(k)\},\ \mathsf{d}_1 := \{d_1(k)\},\ \ldots,\ \mathsf{d}_{\mathsf{M}-1} := \{d_{\mathsf{M}-1}(k)\}$$

are up-sampled by M, followed by filtering with some suitable corresponding FIR filters

$$\mathsf{p} = \{p(k)\},\ \mathsf{q}_1 = \{q_1(k)\},\ \ldots,\ \mathsf{q}_{\mathsf{M}-1} = \{q_{\mathsf{M}-1}(k)\},$$

namely:

$$c^*(s) = \sum_n \left[p(s - n\mathsf{M})c_0(n) + \sum_{\gamma=1}^{\mathsf{M}-1} q_\gamma(s - n\mathsf{M})d_\gamma(n) \right],\ s \in \mathbb{Z}.$$

This process is called *multirate interpolation filtering.*

For more details, the interested reader is referred to the comprehensive book [19] by P.P. Vaidyanathan, where the notion of perfect reconstruction (PR) is also introduced and studied. More precisely, a multirate system, as described above, is said to be a PR system, if when the consideration of output delay is ignored, the specification for the FIR decimation and interpolation filters are constructed to assure that the output c^* agrees with the input c. Since this PR requirement must be met for all bi-infinite bounded sequences c, we have the following mathematical formulation of a *PR multirate system* $(\{\mathsf{p}; \mathsf{q}_1, \ldots, \mathsf{q}_{\mathsf{M}-1}\}, \{\mathsf{a}; \mathsf{b}_1, \ldots, \mathsf{b}_{\mathsf{M}-1}\})$:

$$\sum_n \left[p(s - n\mathsf{M})a(n\mathsf{M} - k) + \sum_{\gamma=1}^{\mathsf{M}-1} q_\gamma(s - n\mathsf{M})b_\gamma(n\mathsf{M} - k) \right] = \delta(s - k),\ s, k \in \mathbb{Z}, \tag{1.1}$$

where $\{\delta(k)\}$ denotes the Kronecker delta sequence.

The birth and rapid advance of "wavelet analysis" in the late 1980s and early 1990s had impact in accelerating the popularity of multirate digital processing (for example, see [19] and the book [18] by B.W. Suter.) The reason is that the computation of the discrete wavelet transform (DWT) shares the same architecture as the basic core of a multirate system, though with emphasis on the decomposition into only $\mathsf{M} = 2$ bands (or channels). Unfortunately, since the objective of the study of multirate signal processing and that of DWT in wavelet analysis were quite different, the impact of the more recent mathematical development of wavelet analysis to the advancement of multirate signal processing has not been as significant as one would desire. In the first place, while multirate signal processing certainly benefits from PR filter bank algorithms, particularly with short filter taps, the research focus of wavelet analysis was mainly concerned with the properties of the wavelet and scaling functions. In addition, with the exception of a few publications such as [3,6,8,13,20] and the book [17] by Q. Sun, N. Bi, and D. Huang, most of other books and papers are concerned with dilation 2 (that is, with only $\mathsf{M} = 2$ bands or channels, when applied to multirate systems). Furthermore, to meet other desirable properties, such as orthogonality, order of smoothness, and symmetry of the wavelet and scaling functions, the decomposition filter lengths have to be significantly longer than desired for a multirate system. To further convey this point of view, let us consider spline-wavelets with dilation $\mathsf{M} = 2$ as an example. If (full) orthogonality is required, then the well-known Battle–Lemarie scaling functions and wavelets have infinite support, so that all of the four (decomposition and reconstruction) filters have infinite length. Giving up the orthogonality property of integer translates, while retaining orthogonality among the multi-level wavelet subspaces, allows us to construct compactly supported synthesis wavelets, but the analysis wavelets still have infinite support. These are the semi-orthogonal wavelets, as discussed in the book [2]. On the other hand, if only biorthogonality is desired, then both analysis and synthesis wavelets with compact supports have been constructed with arbitrarily pre-assigned order of smoothness in [7], although the filter lengths are significantly longer than desired for multirate signal processing. By ignoring the analysis wavelet and corresponding scaling functions, explicit formulations of the shortest decomposition filters were derived in [4] and studied in details in the book [5], where application to curve editing of spline-based subdivision schemes is discussed.

The core mathematical structure of wavelet analysis, particularly in the study of DWT, is the notion of multiresolution approximation/analysis (MRA). In this regard, among all compactly supported scaling functions, with dilation factor $\mathsf{M} = 2$, that possess the MRA property, only the cardinal B-splines have explicit expressions. In addition, since any (polynomial) spline space is a subspace of another spline space with finer knots by arbitrary knot insertion, a cardinal B-spline (of any desirable order) is a scaling function with arbitrary integer dilation $\mathsf{M} \geqslant 2$, and hence, its refinement sequence is a prime candidate for the design of multirate systems. Without loss of generality, we may consider cardinal splines defined on the integer knot sequence $\mathbb{Z}$. Such cardinal splines $f_0(x)$ can be written as B-spline series, namely:

$$f_0(x) = \sum_k c(k) N_m(x-k), \tag{1.2}$$

for some finite, infinite, or bi-infinite coefficient sequence $\mathsf{c} := \{c(k)\}$, where $N_m(x)$ denotes the cardinal B-spline of order $m \geqslant 1$, defined by

$$N_m(x) = \int_0^1 N_{m-1}(x-t)dt, \quad m = 2, 3, \dots,$$

with $N_1 = \chi_{[0,1)}$, the characteristic function of the interval $[0, 1)$. We remark that since $N_m(x)$ has compact support, (pointwise) convergence is not an issue at all, for any infinite or bi-infinite sequence c in (1.2).

In this paper, since one of our main objectives is to apply M-band MRA wavelet decomposition and reconstruction filters to PR multirate signal processing, we will consider spline coefficient sequences, such as c in (1.2), as signals; and therefore, c is always assumed to be a bounded bi-infinite sequence, with symbol (called z-transform), defined by $C(z) = \sum_k c(k)z^k$. Of course, this consideration does not exclude finite and (one-sided) infinite sequences, since they can be padded with zeros. One of the important properties of cardinal B-splines is that the B-spline series representation of $f_0(x)$ in (1.2), with bounded coefficient sequence $\mathsf{c} = \{c(k)\}$, is unique; that is, $c(k) = 0$ for all $k \in \mathbb{Z}$ if $f_0(x) = 0$ is the zero function (see [1, p. 133, Eq. (8)] and [5, p. 63, Theorem 2.5.1]). We remark that the uniqueness property remains valid for unbounded coefficient sequences $\mathsf{c} = \{c(k)\}$, since N_m has linear independent shifts (see [5, p. 56] for the definition of linear independent shifts; p. 62, Corollary 2.4.1; and p. 58, Lemma 2.4.2). Henceforth, we no longer restrict B-spline series to have bounded coefficient sequences (see Theorem 1 below). Of course, the most important property of the cardinal B-spline $N_m(x)$ is its refinability for any dilation integer factor $\mathsf{M} \geqslant 2$, with finite refinement sequence. In the following discussion, we will shift $N_m(x)$ to center at the origin, or close to the origin (for odd order m), by introducing the notation

$$\phi_m(x) = N_m(x + \lfloor m/2 \rfloor).$$

To write out the refinement sequence $\mathsf{p}_m := \{p_m(k)\}$ of $\phi_m(x)$ explicitly in the refinement relation:

$$\phi_m(x) = \sum_k p_m(k)\phi_m(\mathsf{M}x - k), \tag{1.3}$$

it is more convenient to consider its (normalized) symbol

$$P_m(z) = \frac{1}{\mathsf{M}} \sum_k p_m(k) z^k := z^{-(\mathsf{M}-1)\lfloor m/2 \rfloor} \left(\frac{1 + z + \cdots + z^{\mathsf{M}-1}}{\mathsf{M}} \right)^m, \tag{1.4}$$

where the normalization in (1.4) is simply division of the symbol of p_m by M. Observe from (1.4) that the support of the sequence p_m is given by

$$\operatorname{supp} \mathsf{p}_m := \big[-(\mathsf{M}-1)\lfloor m/2 \rfloor, (\mathsf{M}-1)\lfloor (m+1)/2 \rfloor \big] \cap \mathbb{Z}.$$

Hence, when considered as an FIR filter, p_m has $m(\mathsf{M}-1)$ filter taps (or the length of p_m, respectively P_m, is $m(\mathsf{M}-1)$).

For multirate interpolation filtering as discussed earlier, we need another $\mathsf{M}-1$ FIR filters

$$\mathsf{q}_{m,\gamma} := \{q_{m,\gamma}(k)\}, \quad \gamma = 1, \ldots, \mathsf{M}-1,$$

to go with the filter p_m. For these filters, we will also consider their corresponding (normalized) symbols:

$$Q_{m,\gamma}(z) = \frac{1}{\mathsf{M}} \sum_k q_{m,\gamma}(k) z^k, \quad \gamma = 1, \ldots, \mathsf{M}-1, \tag{1.5}$$

so that when the compactly supported cardinal spline $\phi_m(x)$ is considered as a scaling function with dilation factor M, the $\mathsf{M}-1$ FIR filters $\mathsf{q}_{m,\gamma}$ can be used to define the compactly supported functions

$$\psi_m^\gamma(x) = \sum_k q_{m,\gamma}(k)\phi_m(\mathsf{M}x - k), \quad \gamma = 1, \ldots, \mathsf{M}-1, \tag{1.6}$$

called (cardinal spline) wavelets. To assure that the set of M FIR filters

$$\mathsf{p}_m, \mathsf{q}_{m,1}, \ldots, \mathsf{q}_{m,\mathsf{M}-1}$$

constitutes the interpolation filter component of a PR multirate filter bank, the filters $\mathsf{q}_{m,1}, \ldots, \mathsf{q}_{m,\mathsf{M}-1}$ must be so constructed that the wavelets defined in (1.6) are synthesis wavelets in the sense that for any integer j, every cardinal spline

$$f_j(x) := \sum_k c_j(k)\phi_m(\mathsf{M}^j x - k)$$

of order m and with knot sequence $\mathsf{M}^{-j}\mathbb{Z}$ has a unique decomposition as the sum of M components:

$$f_j(x) = f_{j-1}(x) + g_{j-1}^1(x) + \cdots + g_{j-1}^{\mathsf{M}-1}(x), \tag{1.7}$$

where $f_{j-1}(x)$ is also a cardinal spline of the same order, but with the coarser knot sequence $\mathsf{M}^{-j+1}\mathbb{Z}$, and the detailed components $g_{j-1}^1(x), \ldots, g_{j-1}^{\mathsf{M}-1}(x)$ are wavelet series:

$$g_{j-1}^\gamma(x) = \sum_k d_{j-1,\gamma}(k)\psi_m^\gamma(\mathsf{M}^{j-1}x - k), \quad \gamma = 1, \ldots, \mathsf{M}-1,$$

with bounded coefficient sequences $\mathsf{d}_{j-1,\gamma} := \{d_{j-1,\gamma}(k)\}$, $\gamma = 1, \ldots, \mathsf{M}-1$.

Observe that by applying the refinement relation (1.3), the right-hand side of (1.7) can be written as:

$$\begin{aligned} & f_{j-1}(x) + g_{j-1}^1(x) + \cdots + g_{j-1}^{\mathsf{M}-1}(x) \\ & = \sum_k \left[\sum_s p_m(k - s\mathsf{M})c_{j-1}(s) + \sum_{\gamma=1}^{\mathsf{M}-1} \sum_s q_{m,\gamma}(k - s\mathsf{M})d_{j-1,\gamma}(s) \right] \phi_m(\mathsf{M}^j x - k). \end{aligned}$$

Hence, for the left-hand and right-hand sides of (1.7) to agree for all x, it follows from the uniqueness property of B-spline series representations that

$$c_j(k) = \sum_s \left[p_m(k - s\mathsf{M})c_{j-1}(s) + \sum_{\gamma=1}^{\mathsf{M}-1} q_{m,\gamma}(k - s\mathsf{M})d_{j-1,\gamma}(s) \right], \quad k \in \mathbb{Z}.$$

Of course, the FIR wavelet filters $\mathsf{q}_{m,1}, \ldots, \mathsf{q}_{m,\mathsf{M}-1}$ have yet to be constructed. Moreover, we need to construct certain decomposition FIR filters $\mathsf{a}, \mathsf{b}_1, \ldots, \mathsf{b}_{\mathsf{M}-1}$, corresponding to $\mathsf{p}_m, \mathsf{q}_{m,1}, \ldots, \mathsf{q}_{m,\mathsf{M}-1}$, such that the M coefficient sequences $\mathsf{c}_{j-1}, \mathsf{d}_{j-1,1}, \ldots, \mathsf{d}_{j-1,\mathsf{M}-1}$ of $f_{j-1}(x), g_{j-1}^1(x), \ldots, g_{j-1}^{\mathsf{M}-1}(x)$, respectively, can be derived by the multirate decimation filtering, namely:

$$\begin{aligned} c_{j-1}(n) &= \sum_k a(\mathsf{M}n - k)c_j(k), \\ d_{j-1,1}(n) &= \sum_k b_1(\mathsf{M}n - k)c_j(k), \\ &\vdots \\ d_{j-1,\mathsf{M}-1}(n) &= \sum_k b_{\mathsf{M}-1}(\mathsf{M}n - k)c_j(k). \end{aligned}$$

Then, according to the above discussion of PR multirate systems, the desired pair

$$(\{\mathsf{p}_m; \mathsf{q}_{m,1}, \ldots, \mathsf{q}_{m,\mathsf{M}-1}\}, \{\mathsf{a}; \mathsf{b}_1, \ldots, \mathsf{b}_{\mathsf{M}-1}\})$$

of filters, must be constructed according to the specification in (1.1).

To formulate the first specification item for meeting the perfect reconstruction requirement (1.1), we introduce the square matrix $R(z)$ defined by

$$R(z) = \begin{bmatrix} P_m(z) & P_m(\alpha z) & \cdots & P_m(\alpha^{\mathsf{M}-1}z) \\ Q_{m,1}(z) & Q_{m,1}(\alpha z) & \cdots & Q_{m,1}(\alpha^{\mathsf{M}-1}z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m,\mathsf{M}-1}(z) & Q_{m,\mathsf{M}-1}(\alpha z) & \cdots & Q_{m,\mathsf{M}-1}(\alpha^{\mathsf{M}-1}z) \end{bmatrix}, \tag{1.8}$$

which is solely dependent on the known normalized symbol $P_m(z)$ in (1.4) and the symbols $Q_{m,1}(z), \dots, Q_{m,\mathsf{M}-1}(z)$ in (1.5) of the wavelet filters to be constructed in this paper. Here and throughout, the notation

$$\alpha := e^{2\pi i/\mathsf{M}}, \tag{1.9}$$

for the Mth root of unity is used for convenience, and for any Laurent polynomial $T(z) = \sum_k t(k)z^k$, we adopt the polyphase notation

$$T(z) = \sum_{\gamma=0}^{\mathsf{M}-1} z^\gamma T^{[\gamma]}(z^{\mathsf{M}}), \text{ where } T^{[\gamma]}(z) := \sum_k t(\mathsf{M}k+\gamma)z^k, \gamma = 0, \dots \mathsf{M}-1. \tag{1.10}$$

It follows from (1.9) and (1.10) that

$$\sum_{k=0}^{\mathsf{M}-1} \alpha^{-\gamma k} T(\alpha^k z) = \mathsf{M} z^\gamma T^{[\gamma]}(z^{\mathsf{M}}), \qquad \gamma = 0, \dots, \mathsf{M}-1. \tag{1.11}$$

By extending the result in [2, Theorem 5.9], we have the following:

Theorem 1. *Let $R(z)$ be the square matrix defined in (1.8). Then a necessary and sufficient condition for the unique decomposition (1.7) of any spline series $f_j(x)$ is that the matrix $R(z)$ is non-singular on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ of the complex plane. Furthermore, under the non-singularity condition of $R(z)$, there exists a set of filters, $\mathsf{a}, \mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}$, such that the wavelet decomposition relation*

$$\phi_m(\mathsf{M}x-k) = \sum_n a(\mathsf{M}n-k)\phi_m(x-n) + \sum_{\gamma=1}^{\mathsf{M}-1}\sum_n b_\gamma(\mathsf{M}n-k)\psi_m^\gamma(x-n) \tag{1.12}$$

holds for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

In the wavelet literature (see, for example, the derivation of the special case $\mathsf{M} = 2$ on pages 143–144 in [2]), the wavelet decomposition relation (1.12) is obtained by applying the matrix identity:

$$S(z)R(z) = \mathsf{I}_{\mathsf{M}}, \qquad z \in \mathbb{T}, \tag{1.13}$$

where I_{M} denotes the identity matrix of dimension M and $S(z)$ is defined by

$$S(z) = \begin{bmatrix} A(z) & A(\alpha z) & \cdots & A(\alpha^{\mathsf{M}-1}z) \\ B_1(z) & B_1(\alpha z) & \cdots & B_1(\alpha^{\mathsf{M}-1}z) \\ \vdots & \vdots & \ddots & \vdots \\ B_{\mathsf{M}-1}(z) & B_{\mathsf{M}-1}(\alpha z) & \cdots & B_{\mathsf{M}-1}(\alpha^{\mathsf{M}-1}z) \end{bmatrix}^T,$$

by using the symbols $A(z), B_1(z), \dots, B_{\mathsf{M}-1}(z)$ of the sequences $\mathsf{a} = \{a(k)\}, \mathsf{b}_1 = \{b_1(k)\}, \dots, \mathsf{b}_{\mathsf{M}-1} = \{b_{\mathsf{M}-1}(k)\}$, respectively. Here, different from (1.4) and (1.5), these symbols are not normalized in the sense that $A(z) = \sum_k a(k)z^k$, $B_\gamma(z) = \sum_k b_\gamma(k)z^k$, $\gamma = 1, \dots, \mathsf{M}-1$.

Observe that being the left inverse of $R(z)$, the matrix $S(z)$ is also the right inverse of $R(z)$; that is, the matrix identity (1.13) is equivalent to the matrix identity:

$$R(z)S(z) = \mathsf{I}_\mathsf{M}, \quad z \in \mathbb{T}. \tag{1.14}$$

From (1.14), multiplication of the first row of $R(z)$ with the matrix $S(z)$ can be re-written as a set of M equations:

$$\sum_{k=0}^{\mathsf{M}-1} P_m(\alpha^k z)A(\alpha^k z) = 1; \tag{1.15}$$

and

$$\sum_{k=0}^{\mathsf{M}-1} P_m(\alpha^k z)B_\gamma(\alpha^k z) = 0, \quad \gamma = 1, \dots, \mathsf{M}-1. \tag{1.16}$$

The time-domain formulation of (1.15) states that the sequence a is an M-*dual* of the given sequence p_m, namely:

$$\sum_{k\in\mathbb{Z}} p_m(k)a(\mathsf{M}s - k) = \delta(s), \quad s \in \mathbb{Z}, \tag{1.17}$$

while the time-domain formulation of (1.16) states that the sequences $\mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}$ are M-*orthogonal* to the given sequence p_m, namely:

$$\sum_{k\in\mathbb{Z}} p_m(k)b_\gamma(\mathsf{M}s - k) = 0, \quad s \in \mathbb{Z},\ \gamma = 1, \dots, \mathsf{M}-1. \tag{1.18}$$

We remark that Theorem 1 is only concerned with the existence of the decomposition filters $\mathsf{a}, \mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}$, which could be infinite. The main objective of this paper is to construct finite (that is, FIR) filters $\mathsf{a}, \mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}$ with minimum lengths. More precisely, the FIR filter a is constructed according to the M-duality specification relative to the given B-spline refinement sequence p_m in (1.17); and the construction of the other FIR decomposition filters $\mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}$ meets the M-orthogonality requirement stated in (1.18). Of course, the FIR synthesis wavelet filters $\mathsf{q}_{m,1}, \dots, \mathsf{q}_{m,\mathsf{M}-1}$ are then uniquely determined by the decomposition filters $\mathsf{a}, \mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}$, according to the matrix identity (1.14). In addition, it follows from (1.14) that each of the synthesis wavelet filters $\mathsf{q}_{m,1}, \dots, \mathsf{q}_{m,\mathsf{M}-1}$ is M-orthogonal to a, while $\mathsf{q}_{m,\gamma}$ is an M-dual of b_γ for all $\gamma = 1, \dots, \mathsf{M}-1$, and q_{m,γ_1} is M-orthogonal to q_{m,γ_2}, whenever $\gamma_1 \neq \gamma_2$.

We remark that since all polynomials of degree $\leqslant m-1$ can be reproduced locally by some spline series $f_j(x)$, it can be shown by a standard argument that the M-orthogonality property of the decomposition filters $\mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}$ implies that each of these filters has (discrete) vanishing moments of order m. On the other hand, the other decomposition filter a (with minimum-length), as described above, is not necessarily a refinement sequence of some scaling function. Another objective of this paper is to construct the shortest finite filter a that satisfies the additional constraint of M-*sum rule property of order* $\ell \geqslant 1$, defined by

$\sum_s a(\mathsf{M}s+k) = \frac{1}{\mathsf{M}}$, $k = 0, \dots, \mathsf{M}-1$ and

$$\sum_s (\mathsf{M}s)^r a(\mathsf{M}s) = \sum_s (\mathsf{M}s+1)^r a(\mathsf{M}s+1) = \cdots = \sum_s (\mathsf{M}s+\mathsf{M}-1)^r a(\mathsf{M}s+\mathsf{M}-1),$$

for $r = 1, \dots, \ell-1$ (if $\ell \geqslant 2$). In terms of the symbol $A(z) = \sum_k a(k)z^k$, a has M-sum rule property of order ℓ if and only if $A(1) = 1$ and $(1+z+\cdots+z^{\mathsf{M}-1})^\ell \mid A(z)$. The importance of this additional constraint is two-fold: firstly, it assures that the filter a is the refinement sequence of some scaling function (with dilation factor M) with Sobolev smoothness dictated by the order ℓ of the sum-rule property; and secondly it also guarantees that each of the synthesis spline-wavelets $\psi_m^\gamma(x)$, introduced in (1.6), has (integral) *vanishing moments of order* ℓ; i.e., $\int_{\mathbb{R}} x^r \psi_m^\gamma(x)dx = 0$ for $r = 0, \dots, \ell-1$ and $\gamma = 1, \dots, \mathsf{M}-1$.

Since our construction will be in terms of (Laurent polynomial) symbols of the filters and since both matrices $R(z)$ and $S(z)$ are correlated, we will apply polyphase decomposition and representation in our derivations. The interested reader is referred to the book [19] for a detailed discussion of this de-correlation method. The paper is organized as follows. Construction of the shortest M-dual filter $\mathsf{a} = \mathsf{a}_m$, for any order $m \geqslant 2$, is studied in Section 2, with derivation of explicit formula for recursive computation. This discussion will include those filters a_m^ℓ that satisfy the M-sum rule property of order ℓ. In Section 3, the shortest decomposition filters $\mathsf{b}_1 = \mathsf{b}_{m,1}, \dots, \mathsf{b}_{\mathsf{M}-1} = \mathsf{b}_{m,\mathsf{M}-1}$ are formulated. Then the wavelet synthesis filters $\mathsf{q}_{m,1}, \dots, \mathsf{q}_{m,\mathsf{M}-1}$ are derived in terms of the decomposition filters, first without the sum-rule constraint, and finally with the constraint of order $\ell \geqslant 1$, for which their corresponding cardinal spline wavelets ψ_m^γ, $\gamma = 1, \dots, \mathsf{M}-1$ have vanishing moments also of order ℓ. Together with results in Section 2, the results in Section 3 provide the derivation of PR multirate systems $(\{\mathsf{p}; \mathsf{q}_1, \dots, \mathsf{q}_{\mathsf{M}-1}\}, \{\mathsf{a}; \mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}\})$ with shortest spline-wavelet filters whose symbols satisfy (1.14), which is summarized in Algorithm 1 at the end of Section 3. Section 4 is then devoted to the further discussion of biorthogonal wavelet bases in a pair $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ of Sobolev spaces associated with the pairs $(\{\mathsf{p}; \mathsf{q}_1, \dots, \mathsf{q}_{\mathsf{M}-1}\}, \{\mathsf{a}; \mathsf{b}_1, \dots, \mathsf{b}_{\mathsf{M}-1}\})$ of PR multirate systems obtained in Section 3. Several illustrative examples and tables of the filters derived in this paper are included in Section 5 and Appendix A.

2. Laurent polynomial symbol of the M-dual low-pass filter

Let $A(z) = A_m(z)$ be the Laurent polynomial symbol of the FIR filter $\mathsf{a} = \mathsf{a}_m$, which is M-dual to the given B-spline refinement sequence p_m; that is, $A_m(z) = \sum_k a_m(k)z^k$. For even $m = 2n$, an explicit formula of $A_m(z)$ is given in [3, Eq. (1.3)]; namely:

$$A_{2n}(z) = \sum_{j=0}^{n-1} \left(\sum_{j_1+\cdots+j_{\mathsf{M}-1}=j} \prod_{k=1}^{\mathsf{M}-1} \binom{n+j_k-1}{j_k} \sin^{-2j_k}\left(\frac{k\pi}{\mathsf{M}}\right) \right) \left(\frac{1}{2} - \frac{z+z^{-1}}{4}\right)^j, \tag{2.1}$$

which corresponds to the residual polynomial in the Deslauriers–Dubuc family of symmetric interpolatory masks (see, for example, [3]). Observe from (2.1) that $A_2(z) \equiv 1$ for any integer $\mathsf{M} \geq 2$. However, there is no published literature on the study of $A_m(z)$ for odd $m = 2n+1$ and $\mathsf{M} > 2$, although the special case, with dilation factor $\mathsf{M} = 2$ has been thoroughly discussed in the recent book [5, Chapter 6] (see also the earlier work [4]). For $\mathsf{M} = 3$, although a preliminary study for odd order B-spline a_m was carried out in [20], yet the filter a_m obtained in [20] is not the shortest. In this section, for any integer $m \geq 2$, we derive not only the general form of $A(z)$ from the identity (1.15), but also a recursive formula $A_{m+1}(z)$ from $A_m(z)$ for the shortest possible M-dual FIR filter a_m to p_m; thereby providing an alternative and efficient way of computing (2.1) as well as a new family $\{A_{2n+1}(z) : n \in \mathbb{N}\}$ of masks that fills the gap of $\{A_{2n}(z) : n \in \mathbb{N}\}$ for any integer dilation $\mathsf{M} \geqslant 2$.

We first analyze (1.15) for $A(z)$ with the shortest possible support. We shall write Π_k for the space of polynomials of degree $\leq k$, and with real coefficients. Let M and m denote integers, with $\mathsf{M} \geqslant 2$ and $m \geqslant 2$.

It is easily shown that if H is a polynomial satisfying the identity

$$\sum_{k=0}^{\mathsf{M}-1} \alpha^k \left(\frac{1+\alpha^k z + \cdots + (\alpha^k z)^{\mathsf{M}-1}}{\mathsf{M}} \right)^m H(\alpha^k z) = z^{\mathsf{M}\lfloor m/2 \rfloor - 1}, \tag{2.2}$$

then the Laurent polynomial $A(z) := z^{-\lfloor m/2 \rfloor+1} H(z)$ satisfies the identity (1.15). By applying the Leibniz rule for the differentiation of a product to (2.2), we deduce that a necessary condition for a polynomial H to satisfy the identity (2.2) is that

$$H^{(s)}(1) = \beta_m(s), \qquad s = 0, \ldots, m-1, \tag{2.3}$$

where $\{\beta_m(s) : s = 0, \ldots, m-1\}$ is the (unique) solution in $\mathbb{R}^m$ of the $m \times m$ lower-triangular linear system

$$\sum_{j=0}^{s} \binom{s}{j} \left[\left(\frac{d}{dz} \right)^{s-j} \left(\frac{1+z+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}} \right)^m \right] \Bigg|_{z=1} \beta_m(j) = \binom{\mathsf{M}\lfloor \frac{m}{2} \rfloor - 1}{s} s! \tag{2.4}$$

for $s = 0, \ldots, m-1$. Let the polynomial $H_m \in \Pi_{m-1}$ be defined by

$$H_m(z) := \sum_{j=0}^{m-1} \frac{\beta_m(j)}{j!} (z-1)^j. \tag{2.5}$$

It then follows from (2.4) and $\left(1 + \alpha^j + \cdots + (\alpha^j)^{\mathsf{M}-1}\right)/\mathsf{M} = \delta(j)$, that the polynomial

$$G(z) := \sum_{k=0}^{\mathsf{M}-1} \alpha^k \left(\frac{1+\alpha^k z + \cdots + (\alpha^k z)^{\mathsf{M}-1}}{\mathsf{M}} \right)^m H_m(\alpha^k z) - z^{\mathsf{M}\lfloor m/2 \rfloor - 1} \tag{2.6}$$

satisfies $G^{(s)}(\alpha^k) = 0$, $s = 0, \ldots, m-1; k = 0, \ldots, \mathsf{M}-1$, and thus

$$G(z) = (1-z)^m (1+z+\cdots+z^{\mathsf{M}-1})^m K(z),$$

for some polynomial K. Since $H_m \in \Pi_{m-1}$, we note from (2.6) that $G \in \Pi_{\mathsf{M}m-1}$. Hence K, and therefore also G, must be the zero polynomial, which, together with (2.6), shows that H_m is a polynomial solution of the identity (2.2). Moreover, by recalling also that (2.3) is a necessary condition on a polynomial solution H of (2.2), according to which any polynomial solution $H \in \Pi_{m-1}$ of (2.2) must be given by the right-hand-side of (2.5), we deduce that $H = H_m$ is the only solution in Π_{m-1} of (2.2).

We proceed to show that the polynomial H_m satisfies the property $H_m \in \Pi_{m-2}$. To this end, let cz^{m-1} denote the leading term of $H_m(z)$ with respect to its representation in Π_{m-1}, from which we deduce that the leading term of the polynomial in the left hand side of (2.2) is given by $\mathsf{M}^{-m+1} c z^{\mathsf{M}m-1}$. It then follows from (2.2) that we must have $c = 0$, and thus $H_m \in \Pi_{m-2}$ is satisfied.

Next, we prove that H_{2n} is a symmetric polynomial, in the sense that

$$z^{2n-2} H_{2n}(z^{-1}) = H_{2n}(z).$$

We first replace z by z^{-1} in (2.2) to obtain, for $m = 2n$, and with $H = H_{2n}$,

$$\sum_{k=0}^{\mathsf{M}-1} \alpha^k \left(\frac{1+\alpha^k z^{-1} + \cdots + (\alpha^k z^{-1})^{\mathsf{M}-1}}{\mathsf{M}} \right)^{2n} H_{2n}(\alpha^k z^{-1}) = z^{-\mathsf{M}n+1}. \tag{2.7}$$

Multiplying both sides of (2.7) by $z^{2\mathsf{M}n-2}$, and using (1.9), we obtain

$$
\begin{aligned}
z^{\mathsf{M}n-1} &= z^{2\mathsf{M}n-2}\sum_{k=0}^{\mathsf{M}-1}\alpha^k\left(\frac{1+\alpha^k z^{-1}+\cdots+(\alpha^k z^{-1})^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n} H_{2n}(\alpha^k z^{-1})\\
&= z^{2\mathsf{M}n-2}\sum_{k=1}^{\mathsf{M}}\alpha^{\mathsf{M}-k}\left(\frac{1+\alpha^{\mathsf{M}-k} z^{-1}+\cdots+(\alpha^{\mathsf{M}-k} z^{-1})^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n} H_{2n}(\alpha^{\mathsf{M}-k} z^{-1})\\
&= z^{2\mathsf{M}n-2}\sum_{k=1}^{\mathsf{M}}\alpha^{\mathsf{M}-k}\left(\frac{1+\alpha^k z+\cdots+(\alpha^k z)^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n}\cdot(\alpha^k z)^{-(\mathsf{M}-1)2n}\cdot H_{2n}((\alpha^k z)^{-1})\\
&= \sum_{k=1}^{\mathsf{M}}\alpha^{k}\left(\frac{1+\alpha^k z+\cdots+(\alpha^k z)^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n}\cdot(\alpha^k z)^{2n-2}\cdot H_{2n}((\alpha^k z)^{-1})\\
&= \sum_{k=0}^{\mathsf{M}-1}\alpha^{k}\left(\frac{1+\alpha^k z+\cdots+(\alpha^k z)^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n}\tilde{H}_{2n}(\alpha^k z),
\end{aligned}
$$

where $\tilde{H}_{2n}(z) := z^{2n-2}H_{2n}(z^{-1})$. Hence, $\tilde{H}_{2n}\in\Pi_{2n-2}$. By recalling also that H_{2n} is the unique solution in Π_{2n-1} of the polynomial identity (2.2) for $m = 2n$, we deduce that $\tilde{H}_{2n} = H_{2n}$, which yields the desired symmetry result.

By recalling also $A(z) = z^{-\lfloor m/2\rfloor+1}H(z)$, we have therefore now established the following result.

Theorem 2. *Let $\mathsf{M}, m \geqslant 2$ be any integers. Then the following statements hold.*

(a) *There exists precisely one polynomial $H = H_m \in \Pi_{m-2}$ satisfying the identity (2.2), where H_m is given by (2.5) in terms of the unique solution $\{\beta(j) : j = 0,\ldots,m-1\}$ of the $m\times m$ lower-triangular linear system (2.4).*
(b) *The Laurent polynomial $A = A_m$, as defined by*

$$A_m(z) := z^{-\lfloor m/2\rfloor+1}H_m(z), \tag{2.8}$$

is a Laurent polynomial solution of the identity (1.15) with the shortest possible length, and satisfies, for $m = 2n$, the symmetry condition $A_{2n}(z^{-1}) = A_{2n}(z)$.

For the case $m = 2n$, we recall the explicit formulation (2.1) of the Laurent polynomial A_{2n}. The following result provides an alternative and efficient way of computing A_{2n} as well as the family $\{A_{2n+1} : n\in\mathbb{N}\}$ of masks in terms of a recursive formula that fills the gap of the family $\{A_{2n} : n \in \mathbb{N}\}$ of masks, which are M-dual to p_m with the shortest possible length.

Theorem 3. *In Theorem 2, A_m with the shortest possible length that satisfies (1.15) is given by the following recursive formulas*

$$
\begin{cases}
A_2 = 1;\\
A_{2n+1} = \dfrac{A_{2n}(z) + z^{-n+1}(1-z)^{2n}W_{2n}(z)}{(1+z+\cdots+z^{\mathsf{M}-1})/\mathsf{M}};\\
A_{2n+2}(z) = \dfrac{z^{-1}A_{2n+1}(z) + z^{-n}(1-z)^{2n+1}W_{2n+1}(z)}{(1+z+\cdots+z^{\mathsf{M}-1})/\mathsf{M}},
\end{cases} \tag{2.9}
$$

where $n\in\mathbb{N}$ and $W_m \in \Pi_{\mathsf{M}-2}$ is uniquely determined by

$$W_m(\alpha^k) = -(1-\alpha^k)^{-m}(\alpha^k)^{\lfloor m/2\rfloor-1}A_m(\alpha^k),\ k = 1,\ldots,\mathsf{M}-1.$$

Proof. By successively setting $H = H_m$, $H = H_{m+1}$, in (2.2), we obtain

$$\sum_{k=0}^{\mathsf{M}-1} \alpha^k \left(\frac{1 + \alpha^k z + \cdots + (\alpha^k z)^{\mathsf{M}-1}}{\mathsf{M}} \right)^m D_m(\alpha^k z) = 0,$$

where

$$D_m(z) := \left(\frac{1 + z + \cdots + z^{\mathsf{M}-1}}{\mathsf{M}} \right) H_{m+1}(z) - H_m(z).$$

Analogously to the argument in (2.2) and (2.3), we deduce that

$$D_m^{(s)}(1) = 0, \quad s = 0, \ldots, m-1,$$

and thus

$$D_m(z) = (1-z)^m W_m(z),$$

for some polynomial W_m. Together with $D_m \in \Pi_{m+\mathsf{M}-2}$, we deduce that $W_m \in \Pi_{\mathsf{M}-2}$. Consequently, we have

$$\left(\frac{1 + z + \cdots + z^{\mathsf{M}-1}}{\mathsf{M}} \right) H_{m+1}(z) = H_m(z) + (1-z)^m W_m(z),$$

in which, we may set $z = \alpha^k$, $k = 1, \ldots, \mathsf{M}-1$, to obtain,

$$W_m(\alpha^k) = -(1-\alpha^k)^{-m} H_m(\alpha^k), \quad k = 1, \ldots, \mathsf{M}-1,$$

which uniquely determines the polynomial $W_m \in \Pi_{\mathsf{M}-2}$. Therefore,

$$H_{m+1}(z) = \frac{H_m(z) + (1-z)^m W_m(z)}{(1 + z + \cdots + z^{\mathsf{M}-1})/\mathsf{M}}.$$

The conclusions of the theorem then follows from (2.8). We are done. □

If one only concerns about the efficient derivation of A_{2n+1}, we further provide the following formulation of A_{2n+1} in terms of A_{2n}, which, apart from polynomial multiplication and division, requires merely the calculation of a polynomial interpolant in $\Pi_{(\mathsf{M}-2)/2}$ if M is even, and in $\Pi_{(\mathsf{M}-3)/2}$ if M is odd.

Corollary 1. *Let A_{2n} be given as in (2.1). In Theorem 2, for any $n \in \mathbb{N}$, it holds that*

$$A_{2n+1}(z) = \frac{A_{2n}(z) + z^{\lfloor \mathsf{M}/2 \rfloor - n}(1-z)^{2n} U_n(\frac{1}{2} - \frac{z+z^{-1}}{4})}{(1 + z + \cdots + z^{\mathsf{M}-1})/\mathsf{M}}, \tag{2.10}$$

where $U_n \in \Pi_{\lfloor \mathsf{M}/2 \rfloor - 1}$ is defined, with the notation $\zeta_k := \sin^2\left(\frac{\pi k}{\mathsf{M}}\right)$ and

$$\eta_{n,s} := \frac{(-1)^{n+s+1}}{2^{2n}} \sum_{j=0}^{n-1} \left[\sum_{j_1 + \cdots + j_{\mathsf{M}-1} = j} \prod_{k=1}^{\mathsf{M}-1} \binom{n + j_k - 1}{j_k} \zeta_k^{-j_k} \right] \zeta_s^{j-n}, \quad s \in \mathbb{N}, \tag{2.11}$$

by:

(i) *if* M *is even, then* U_n *is the polynomial in* $\Pi_{(\mathsf{M}-2)/2}$ *determined by the interpolation conditions*

$$U_n(\zeta_s) = \eta_{n,s}, \quad s = 1, \dots, \mathsf{M}/2; \tag{2.12}$$

(ii) *if* M *is odd, then*

$$U_n(z) = (1+z)V_n(z), \tag{2.13}$$

where V_n *is the polynomial in* $\Pi_{(\mathsf{M}-3)/2}$ *determined by the interpolation conditions*

$$V_n(\zeta_s) = \frac{\eta_{n,s}}{2\sqrt{1-\zeta_s}}, \quad s = 1, \dots, (\mathsf{M}-1)/2. \tag{2.14}$$

Proof. By (2.9) in Theorem 3, we have

$$\left(\frac{1+z+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}}\right) A_{2n+1}(z) = A_{2n}(z) + z^{-n+1}(1-z)^{2n}W_{2n}(z), \tag{2.15}$$

with $W_{2n} \in \Pi_{\mathsf{M}-2}$ uniquely determined by

$$W_{2n}(\alpha^j) = -(\alpha^j)^{n-1}(1-\alpha^j)^{-2n}A_{2n}(\alpha^j), \quad j = 1, \dots, \mathsf{M}-1,$$

which together with the symmetry of A_{2n} implies

$$W_{2n}(\alpha^{-j}) = \overline{W_{2n}(\alpha^j)} = (\alpha^j)^{2-\mathsf{M}}W_{2n}(\alpha^j),$$

and thus

$$(\alpha^j)^{\mathsf{M}-2}W_{2n}(\alpha^{-j}) = W_{2n}(\alpha^j), \quad j = 1, \dots, \mathsf{M}-1.$$

Noting that $W_{2n} \in \Pi_{\mathsf{M}-2}$, the definition $J(z) := z^{\mathsf{M}-2}W_{2n}(z^{-1}) - W_{2n}(z)$ implies $J \in \Pi_{\mathsf{M}-2}$ and $J(\alpha^j) = 0$, $j = 1, \dots, \mathsf{M}-1$, and thus, since (1.9) implies that $\{\alpha^j : \ j = 1, \dots, \mathsf{M}-1\}$ are $\mathsf{M}-1$ distinct points in the complex plane $\mathbb{C}$, it follows that J must be the zero polynomial, or equivalently, the polynomial W_{2n} satisfies the symmetry condition $z^{\mathsf{M}-2}W_{2n}(z^{-1}) = W_{2n}(z)$.

For odd M, we may set $z = -1$ to obtain $W_{2n}(-1) = 0$. It follows that $W_{2n}(z) = (1+z)F_n(z)$ for some $F \in \Pi_{\mathsf{M}-3}$. Also, we have $z^{\mathsf{M}-3}F_n(z^{-1}) = F_n(z)$. Let the Laurent polynomial G_n be defined by

$$G_n(z) := \begin{cases} z^{-(\mathsf{M}-2)/2}W_{2n}(z), & \text{if } \mathsf{M} \text{ is even;} \\ z^{-(\mathsf{M}-3)/2}F_n(z), & \text{if } \mathsf{M} \text{ is odd.} \end{cases} \tag{2.16}$$

By the symmetry property of W_{2n}, we have $G_n(z^{-1}) = G_n(z)$, and thus, there exist polynomials $U_n \in \Pi_{(\mathsf{M}-2)/2}$ and $V_n \in \Pi_{\mathsf{M}-3)/2}$ such that

$$G_n(z) = \begin{cases} U_n(\zeta), & \text{if} \quad \mathsf{M} \quad \text{is even;} \\ V_n(\zeta), & \text{if} \quad \mathsf{M} \quad \text{is odd,} \end{cases} \tag{2.17}$$

where $\zeta := \frac{1}{2} - \frac{z+z^{-1}}{4}$. Observe that $\zeta\big|_{z=\alpha^j} = \zeta_j, j \in \mathbb{N}$. Also, note from (2.1) that $A_{2n}(\alpha^j) = \sum_{k=0}^{n-1}\left[\sum_{k_1+\cdots+k_{\mathsf{M}-1}=k}\prod_{\ell=1}^{\mathsf{M}-1}\binom{n+k_\ell-1}{k_\ell}\zeta_\ell^{-k_\ell}\right]\zeta_j^k, j \in \mathbb{N}$, and by the definition of ζ we deduce that $(1-z)^{2n} = z^n(-1)^n 2^{2n}\zeta^n$.

If M is even, then by the above relations and $(\alpha^j)^{\mathsf{M}/2} = (-1)^j, j \in \mathbb{N}$, it follows that the polynomial U_n satisfies the interpolation conditions (2.12). The identity (2.10) is an immediate consequence of (2.15), (2.16), and (2.17).

If M is odd, then by the above relations and $\frac{(\alpha^j)^{\frac{1}{2}}}{1+\alpha^j} = \frac{1}{2\cos(\frac{\pi j}{\mathsf{M}})} = \frac{1}{2\sqrt{1-\zeta_j}}$, $j = 1, \dots, (\mathsf{M}-1)/2$, we deduce that the polynomial V_n satisfies the interpolation conditions (2.14). The identity (2.10) is obtained by combining (2.15), (2.16), and (2.17). □

We proceed to calculate the following examples for $\mathsf{M} = 2, 3$ and 4 by means of (2.1) and Corollary 1, which can be also directly verified by using (2.9) as in Theorem 3.

Examples. (i) *The case* $\mathsf{M} = 2$. According to (2.1), we have here the formula

$$A_{2n}(z) = \sum_{j=0}^{n-1} \binom{n+j-1}{j} \left(\frac{1}{2} - \frac{z+z^{-1}}{4}\right)^j. \tag{2.18}$$

Also, in Theorem 2.2, U_n is the constant polynomial $U_n(z) = \eta_{n,1}$, where, from (2.11) and (2.12),

$$\eta_{n,1} = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n-1} \binom{n+k-1}{k} = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n-1} \left[\binom{n+k}{k} - \binom{n+k-1}{k-1}\right] = \frac{(-1)^n}{2^{2n}} \binom{2n-1}{n-1},$$

and thus, from (2.10),

$$A_{2n+1}(z) = \frac{A_{2n}(z) + \frac{(-1)^n}{2^{2n}}\binom{2n-1}{n-1} z^{1-n}(1-z)^{2n}}{(1+z)/2}. \tag{2.19}$$

Calculating by means of (2.18) and (2.19), we obtain Table 5.1 (see Appendix A), which agrees with the corresponding tabled values in [6, Section 9.4].

(ii) *The case* $\mathsf{M} = 3$. According to Corollary 1, we have here that V_n is a constant polynomial, that is, $V_n(z) = \eta_{n,1}$, where $\zeta_1 = \zeta_2 = \frac{2}{3}$, we may apply (2.11) and (2.1) to deduce that $\eta_{n,1} = \frac{(-1)^n}{3^n} A_{2n}(\alpha)$. Now use the polyphase notation (1.10), together with (1.9), to obtain

$$A_{2n}(\alpha) = A_{2n}^{[0]}(1) + \alpha A_{2n}^{[1]}(1) + \alpha^2 A_{2n}^{[2]}(1),$$

and thus

$$\overline{A_{2n}(\alpha)} = A_{2n}^{[0]}(1) + \alpha^2 A_{2n}^{[1]}(1) + \alpha A_{2n}^{[2]}(1).$$

By the symmetry property $A_{2n}(z^{-1}) = A_{2n}(z)$, we have $A_{2n}(\alpha) \in \mathbb{R}$, according to which $\overline{A_{2n}(\alpha)} = A_{2n}(\alpha)$, so that we may add the above two equations for $A_{2n}(\alpha)$ and $\overline{A_{2n}(\alpha)}$ to obtain

$$A_{2n}(\alpha) = A_{2n}^{[0]}(1) - \frac{1}{2}\left[A_{2n}^{[1]}(1) + A_{2n}^{[2]}(1)\right].$$

But

$$1 = A_{2n}(1) = A_{2n}^{[0]}(1) + A_{2n}^{[1]}(1) + A_{2n}^{[2]}(1),$$

as follows from (1.15) and (1.4). Hence, $\eta_{n,1} = \frac{(-1)^n}{3^n}\,\frac{1}{2}\left[3A_{2n}^{[0]}(1) - 1\right]$, and thus, from (2.10) and (2.13), we deduce

$$A_{2n+1}(z) = \frac{A_{2n}(z) + \frac{(-1)^n}{3^n}\frac{1}{2}z^{1-n}\left[3A_{2n}^{[0]}(1) - 1\right](1-z)^{2n}(1+z)}{(1+z+z^2)/3}. \tag{2.20}$$

Calculating by means of (2.1) and (2.20), we obtain Table 5.2 (see Appendix A).

(iii) *The case* $\mathsf{M} = 4$. By using (2.11), (2.12) in Corollary 1, we obtain the formulas

$$U_1(\zeta) = \frac{1}{4}(-5+6\zeta), \quad U_2(\zeta) = \frac{1}{16}(59-70\zeta), \quad U_3(\zeta) = -\frac{1}{32}(391-462\zeta),$$

and thus, from (2.10),

$$A_3(z) = \frac{A_2(z) + \frac{1}{4}z(1-z)^2\left[-5+6(\frac{1}{2} - \frac{z+z^{-1}}{4})\right]}{(1+z+z^2+z^3)/4},$$

$$A_5(z) = \frac{A_4(z) + \frac{1}{16}(1-z)^4\left[59-70(\frac{1}{2} - \frac{z+z^{-1}}{4})\right]}{(1+z+z^2+z^3)/4},$$

$$A_7(z) = \frac{A_6(z) + \frac{1}{64}z^{-1}(1-z)^6\left[-391+462(\frac{1}{2} - \frac{z+z^{-1}}{4})\right]}{(1+z+z^2+z^3)/4}. \tag{2.21}$$

Calculating by means of (2.1) and (2.21), we obtain Table 5.3 (see Appendix A). ■

The above results provide recursive formulas for efficient computation of a_m that is M-dual to p_m with the shortest possible length. However, a_m does not have any order of M-sum rule property. Our following result, which is a direct consequence of Theorem 2, provides a family of masks that are M-dual to p_m with arbitrary preassigned M-sum rule order ℓ. Together with (1.4), such a family of masks will be required in Section 3.

Corollary 2. *For any integers* $\mathsf{M} \ge 2$, $m \ge 2$ *and* $\ell \in \mathbb{N}$, *the Laurent polynomial* $A = A_m^\ell$ *defined by*

$$A_m^\ell(z) := z^{-(\mathsf{M}-1)(\lfloor (m+\ell)/2 \rfloor - \lfloor m/2 \rfloor)}\left(\frac{1+z+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}}\right)^\ell A_{m+\ell}(z), \tag{2.22}$$

with $A_{m+\ell}$ *obtained as in* (2.9), *satisfies the identity* (1.15), *and the corresponding low-pass filter sequence* $\mathsf{a}_m^\ell := \{a_m^\ell(k) : k \in \mathbb{Z}\}$, *as defined by* $\sum_j a_m^\ell(k)z^k := A_m^\ell(z)$, *satisfies the* M*-sum rule property of order* ℓ.

Proof. It is easily seen that $P_m(z)A_m^\ell(z) = P_{m+\ell}(z)A_{m+\ell}(z)$. The conclusions then follow from Theorem 2 and the definition of M-sum rule property. □

Finally in this section, we derive a formulation for the general Laurent polynomial solution A of the identity (1.15). Our first step is to establish the following result concerning the general form of filters that are M-orthogonal to p_m, as will be required also in Section 3.

Lemma 1. *For integers* $\mathsf{M} \ge 2$ *and* $m \ge 2$, *let* F *denote any Laurent polynomial solution of the identity*

$$\sum_{k=0}^{\mathsf{M}-1} P_m(\alpha^k z)F(\alpha^k z) = 0. \tag{2.23}$$

Then F *satisfies the formulation*

$$F(z) = z^{-\lfloor m/2 \rfloor}(1-z)^m G(z), \tag{2.24}$$

with G denoting any Laurent polynomial satisfying the identity

$$G^{[0]}(z^{\mathsf{M}}) = \frac{1}{\mathsf{M}} \sum_{k=0}^{\mathsf{M}-1} G(\alpha^k z) = 0. \tag{2.25}$$

Proof. According to (2.23) and (1.4),

$$F^{(s)}(1) = 0, \quad s = 0, \ldots, m-1,$$

and it follows that (2.24) is satisfied for some Laurent polynomial G. By substituting (2.24) into (2.23), and using (1.4), we obtain the second equation in (2.25). Finally, note that the first equation in (2.25) is obtained by setting $\gamma = 0$ in (1.11). □

The following result is then an immediate consequence of Lemma 1.

Theorem 4. *The general Laurent polynomial solution A of the identity* (1.15) *is given by*

$$A(z) = A^*(z) + z^{-\lfloor m/2 \rfloor}(1-z)^m \tilde{A}(z),$$

where $A = A^*$ is any particular solution of (1.15)*, and with $\tilde{A}$ denoting any Laurent polynomial satisfying*

$$\tilde{A}^{[0]}(z) = \frac{1}{\mathsf{M}} \sum_{k=0}^{\mathsf{M}-1} \tilde{A}(\alpha^k z) = 0.$$

Observe from Theorem 2 and Corollary 2 that $A^* = A_m$ in (2.9) and $A^* = A_m^{\ell}$ in (2.22) are both admissible choices in Theorem 4.

3. The Laurent polynomial symbols of wavelet high-pass filters

For any given Laurent polynomial A satisfying the identity (1.15), we proceed in this section to obtain Laurent polynomials $Q_1, \ldots, Q_{\mathsf{M}-1}$ and $B_1, \ldots, B_{\mathsf{M}-1}$ satisfying (1.13), or equivalently, (1.14). In the synthesis side, $P = P_m$ is with respect to the B-spline low-pass filter p_m, and $Q_1, \ldots, Q_{\mathsf{M}-1}$ are corresponding to the spline-wavelet high-pass filters $\mathsf{q}_1, \ldots, \mathsf{q}_{\mathsf{M}-1}$, which are all M-orthogonal to the analysis low-pass filter a, while $B_1, \ldots, B_{\mathsf{M}-1}$ are corresponding to the analysis wavelet high-pass filters $\mathsf{b}_1, \ldots, \mathsf{b}_{\mathsf{M}-1}$. The pair $(\{\mathsf{p}; \mathsf{q}_1, \ldots, \mathsf{q}_{\mathsf{M}-1}\}, \{\mathsf{a}; \mathsf{b}_1, \ldots, \mathsf{b}_{\mathsf{M}-1})$ forms a PR multirate system, which shall be further studied in next section in connection with the biorthogonal wavelet bases in a pair of Sobolev spaces. In applications, vanishing moments and short support of wavelet filters play an important role in signal/image processing since these properties provide efficient and sparse representation of the signals/images. In this section, we shall show that once the M-dual pair $(\mathsf{p}_m, \mathsf{a})$ is given, the high-pass filters $\mathsf{b}_1, \ldots, \mathsf{b}_{\mathsf{M}-1}$ with shortest lengths can be derived, and it follows that the spline-wavelet filter $\mathsf{q}_1, \ldots, \mathsf{q}_{\mathsf{M}-1}$ are then uniquely determined; moreover, similar to Section 2, recursive formula of $Q_1, \ldots, Q_{\mathsf{M}-1}$ can be derived as well.

We first simplify the matrix equation in (1.14) in terms of polyphase notation. It follows from (1.16), together with Lemma 1, that we must have

$$B_\gamma(z) = z^{-\lfloor m/2 \rfloor}(1-z)^m C_\gamma(z), \quad \gamma = 1, \ldots, \mathsf{M}-1, \tag{3.1}$$

for some Laurent polynomials $C_1, \ldots, C_{\mathsf{M}-1}$ such that

$$\frac{1}{\mathsf{M}}\sum_{k=0}^{\mathsf{M}-1} C_\gamma(\alpha^k z) = C_\gamma^{[0]}(z^{\mathsf{M}}) = 0, \quad \gamma = 1,\dots,\mathsf{M}-1. \tag{3.2}$$

Let the Laurent polynomial A be any solution of the identity (1.15). It follows from (3.1) that the Laurent polynomials $Q_1,\dots,Q_{\mathsf{M}-1}$ and $B_1,\dots,B_{\mathsf{M}-1}$ satisfy the identities (1.14) if and only if the Laurent polynomials $Q_1,\dots,Q_{\mathsf{M}-1}$ and $C_1,\dots,C_{\mathsf{M}-1}$ satisfy the identity

$$\begin{bmatrix} C_1(z) & C_2(z) & \cdots & C_{\mathsf{M}-1}(z) \\ C_1(\alpha z) & C_2(\alpha z) & \cdots & C_{\mathsf{M}-1}(\alpha z) \\ \vdots & \vdots & \ddots & \vdots \\ C_1(\alpha^{\mathsf{M}-1}z) & C_2(\alpha^{\mathsf{M}-1}z) & \cdots & C_{\mathsf{M}-1}(\alpha^{\mathsf{M}-1}z) \end{bmatrix} \begin{bmatrix} Q_1(z) \\ Q_2(z) \\ \vdots \\ Q_{\mathsf{M}-1}(z) \end{bmatrix} = \begin{bmatrix} \dfrac{1-P_m(z)A(z)}{z^{-\lfloor m/2\rfloor}(1-z)^m} \\ \dfrac{-P_m(z)A(\alpha z)}{(\alpha z)^{-\lfloor m/2\rfloor}(1-\alpha z)^m} \\ \vdots \\ \dfrac{-P_m(z)A(\alpha^{\mathsf{M}-1}z)}{(\alpha^{\mathsf{M}-1}z)^{-\lfloor m/2\rfloor}(1-\alpha^{\mathsf{M}-1}z)^m} \end{bmatrix}, \tag{3.3}$$

as well as the condition (3.2), and where the Laurent polynomials $B_1,\dots,B_{\mathsf{M}-1}$ are then given by (3.1). By multiplying the left hand side of (3.3) from the left by the matrix

$$\mathcal{T}(z) := \frac{1}{\mathsf{M}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z^{-1} & (\alpha z)^{-1} & \cdots & (\alpha^{\mathsf{M}-1}z)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ z^{-(\mathsf{M}-1)} & (\alpha z)^{-(\mathsf{M}-1)} & \cdots & (\alpha^{\mathsf{M}-1}z)^{-(\mathsf{M}-1)} \end{bmatrix}, \tag{3.4}$$

and using (1.10), (1.11), as well as (3.2), we obtain

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_1^{[1]}(z^{\mathsf{M}}) & C_2^{[1]}(z^{\mathsf{M}}) & \cdots & C_{\mathsf{M}-1}^{[1]}(z^{\mathsf{M}}) \\ \vdots & \vdots & \ddots & \vdots \\ C_1^{[\mathsf{M}-1]}(z^{\mathsf{M}}) & C_2^{[\mathsf{M}-1]}(z^{\mathsf{M}}) & \cdots & C_{\mathsf{M}-1}^{[\mathsf{M}-1]}(z^{\mathsf{M}}) \end{bmatrix} \begin{bmatrix} Q_1(z) \\ Q_2(z) \\ \vdots \\ Q_{\mathsf{M}-1}(z) \end{bmatrix} = \begin{bmatrix} T_0(z) \\ T_1(z) \\ \vdots \\ T_{\mathsf{M}-1}(z) \end{bmatrix}. \tag{3.5}$$

Here the vector

$$[T_0(z) \quad T_1(z) \quad \cdots \quad T_{\mathsf{M}-1}(z)]^T \tag{3.6}$$

is obtained by the multiplication from the left of the right hand side of (3.3) with the matrix $\mathcal{T}(z)$. Since (1.4) implies

$$P_m(\alpha^k z)(\alpha^k z)^{-\lfloor m/2\rfloor}(1-\alpha^k z)^m = z^{-\mathsf{M}\lfloor m/2\rfloor}\left(\frac{1-z^{\mathsf{M}}}{\mathsf{M}}\right)^m, \quad k = 0,\dots,\mathsf{M}-1,$$

by using also (1.15), we have

$$\begin{aligned} T_0(z) &= \frac{P_m(z)\left[(1-P_m(z)A(z)) - \sum_{k=1}^{\mathsf{M}-1} P_m(\alpha^k z)A(\alpha^k z)\right]}{z^{-\mathsf{M}\lfloor m/2\rfloor}((1-z^{\mathsf{M}})/\mathsf{M})^m} \\ &= \frac{P_m(z)\left[1 - \sum_{k=0}^{\mathsf{M}-1} P_m(\alpha^k z)A(\alpha^k z)\right]}{z^{-\mathsf{M}\lfloor m/2\rfloor}((1-z^{\mathsf{M}})/\mathsf{M})^m} = 0. \end{aligned} \tag{3.7}$$

Similarly, for $\gamma = 1, \ldots, \mathsf{M}-1$, we have

$$T_\gamma(z) = \frac{1}{\mathsf{M}z^\gamma} \; \frac{P_m(z)\left[(1-P_m(z)A(z)) - \sum_{k=1}^{\mathsf{M}-1} \alpha^{-\gamma k} P_m(\alpha^k z)A(\alpha^k z)\right]}{z^{-\mathsf{M}\lfloor m/2\rfloor}((1-z^{\mathsf{M}})/\mathsf{M})^m}$$
$$= \frac{1}{\mathsf{M}z^\gamma} \; \frac{1 - \sum_{k=0}^{\mathsf{M}-1} \alpha^{-\gamma k} P_m(\alpha^k z)A(\alpha^k z)}{z^{-\lfloor m/2\rfloor}(1-z)^m}. \tag{3.8}$$

By using also (1.15), we have

$$T_\gamma(z) = \frac{1}{\mathsf{M}} \frac{z^{\lfloor m/2\rfloor-\gamma}}{(1-z)^m} \sum_{k=1}^{\mathsf{M}-1} (1-\alpha^{-\gamma k}) P_m(\alpha^k z)A(\alpha^k z), \quad \gamma = 1, \ldots \mathsf{M}-1. \tag{3.9}$$

According to (1.4), we have that $(1-z)^m$ divides $P_m(\alpha^k z)$ for $k = 1, \ldots, \mathsf{M}-1$, and it follows from (3.9) that T_γ is a Laurent polynomial for each $\gamma \in \{1, \ldots, \mathsf{M}-1\}$. Moreover, (3.8) and (1.11) imply the formulation

$$T_\gamma(z) = \frac{1}{\mathsf{M}} z^{\lfloor m/2\rfloor-\gamma} \frac{1 - \mathsf{M}z^\gamma (P_m A)^{[\gamma]}(z^{\mathsf{M}})}{(1-z)^m}, \quad \gamma = 1, \ldots, \mathsf{M}, \tag{3.10}$$

according to which $T_1, \ldots, T_{\mathsf{M}-1}$ are Laurent polynomials with real coefficients.

Now observe from (3.4) that $\mathcal{T}(z) = [V(z)]^T$, where $V(z)$ is the $\mathsf{M}\times\mathsf{M}$ Vandermonde matrix with respect to the M distinct points $\{(\alpha^\gamma z)^{-1} : \ \gamma = 0, \ldots, \mathsf{M}-1\}$. Hence $V(z)$ is an invertible matrix, according to which its transpose $\mathcal{T}(z)$ is also an invertible matrix. We may thus deduce from (3.5), (3.6), (3.7), and (3.8) that the Laurent polynomials $Q_1, \ldots, Q_{\mathsf{M}-1}$ and $B_1, \ldots, B_{\mathsf{M}-1}$ satisfy (1.14) if and only if the Laurent polynomials $Q_1, \ldots, Q_{\mathsf{M}-1}$ and $C_\gamma^{[k]}, \gamma, k = 1, \ldots, \mathsf{M}-1$, satisfy the identity

$$\begin{bmatrix} C_1^{[1]}(z^{\mathsf{M}}) & C_2^{[1]}(z^{\mathsf{M}}) & \cdots & C_{\mathsf{M}-1}^{[1]}(z^{\mathsf{M}}) \\ C_1^{[2]}(z^{\mathsf{M}}) & C_2^{[2]}(z^{\mathsf{M}}) & \cdots & C_{\mathsf{M}-1}^{[2]}(z^{\mathsf{M}}) \\ \vdots & \vdots & \ddots & \vdots \\ C_1^{[\mathsf{M}-1]}(z^{\mathsf{M}}) & C_2^{[\mathsf{M}-1]}(z^{\mathsf{M}}) & \cdots & C_{\mathsf{M}-1}^{[\mathsf{M}-1]}(z^{\mathsf{M}}) \end{bmatrix} \begin{bmatrix} Q_1(z) \\ Q_2(z) \\ \vdots \\ Q_{\mathsf{M}-1}(z) \end{bmatrix} = \begin{bmatrix} T_1(z) \\ T_2(z) \\ \vdots \\ T_{\mathsf{M}-1}(z) \end{bmatrix} \tag{3.11}$$

with the Laurent polynomials $T_1, \ldots, T_{\mathsf{M}-1}$ given by (3.10). From (3.1), (1.10) and (3.2), the Laurent polynomials $B_1, \ldots, B_{\mathsf{M}-1}$ are then given by

$$B_\gamma(z) = z^{-\lfloor m/2\rfloor}(1-z)^m \sum_{k=1}^{\mathsf{M}-1} z^k C_\gamma^{[k]}(z^{\mathsf{M}}), \quad \gamma = 1, \ldots, \mathsf{M}-1.$$

By observing that the Laurent polynomials

$$C_\gamma^{[k]}(z) := \delta(k-\gamma); \quad Q_\gamma(z) := T_\gamma(z), \quad \gamma, k = 1, \ldots, \mathsf{M}-1, \tag{3.12}$$

satisfy the identity (3.11), we have therefore now established the following result.

Theorem 5. *For integers $\mathsf{M} \geq 2$ and $m \geq 2$, let P_m be given as in (1.4) and A denote any Laurent polynomial satisfying the identity (1.15). Then the definitions*

$$Q_\gamma(z) := \frac{1}{\mathsf{M}} z^{\lfloor m/2 \rfloor - \gamma} \frac{1 - \mathsf{M} z^\gamma (P_m A)^{[\gamma]}(z^{\mathsf{M}})}{(1-z)^m}, \quad \gamma = 1, \dots, \mathsf{M}-1; \tag{3.13}$$

and

$$B_\gamma(z) := z^{-\lfloor m/2 \rfloor + \gamma}(1-z)^m, \quad \gamma = 1, \dots, \mathsf{M}-1, \tag{3.14}$$

yield Laurent polynomials satisfying (1.14). Moreover, for each $\gamma \in \{1, \dots, \mathsf{M}-1\}$, B_γ is a Laurent polynomial of shortest length satisfying the identity (1.16), and, for these given choices of $B_1, \dots, B_{\mathsf{M}-1}$, we have that $Q_1, \dots, Q_{\mathsf{M}-1}$ are the unique Laurent polynomials satisfying (1.14).

According to Theorem 2 (or Theorem 3) and Corollary 2, we may choose $A = A_m$ or $A = A_m^\ell$ in Theorem 5 to yield, in (3.13), the Laurent polynomials

$$Q_\gamma(z) = Q_{m,\gamma}(z) := \frac{1}{\mathsf{M}} z^{\lfloor m/2 \rfloor - \gamma} \frac{1 - \mathsf{M} z^\gamma (P_m A_m)^{[\gamma]}(z^{\mathsf{M}})}{(1-z)^m}, \quad \gamma = 1, \dots, \mathsf{M}-1, \tag{3.15}$$

and

$$Q_\gamma(z) = Q^\ell_{m,\gamma}(z) := \frac{1}{\mathsf{M}} z^{\lfloor m/2 \rfloor - \gamma} \frac{1 - \mathsf{M} z^\gamma (P_m A^\ell_m)^{[\gamma]}(z^{\mathsf{M}})}{(1-z)^m}, \quad \gamma = 1, \dots, \mathsf{M}-1, \tag{3.16}$$

for which we can establish the following properties.

Corollary 3. *Let $Q_{m,\gamma}$ and $Q^\ell_{m,\gamma}$ be defined as in (3.15) and (3.16) for integers $m \geqslant 2$, $\ell \geqslant 0$, and $\gamma = 1, \dots, \mathsf{M}-1$. Then the following statements hold.*

(a) *$Q_{m,\gamma}(z) = z^{-(\mathsf{M}-1)\lfloor \frac{m}{2} \rfloor} R_{m,\gamma}(z)$, where $R_{m,\gamma} \in \Pi_{m(\mathsf{M}-1)-\mathsf{M}}$, $\gamma = 1, \dots, \mathsf{M}-1$.*
(b) *For $m = 2n$, the sequential symmetry condition $Q_{2n,\mathsf{M}-\gamma}(z^{-1}) = z^{\mathsf{M}} Q_{2n,\gamma}(z)$, $\gamma = 1, \dots, \mathsf{M}-1$, holds.*
(c) *$Q_{m,\gamma}$ and $Q^\ell_{m,\gamma}(z)$ are related by*

$$Q^\ell_{m,\gamma}(z) = z^{\lfloor m/2 \rfloor - \lfloor (m+\ell)/2 \rfloor}(1-z)^\ell Q_{m+\ell,\gamma}(z). \tag{3.17}$$

Proof. (a) By applying (1.4), (2.8), (1.10) and $H_m \in \Pi_{m-2}$, we deduce that

$$1 - \mathsf{M} z^\gamma (P_m A_m)^{[\gamma]}(z^{\mathsf{M}}) = z^{-\mathsf{M}\lfloor m/2 \rfloor + \gamma} T_{m,\gamma}(z), \quad \gamma = 1, \dots, \mathsf{M}-1, \tag{3.18}$$

where $T_{m,\gamma} \in \Pi_{\mathsf{M}(m-1)}$, $\gamma = 1, \dots, \mathsf{M}-1$. The desired results then immediately follow from (3.15) and (3.18), together with the divisibility condition

$$(1-z)^m | (1 - \mathsf{M} z^\gamma (P_m A_m)^{[\gamma]}(z^{\mathsf{M}})), \quad \gamma = 1, \dots, \mathsf{M}-1,$$

as follows from the equivalent formulations (3.9) and (3.10).

(b) Observe from (1.4) that $P_{2n}(z^{-1}) = P_{2n}(z)$. Now apply (3.12) and (3.9) with $A = A_{2n}$, together with (1.9) and $A_{2n}(z^{-1}) = A_{2n}(z)$, to deduce that, for any $\gamma \in \{1, \dots, \mathsf{M}-1\}$,

$$\begin{aligned} \mathsf{M} z^{-\mathsf{M}-n+\gamma}(1-z)^{2n} Q_{2n,\mathsf{M}-\gamma}(z^{-1}) &= \sum_{k=1}^{\mathsf{M}-1} (1-\alpha^{\gamma k}) P_{2n}(\alpha^{-(\mathsf{M}-k)} z^{-1}) A_{2n}(\alpha^{-(\mathsf{M}-k)} z^{-1}) \\ &= \sum_{k=1}^{\mathsf{M}-1} (1-\alpha^{\gamma(\mathsf{M}-k)}) P_{2n}((\alpha^k z)^{-1}) A_{2n}((\alpha^k z)^{-1}) \end{aligned}$$

$$= \sum_{k=1}^{\mathsf{M}-1}(1-\alpha^{-\gamma k})P_{2n}(\alpha^k z)A_{2n}(\alpha^k z)$$
$$= \mathsf{M}z^{-n+\gamma}(1-z)^{2n}Q_{2n,\gamma}(z),$$

which proves the desired symmetry result.

(c) The relation (3.17) is a direct consequence of the relation $P_m A_m^\ell = P_{m+\ell}A_{m+\ell}$. □

Analogously to the family $\{A_m : \ m = 2,3,\ldots\}$, we proceed to deduce recursive formulations for the families $\{Q_{m,\gamma} : m = 2,3,\ldots\}$, $\gamma = 1,\ldots,\mathsf{M}-1$ as well with respect to m.

Theorem 6. *For any integers $\mathsf{M} \ge 2, m \ge 2$ and $\gamma \in \{1,\ldots,\mathsf{M}-1\}$, the Laurent polynomials $\{Q_{m,\gamma} : m = 2,3,\ldots\}$, $\gamma = 1,\ldots,\mathsf{M}-1$ in (3.15) satisfy the recursive formulation*

$$\begin{cases} Q_{2,\gamma}(z) = \dfrac{1}{\mathsf{M}^2}\left[-\displaystyle\sum_{k=-\mathsf{M}+1}^{-\gamma-1}(\mathsf{M}+k)z^k + (\mathsf{M}-\gamma)\sum_{k=-\gamma}^{-1}kz^k\right]; \\ Q_{2n+1,\gamma}(z) = \dfrac{Q_{2n,\gamma}(z) - z^{-(\mathsf{M}-1)n}Q_{2n,\gamma}(1)\left(\dfrac{1+z+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n}}{1-z}; \\ Q_{2n+2,\gamma}(z) = \dfrac{zQ_{2n+1,\gamma}(z) - z^{-(\mathsf{M}-1)(n+1)}Q_{2n+1,\gamma}(1)\left(\dfrac{1+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n+1}}{1-z}. \end{cases} \tag{3.19}$$

Proof. First, for $m = 2$, we apply (3.15) and $A_2 \equiv 1$ to obtain

$$Q_{2,\gamma}(z) = \frac{1}{\mathsf{M}}z^{1-\gamma}\frac{1-\mathsf{M}z^\gamma(P_2)^{[\gamma]}(z^\mathsf{M})}{(1-z)^2}, \quad \gamma = 1,\ldots,\mathsf{M}-1.$$

Now observe from (1.4) that

$$P_2(z) = \frac{1}{\mathsf{M}^2}\left[\sum_{k=-\mathsf{M}+1}^{-1}(\mathsf{M}+k)z^k + \mathsf{M} + \sum_{k=1}^{\mathsf{M}-1}(\mathsf{M}-k)z^k\right],$$

from which, together with (1.10), we deduce that $(P_2)^{[\gamma]}(z^\mathsf{M}) = \frac{1}{\mathsf{M}^2}\left[\gamma z^{-\mathsf{M}} + (\mathsf{M}-\gamma)\right]$, $\gamma = 0,\ldots,\mathsf{M}-1$. It follows that

$$Q_{2,\gamma}(z) = -\frac{1}{\mathsf{M}^2}z^{1-\mathsf{M}}\frac{(\mathsf{M}-\gamma)z^\mathsf{M} - \mathsf{M}z^{\mathsf{M}-\gamma}+\gamma}{(1-z)^2}, \quad \gamma = 1,\ldots \mathsf{M}-1.$$

For any $\gamma \in \{1,\ldots \mathsf{M}-1\}$, we have

$$\begin{aligned}\frac{(\mathsf{M}-\gamma)z^\mathsf{M} - \mathsf{M}z^{\mathsf{M}-\gamma}+\gamma}{(z-1)^2} &= \frac{1}{z-1}\left[-\gamma\sum_{k=0}^{\mathsf{M}-1-\gamma}z^k + (\mathsf{M}-\gamma)\sum_{k=\mathsf{M}-\gamma}^{\mathsf{M}-1}z^k\right] \\ &= \gamma\sum_{k=0}^{\mathsf{M}-2-\gamma}(k+1)z^k + (\mathsf{M}-\gamma)\sum_{k=\mathsf{M}-1-\gamma}^{\mathsf{M}-2}(\mathsf{M}-1-k)z^k,\end{aligned}$$

with the convention $\sum_{\gamma=\gamma_1}^{\gamma_2}F_\gamma := 0$, if $\gamma_2 < \gamma_1$. Consequently, we obtain the explicit formulation

$$Q_{2,\gamma}(z) = \frac{1}{\mathsf{M}^2}\left[-\sum_{k=-\mathsf{M}+1}^{-\gamma-1}(\mathsf{M}+k)z^k + (\mathsf{M}-\gamma)\sum_{k=-\gamma}^{-1} kz^k\right], \quad \gamma = 1,\dots,\mathsf{M}-1. \tag{3.20}$$

Next, we use the relation for Q_γ and B_γ in (1.14); that is,

$$\sum_{k=0}^{\mathsf{M}-1} B_j(\alpha^k z)Q_\gamma(\alpha^k z) = \delta(j-\gamma), \quad j,\gamma = 1,\dots,\mathsf{M}-1,$$

and (3.14) to obtain

$$\sum_{k=0}^{\mathsf{M}-1}(\alpha^k z)^{-\lfloor m/2\rfloor+j}(1-\alpha^k z)^m Q_{m,\gamma}(\alpha^k z) = \delta(j-\gamma), \quad j,\gamma = 1,\dots,\mathsf{M}-1.$$

By successively setting $m = 2n$, $m = 2n+1$ and $m = 2n+2$ in the above identities, and subtracting the resulting identities, we obtain

$$\sum_{k=0}^{\mathsf{M}-1}(\alpha^k z)^{-n+j}(1-\alpha^k z)^{2n+r} X^r_{n,\gamma}(\alpha^k z) = 0, \quad j,\gamma = 1,\dots,\mathsf{M}-1, \tag{3.21}$$

for $r = 0, 1$, where

$$X^r_{n,\gamma}(z) := z^{-r}(1-z)Q_{2n+1+r,\gamma}(z) - Q_{2n+r,\gamma}(z), \quad \gamma = 1,\dots,\mathsf{M}-1; r = 0, 1. \tag{3.22}$$

Let $r \in \{0, 1\}$ and $\gamma \in \{1,\dots,\mathsf{M}-1\}$ be fixed. By setting $z = 1$ in (3.21), we deduce that the sequence

$$g^r_k := X^r_{n,\gamma}(\alpha^k), \quad k = 1,\dots,\mathsf{M}-1,$$

satisfies the $(\mathsf{M}-1)\times(\mathsf{M}-1)$ homogeneous linear system

$$S_r : \sum_{k=1}^{\mathsf{M}-1} \sigma^r_{j,k} g^r_k = 0, \quad j = 1,\dots,\mathsf{M}-1,$$

where $\sigma^r_{j,k} := (\alpha^k)^{-n+j}(1-\alpha^k)^{2n+r}$, $j,k = 1,\dots,\mathsf{M}-1$. Hence, we have

$$\sigma^r_{j,k} = \begin{cases} \tau^0_k \alpha^{jk}, & \text{if} \quad r = 0; \\ \tau^1_k \alpha^{(j+\frac{1}{2})k}, & \text{if} \quad r = 1, \end{cases}$$

where $\tau^0_k := (-1)^n 2^{2n}\sin^{2n}\left(\frac{\pi k}{\mathsf{M}}\right)$ and $\tau^1_k := (-1)^{n+1}2^{2n+1}i\sin^{2n+1}\left(\frac{\pi k}{\mathsf{M}}\right)$, $k = 1,\dots,\mathsf{M}-1$. Note that $\tau^r_k \neq 0$ for $k = 1,\dots,\mathsf{M}-1$.

Now observe that the coefficient matrix

$$S_r := \left[\sigma^r_{j,k}\right]_{j,k=1,\dots,\mathsf{M}-1}$$

corresponding to the linear system S_r has the decomposition

$$S_r = Y_r Z_r, \tag{3.23}$$

where

$$Y_0 := \begin{bmatrix} \alpha & \alpha^2 & \cdots & \alpha^{\mathsf{M}-1} \\ \alpha^2 & (\alpha^2)^2 & \cdots & (\alpha^2)^{\mathsf{M}-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{\mathsf{M}-1} & (\alpha^{\mathsf{M}-1})^2 & \cdots & (\alpha^{\mathsf{M}-1})^{\mathsf{M}-1} \end{bmatrix},$$

$$Y_1 := \begin{bmatrix} \alpha^{\frac{3}{2}} & (\alpha^{\frac{3}{2}})^2 & \cdots & (\alpha^{\frac{3}{2}})^{\mathsf{M}-1} \\ \alpha^{\frac{5}{2}} & (\alpha^{\frac{5}{2}})^2 & \cdots & (\alpha^{\frac{5}{2}})^{\mathsf{M}-1} \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha^{\mathsf{M}-\frac{1}{2}}) & (\alpha^{\mathsf{M}-\frac{1}{2}})^2 & \cdots & (\alpha^{\mathsf{M}-\frac{1}{2}})^{\mathsf{M}-1} \end{bmatrix},$$

and where Z_r is the diagonal matrix with entries $\tau_1^r, \tau_2^r, \dots, \tau_{\mathsf{M}-1}^r$ on its main diagonal. Note that $\det(Y_r) = \alpha^{\nu_r}\det(V)$, where $\nu_0 := 1 + 2 + \cdots + (\mathsf{M}-1) = \frac{1}{2}(\mathsf{M}-1)\mathsf{M}$, $\nu_1 := \frac{3}{2} + \frac{5}{2} + \cdots + \frac{2\mathsf{M}-1}{2} = \frac{1}{2}(\mathsf{M}^2 - 1)$, and

$$V := \begin{bmatrix} 1 & \alpha & \cdots & \alpha^{\mathsf{M}-2} \\ 1 & \alpha^2 & \cdots & (\alpha^2)^{\mathsf{M}-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{\mathsf{M}-1} & \cdots & (\alpha^{\mathsf{M}-1})^{\mathsf{M}-2} \end{bmatrix}. \tag{3.24}$$

Hence, we can deduce that $\det(Y_0) = \det(V)$ and $\det(Y_1) = (-1)^{\mathsf{M}}\alpha^{-\frac{1}{2}}\det(V)$. Now observe from (3.24) that V is the $(\mathsf{M}-1) \times (\mathsf{M}-1)$ Vandermonde matrix with respect to the $\mathsf{M}-1$ distinct points $\{\alpha^k : k = 1, \dots, \mathsf{M}-1\}$, according to which V is an invertible matrix, and thus $\det(V) \neq 0$, which yields $\det(Y_r) \neq 0$, thereby establishing the invertibility of Y_r. Also, it follows from $\tau_k^r \neq 0, k = 1, \dots, \mathsf{M}-1$ that the diagonal matrix Z_r is invertible. Hence the matrix S_r in (3.23) is invertible.

Consequently, $g_k^r = 0, k = 1, \dots, \mathsf{M}-1, r = 0, 1$ and hence $X_{n,\gamma}^r(\alpha^k) = 0, k = 1, \dots, \mathsf{M}-1$. It follows that

$$X_{n,\gamma}^r(z) = (1 + z + \cdots + z^{\mathsf{M}-1})\tilde{X}_{n,\gamma}^r(z) \tag{3.25}$$

for some Laurent polynomial $\tilde{X}_{n,\gamma}^r$. By substituting (3.25) into (3.21), and using the fact that

$$(1 - \alpha^k z)(1 + \alpha^k z + \cdots + (\alpha^k z)^{\mathsf{M}-1}) = (1 - z)^{\mathsf{M}}, \quad k = 0, \dots, \mathsf{M}-1,$$

we deduce that

$$\sum_{k=0}^{\mathsf{M}-1} (\alpha^k z)^{-n+j}(1 - \alpha^k z)^{2n+r-1}\tilde{X}_{n,\gamma}^r(\alpha^k z) = 0, \quad j = 1, \dots, \mathsf{M}-1. \tag{3.26}$$

Repeating the argument from (3.21) to (3.26) eventually yields

$$X_{n,\gamma}^r(z) = (1 + z + \cdots + z^{\mathsf{M}-1})^{2n+r}W_{n,\gamma}^r(z),$$

for some Laurent polynomial $W_{n,\gamma}^r$ satisfying the identity

$$\sum_{k=0}^{\mathsf{M}-1} \alpha^{(-n+j)k}W_{n,\gamma}^r(\alpha^k z) = 0, \quad j = 1, \dots, \mathsf{M}-1.$$

By (3.22) and the length property of $Q_{m,\gamma}$ in Corollary 3(a), we deduce that $W_{n,\gamma}^r$ must be a monomial.

Hence,

$$W^r_{n,\gamma}(z) = z^{-r-(\mathsf{M}-1)\lfloor(2n+1+r)/2\rfloor} c^r_{n,\gamma},\ j = 1, \dots, \mathsf{M}-1, \tag{3.27}$$

for some constants $\{c^r_{n,\gamma}:\ \gamma = 1, \dots, \mathsf{M}-1\}$. By combining (3.22) and (3.27), we obtain

$$(1-z)Q_{2n+1+r,\gamma}(z) = z^r Q_{2n+r,\gamma}(z) + c^r_{n,\gamma} z^{-(\mathsf{M}-1)\lfloor(2n+1+r)/2\rfloor}(1 + z + \dots + z^{\mathsf{M}-1})^{2n+r},$$

in which we may now set $z = 1$ to obtain $c^r_{n,\gamma} = -\mathsf{M}^{-2n-r} Q_{2n+r,\gamma}(1)$, and thus

$$\begin{aligned}&(1-z)Q_{2n+1+r,\gamma}(z)\\ &\quad = z^r Q_{2n+r,\gamma}(z) - z^{-(\mathsf{M}-1)\lfloor(2n+1+r)/2\rfloor} Q_{2n+r,\gamma}(1) \left(\frac{1 + z + \dots + z^{\mathsf{M}-1}}{\mathsf{M}}\right)^{2n+r}.\end{aligned} \tag{3.28}$$

The recursive formulation (3.19) now follows from (3.20), and by successively setting $r = 0$ and $r = 1$ in (3.28). □

For $\mathsf{M} = 2$, note that $\alpha = -1$, so that we may apply the formulation (3.9) to deduce from (3.12), (1.4), and (2.8) that

$$Q_{m,1}(z) = \frac{(-1)^{\lfloor m/2\rfloor}}{2^m} z^{-1} A_m(-z) = -\frac{1}{2^m} z^{-\lfloor m/2\rfloor} H_m(-z), \quad \text{if } \mathsf{M} = 2,$$

with A_m and $H_m \in \Pi_{m-2}$ as in Theorem 2. Calculating by means of Theorem 6, alternatively (3.15), we obtain the Laurent polynomials $\{Q_{m,\gamma}:\ \gamma = 1, \dots, \mathsf{M}-1\}$ for $\mathsf{M} = 2, 3, 4$ and $m = 2, 3, 4$ in Tables 5.4–5.6 in Appendix A.

In summary, we have the following result for

$$(\{P_m; Q^\ell_{m,1}, \dots, Q^\ell_{m,\mathsf{M}-1}\}, \{A^\ell_m; B_{m,1}, \dots, B_{m,\mathsf{M}-1}\})$$

of the PR multirate system (in terms of symbols), from which we also provide an algorithm for their efficient computations (see Algorithm 1).

Corollary 4. *For any integers $\mathsf{M} \geq 2, m \geq 2$, and $\ell \geq 0$, the Laurent polynomials*

$$\begin{aligned}
P(z) &= P_m(z) := z^{-(\mathsf{M}-1)\lfloor m/2\rfloor} \left(\frac{1 + z + \dots + z^{\mathsf{M}-1}}{\mathsf{M}}\right)^m;\\
A(z) &= A^\ell_m(z) := z^{-(\mathsf{M}-1)(\lfloor(m+\ell)/2\rfloor - \lfloor m/2\rfloor)} \left(\frac{1 + z + \dots + z^{\mathsf{M}-1}}{\mathsf{M}}\right)^\ell A_{m+\ell}(z);\\
Q_\gamma(z) &= Q^\ell_{m,\gamma}(z) := z^{\lfloor m/2\rfloor - \lfloor(m+\ell)/2\rfloor}(1-z)^\ell Q_{m+\ell,\gamma}(z), \quad \gamma = 1, \dots, \mathsf{M}-1;\\
B_\gamma(z) &= B_{m,\gamma}(z) := z^{-\lfloor m/2\rfloor+\gamma}(1-z)^m, \quad \gamma = 1, \dots, \mathsf{M}-1,
\end{aligned}$$

with $A_{m+\ell}$ and $Q_{m+\ell}$ as in, respectively, Theorem 3 (Eq. (2.9)) and Theorem 6 (Eq. (3.19)), satisfy the matrix identity (1.14).

4. Biorthogonal spline wavelets in a pair of Sobolev spaces

With the condition that $P(1) = A(1) = 1$, it is shown in [10] that the pair

$$(\{P; Q_1, \dots, Q_{\mathsf{M}-1}\}, \{A; B_1, \dots, B_{\mathsf{M}-1}\})$$

Algorithm 1 PR multirate system with shortest spline-wavelet filters.

(a) **Input:** (M, m, ℓ): dilation factor $\mathsf{M} \geqslant 2$, B-spline order $m \geqslant 2$, and sum rule order $\ell \geqslant 0$.

(b) **Initialization:** Initialize P_m, B_m, A_2, and $Q_{2,\gamma}$, $\gamma = 1, \dots, \mathsf{M}-1$.

$P_m(z) \leftarrow z^{-(\mathsf{M}-1)\lfloor m/2 \rfloor} \left(\frac{1+z+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}}\right)^m$ and $A_2(z) \leftarrow 1$.

for γ from 1 to $\mathsf{M}-1$ **do**

$\quad B_{m,\gamma}(z) \leftarrow z^{-\lfloor m/2 \rfloor + \gamma}(1-z)^m$.

$$\quad Q_{2,\gamma}(z) \leftarrow \frac{1}{\mathsf{M}^2}\left[-\sum_{k=-\mathsf{M}+1}^{-\gamma-1}(\mathsf{M}+k)z^k + (\mathsf{M}-\gamma)\sum_{k=-\gamma}^{-1} kz^k\right].$$

end for

(c) **Recursive Computation:** Compute $A_{m+\ell}, Q_{m+\ell,\gamma}$, $\gamma = 1, \dots, \mathsf{M}-1$.

for $\tilde{m}$ from 2 to $m+\ell-1$ **do**

$\quad$ Determine $W_{\tilde{m}}(z) \in \Pi_{\mathsf{M}-2}$ uniquely by

$$W_{\tilde{m}}(\alpha^k) = -(1-\alpha^k)^{-\tilde{m}}(\alpha^k)^{\lfloor \tilde{m}/2 \rfloor - 1} A_{\tilde{m}}(\alpha^k),\ k = 1, \dots, \mathsf{M}-1.$$

$$\quad A_{\tilde{m}+1}(z) \leftarrow \frac{z^{\lfloor \tilde{m}/2 \rfloor - \lfloor (\tilde{m}+1)/2 \rfloor} A_{\tilde{m}}(z) + z^{-\lfloor (\tilde{m}+1)/2 \rfloor + 1}(1-z)^{\tilde{m}} W_{\tilde{m}}(z)}{(1+z+\cdots+z^{\mathsf{M}-1})/\mathsf{M}}.$$

$\quad$ **for** γ from 1 to $\mathsf{M}-1$ **do**

$$\qquad Q_{\tilde{m}+1,\gamma} \leftarrow \frac{z^{\lfloor \frac{\tilde{m}+1}{2} \rfloor - \lfloor \frac{\tilde{m}}{2} \rfloor} Q_{\tilde{m},\gamma}(z) - z^{-(\mathsf{M}-1)\lfloor \frac{\tilde{m}+1}{2} \rfloor} Q_{\tilde{m},\gamma}(1)\left(\dfrac{1+z+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}}\right)^{\tilde{m}}}{1-z}.$$

$\quad$ **end for**

end for

(d) **Finalization:** Compute $A_m^\ell, Q_{m,\gamma}^\ell$, $\gamma = 1, \dots, \mathsf{M}-1$.

$A_m^\ell(z) \leftarrow z^{-(\mathsf{M}-1)(\lfloor (m+\ell)/2 \rfloor - \lfloor m/2 \rfloor)} \left(\frac{1+z+\cdots+z^{\mathsf{M}-1}}{\mathsf{M}}\right)^\ell A_{m+\ell}(z)$.

for γ from 1 to $\mathsf{M}-1$ **do**

$\quad Q_{m,\gamma}^\ell(z) \leftarrow z^{\lfloor m/2 \rfloor - \lfloor (m+\ell)/2 \rfloor}(1-z)^\ell Q_{m+\ell,\gamma}(z)$.

end for

(e) **Output:** The PR multirate system with the shortest spline-wavelet filters

$$(\{P_m; Q_{m,1}^\ell, \dots, Q_{m,\mathsf{M}-1}^\ell\}, \{A_m^\ell; B_{m,1}, \dots, B_{m,\mathsf{M}-1}\}).$$

---

of PR multirate system is always associated with an underlying pair of frequency-based dual M-framelets in the distribution space. In this section, we shall discuss biorthogonal wavelets in $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ that are associated with such pairs of PR multirate systems.

Let us first recall some notations and definitions. For a function $f : \mathbb{R} \to \mathbb{C}$, we define

$$f_{\lambda;k} := |\lambda|^{1/2} f(\lambda \cdot -k), \qquad \lambda \in \mathbb{R}\backslash\{0\},\ k \in \mathbb{R}.$$

For any $\tau \in \mathbb{R}$, we denote by $H^\tau(\mathbb{R})$ the Sobolev space consisting of all tempered distributions f such that

$$\|f\|_{H^\tau(\mathbb{R})}^2 := \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2(1+|\xi|^2)^\tau d\xi < \infty,$$

where the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$, $\xi \in \mathbb{R}$. Note that $H^\tau(\mathbb{R})$ is a Hilbert space under the inner product

$$\langle f,g\rangle_{H^\tau(\mathbb{R})} := \frac{1}{2\pi}\int\limits_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1+|\xi|^2)^\tau d\xi,\ f,g\in H^\tau(\mathbb{R}).$$

Moreover, for each $g \in H^{-\tau}(\mathbb{R})$,

$$\langle f,g\rangle := \frac{1}{2\pi}\int\limits_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,\ f\in H^\tau(\mathbb{R}),$$

defines a linear functional on $H^\tau(\mathbb{R})$. The spaces $H^\tau(\mathbb{R})$ and $H^{-\tau}(\mathbb{R})$ form a pair of dual spaces.

Let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. The functions $\phi,\psi^1,\dots,\psi^s$ in $H^\tau(\mathbb{R})$ are said to generate the *nonhomogeneous* M-*wavelet system*

$$\mathrm{WS}_0(\phi;\psi^1,\dots,\psi^s) := \{\phi(\cdot-k) : k\in\mathbb{Z}\}\cup\{\psi^\ell_{\mathsf{M}^j;k} : j\in\mathbb{N}_0, k\in\mathbb{Z}, \ell=1,\dots,s\},$$

(see [10]). For functions $\phi,\psi^1,\dots,\psi^s\in H^\tau(\mathbb{R})$ and $\tilde\phi,\tilde\psi^1,\dots,\tilde\psi^s\in H^{-\tau}(\mathbb{R})$, we say that

$$(\{\phi;\psi^1,\dots,\psi^s\},\{\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^s\}) \tag{4.1}$$

generates a pair of biorthogonal M-wavelet bases in $(H^\tau(\mathbb{R}),H^{-\tau}(\mathbb{R}))$, if

$$(\mathrm{WS}_0(\phi;\psi^1,\dots,\psi^s),\mathrm{WS}_0(\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^s))$$

is a pair of biorthogonal bases in $(H^\tau(\mathbb{R}),H^{-\tau}(\mathbb{R}))$; i.e., each of $\mathrm{WS}_0(\phi;\psi^1,\dots,\psi^s)$ and $\mathrm{WS}_0(\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^s)$ is a Riesz basis in $H^\tau(\mathbb{R})$ and $H^{-\tau}(\mathbb{R})$, respectively, and the two systems are biorthogonal to each other; that is

$$\begin{aligned}&\langle\phi_{0;k},\tilde\phi_{0;k'}\rangle=\delta_{k,k'}, \quad \langle\psi^\ell_{\mathsf{M}^j;k},\tilde\psi^{\ell'}_{\mathsf{M}^{j'};k'}\rangle=\delta_{j,j'}\delta_{\ell,\ell'}\delta_{k,k'},\\ &\langle\phi_{0;k},\tilde\psi^{\ell'}_{\mathsf{M}^{j'};k'}\rangle=0, \qquad \langle\psi^\ell_{\mathsf{M}^j;k},\tilde\phi_{0;k'}\rangle=0.\end{aligned}$$

A sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a Riesz basis of $\mathcal{H}$ if the span of $\{f_n\}$ is dense in $\mathcal{H}$ and there exists two positive constants $0<A_1\leqslant A_2<+\infty$ such that

$$A_1\sum_n|c_n|^2\leqslant\left\|\sum_n c_nf_n\right\|^2_{\mathcal{H}}\leqslant A_2\sum_n|c_n|^2$$

for all finite sequences $\{c_n\}$. It follows that the identity

$$\langle f,g\rangle=\sum_{k\in\mathbb{Z}}\langle f,\phi(\cdot-k)\rangle\langle\tilde\phi(\cdot-k),g\rangle+\sum_{j=0}^\infty\sum_{\ell=1}^s\sum_{k\in\mathbb{Z}}\langle f,\psi^\ell_{\mathsf{M}^j;k}\rangle\langle\tilde\psi^\ell_{\mathsf{M}^j;k},g\rangle$$

for all $f\in H^\tau(\mathbb{R})$, $g\in H^{-\tau}(\mathbb{R})$ is satisfied. It has been shown in [10] that if the pair in (4.1) generates a pair of biorthogonal M-wavelet bases in $(H^\tau(\mathbb{R}),H^{-\tau}(\mathbb{R}))$, then we must have $s=\mathsf{M}-1$. See [7,11,17] for biorthogonal wavelets in $L_2(\mathbb{R})$.

For $0<\alpha\leqslant1$ and $1\leqslant p\leqslant\infty$, we say that $f\in\mathrm{Lip}(\alpha,L_p(\mathbb{R}))$ if there is a constant C such that $\|f-f(\cdot-h)\|_{L_p(\mathbb{R})}\leqslant Ch^\alpha$ for all $h>0$. The smoothness of a function f in $L_p(\mathbb{R})$ is measured by

$$\nu_p(f):=\sup\{n+\alpha \mid n\in\mathbb{N}_0, 0<\alpha\leqslant1, f^{(n)}\in\mathrm{Lip}(\alpha,L_p(\mathbb{R}))\},$$

where $f^{(n)}$ denotes the nth derivative of f. In order to state the result on biorthogonal M-wavelets in $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ associated with biorthogonal M-wavelet filter banks, we also need to recall a quantity $\nu_p(F, \mathsf{M})$ for a low-pass filter Laurent polynomial F and $1 \leqslant p \leqslant \infty$. We can write $F(z) = (1 + z + \cdots + z^{\mathsf{M}-1})^m G(z)$ for some Laurent polynomial G such that $(1 + z + \cdots + z^{\mathsf{M}-1}) \nmid G(z)$. Following [9, p. 61 and Proposition 7.2], we may define

$$\nu_p(F, \mathsf{M}) := 1/p - 1 - \log_{\mathsf{M}} \left(\limsup_{n\to\infty} \|G_n\|_{\ell_p(\mathbb{Z})}^{1/n} \right), \quad 1 \leqslant p \leqslant \infty,$$

where $\|G_n\|_{\ell_p(\mathbb{Z})}^p := \sum_{k\in\mathbb{Z}} |g_n(k)|^p$ and $\sum_{k\in\mathbb{Z}} g_n(k) z^k := G(z)G(z^{\mathsf{M}}) \cdots G(z^{\mathsf{M}^{n-1}})$. It has been proved in [9, Theorem 4.3] that the cascade algorithm with some mask (low-pass filter) F and a dilation factor M converges in $L_p(\mathbb{R})$ (as well as $C(\mathbb{R})$ when $p = \infty$) if and only if $\nu_p(F, \mathsf{M}) > 0$.

Let ϕ be the compactly supported normalized M-refinable distribution with mask F and dilation M such that $\hat{\phi}(\xi) := \prod_{j=1}^\infty F(e^{-i\mathsf{M}^{-j}\xi})$. In general, we have $\nu_p(F, \mathsf{M}) \leqslant \nu_p(\phi)$. If the integer shifts of ϕ form a Riesz system, then $\nu_p(F, \mathsf{M}) = \nu_p(\phi)$. The quantity $\nu_p(F, \mathsf{M})$ plays an important role in the study of the convergence of cascade algorithms and smoothness of refinable functions, see [9] and the references therein on these topics. Moreover, when $p = 2$, we also have

$$\nu_2(F, \mathsf{M}) = -1/2 - \log_{\mathsf{M}} \sqrt{\rho(F, \mathsf{M})}, \tag{4.2}$$

where $\rho(F, \mathsf{M})$ denotes the spectral radius of the square matrix $[g(\mathsf{M}j - k)]_{-n_2 \leqslant j,k \leqslant n_2}$, where $n_2 := \lceil \frac{n_1}{\mathsf{M}-1} \rceil$ and $G(z)G(z^{-1}) =: \sum_{k=-n_1}^{k=n_1} g(k) z^k$ (see [8, Theorem 2.1]).

Let $(\{P; Q_1, \ldots, Q_{\mathsf{M}-1}\}, \{A; B_1, \ldots, B_{\mathsf{M}-1}\})$ be a pair of Laurent polynomials and suppose

$$\frac{1}{\mathsf{M}} \sum_k q_\gamma(k) z^k := Q_\gamma(z); \quad \sum_k b_\gamma(k) z^k := B_\gamma(z), \quad \gamma = 1, \ldots, \mathsf{M} - 1.$$

One can define a pair of generators $(\{\phi; \psi_1, \ldots, \psi_{\mathsf{M}-1}\}, \{\tilde{\phi}; \tilde{\psi}_1, \ldots, \tilde{\psi}_{\mathsf{M}-1}\})$ of distributions associated with $(\{P; Q_1, \ldots, Q_{\mathsf{M}-1}\}, \{A; B_1, \ldots, B_{\mathsf{M}-1}\})$ by

$$\hat{\phi}(\xi) := \prod_{j=1}^\infty P(e^{-i\mathsf{M}^{-j}\xi}); \quad \hat{\tilde{\phi}}(\xi) := \prod_{j=1}^\infty A(e^{i\mathsf{M}^{-j}\xi}), \qquad \xi \in \mathbb{R}, \tag{4.3}$$

and

$$\psi^\gamma(x) := \sum_{k\in\mathbb{Z}} q_\gamma(k) \phi(\mathsf{M}x - k) \quad \text{and} \quad \tilde{\psi}^\gamma(x) := \sum_{k\in\mathbb{Z}} b_\gamma(-k) \tilde{\phi}(\mathsf{M}x - k), \tag{4.4}$$

when $P = P_m$, $\phi = \phi_m$ is the mth order (centered) cardinal B-spline and the wavelets $\psi^\gamma = \psi_m^\gamma, \gamma = 1, \ldots, \mathsf{M} - 1$ are the corresponding cardinal spline wavelets. We have the following result.

Theorem 7. *Let*

$$(\{P; Q_1, \ldots, Q_{\mathsf{M}-1}\}, \{A; B_1, \ldots, B_{\mathsf{M}-1}\})$$

be a pair of PR multirate systems, i.e., (1.14) holds. If $P(1) = A(1) = 1$ (that is, P and A are low-pass filters), and $\nu_2(P, \mathsf{M}) > \tau$, $\nu_2(A, \mathsf{M}) > -\tau$ for some $\tau \in \mathbb{R}$, then the pair

$$(\{\phi; \psi^1, \ldots, \psi^{\mathsf{M}-1}\}, \{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^{\mathsf{M}-1}\}) \tag{4.5}$$

defined as in (4.3) and (4.4) generates a pair of biorthogonal M*-wavelet bases in* $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$*. If, in addition,* $\nu_2(P, \mathsf{M}) > 0$ *(respectively,* $\nu_2(A, \mathsf{M}) > 0$*) and*

$$(1-z)^\ell \mid Q_\gamma \quad (\textit{respectively, } (1-z)^{\tilde{\ell}} \mid B_\gamma), \quad \gamma = 1, \ldots, \mathsf{M}-1, \tag{4.6}$$

then the wavelets $\psi^1, \ldots, \psi^{\mathsf{M}-1}$ *(respectively,* $\tilde{\psi}^1, \ldots, \tilde{\psi}^{\mathsf{M}-1}$*) have vanishing moments of order* ℓ *(respectively,* $\tilde{\ell}$*).*

Proof. Since $\nu_2(P, \mathsf{M}) > \tau$ and $\nu_2(A, \mathsf{M}) > -\tau$, by applying [11, Theorem 3.1], (4.5) generates a pair of biorthogonal M-wavelet bases in $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$. By [9], when $\nu_2(P, \mathsf{M}) > 0$, ϕ is a compactly supported refinable function in $L_2(\mathbb{R})$.

By (4.6) and $Q_\gamma(z) = \sum_k q_\gamma(k) z^k$, we have

$$Q_\gamma^{(r)}(1) = \sum_k \binom{k}{r} q_\gamma(k) = 0, \quad r = 0, \ldots, \ell-1, \quad \gamma = 1, \ldots \mathsf{M}-1, \tag{4.7}$$

from which it then inductively follows that

$$\sum_k k^r q_\gamma(k) = 0, \quad r = 0, \ldots, \ell-1, \quad \gamma = 1, \ldots, \mathsf{M}-1. \tag{4.8}$$

For any $\gamma \in \{1, \ldots, \mathsf{M}-1\}$ and $r \in \{0, \ldots, \ell-1\}$, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} x^r \psi^\gamma(x) dx &= \sum_k q_\gamma(k) \int_{-\infty}^{\infty} \left(\frac{k+x}{\mathsf{M}}\right)^r \phi_m(x) dx \\
&= \frac{1}{\mathsf{M}^r} \sum_k q_\gamma(k) \int_{-\infty}^{\infty} \left[\sum_{\nu=0}^{r} \binom{r}{\nu} k^\nu x^{r-\nu}\right] \phi_m(x) dx \\
&= \frac{1}{d^r} \sum_{\nu=0}^{r} \binom{r}{\nu} \left\{\int_{-\infty}^{\infty} x^{r-\nu} \phi_m(x) dx\right\} \left\{\sum_k k^\nu q_\gamma(k)\right\} = 0,
\end{aligned}$$

by virtue of (4.8). Hence, $\psi^1, \ldots, \psi^{\mathsf{M}-1}$ have vanishing moments of order ℓ. A similar proof yields vanishing moments of order $\tilde{\ell}$ for the $\mathsf{M}-1$ wavelets $\tilde{\psi}^1, \ldots, \tilde{\psi}^{\mathsf{M}-1}$. $\square$

It is easy to show that $\nu_2(P_m, \mathsf{M}) = m - 0.5 > 0$ for $m \geqslant 1$. Recalling from Corollary 4 that $(1-z)^m \mid B_\gamma$ and $(1-z)^\ell \mid Q_\gamma$ when $A = A_m^\ell$, we immediately have the following result from Theorem 7.

Corollary 5. *Let* $\mathsf{M} \geq 2, m \geqslant 2,$ *and* $\ell \geqslant 0$ *be integers. Then the pair*

$$(\{\phi; \psi^1, \ldots, \psi^{\mathsf{M}-1}\}, \{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^{\mathsf{M}-1}\}) = (\{\phi_m; \psi_m^1, \ldots, \psi_m^{\mathsf{M}-1}\}, \{\tilde{\phi}_m; \tilde{\psi}_m^1, \ldots, \tilde{\psi}_m^{\mathsf{M}-1})$$

as defined by (4.3) and (4.4) with respect to the pair

$$(\{P; Q_1, \ldots, Q_{\mathsf{M}-1}\}, \{A; B_1, \ldots, B_{\mathsf{M}-1}\}) = (\{P_m; Q_{m,1}^\ell, \ldots, Q_{m,\mathsf{M}-1}^\ell\}, \{A_m^\ell; B_{m,1}, \ldots, B_{m,\mathsf{M}-1}\})$$

of PR multirate system given as in Corollary 4, generates a pair of biorthogonal cardinal spline M*-wavelet bases in* $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ *for some* $\tau \in \mathbb{R}$ *such that* $\nu_2(P_m, \mathsf{M}) = m - 0.5 > \tau$ *and* $\nu_2(A_m^\ell) > -\tau$*. Moreover, the cardinal spline wavelets* $\psi_m^1, \ldots, \psi_m^{\mathsf{M}-1}$ *have vanishing moments of order* ℓ*. If in addition,* $\nu_2(A_m^\ell, \mathsf{M}) > 0$*, then the wavelets* $\tilde{\psi}_m^1, \ldots, \tilde{\psi}_m^{\mathsf{M}-1}$ *have vanishing moments of order* m*.*

The interested reader is referred to [12] for general discussion on the symmetry property of filter banks. For more references on biorthogonal and orthogonal wavelets in $L_2(\mathbb{R})$, see [2,6,8,14–16,21] and references therein.

5. Illustrative examples

In this section, we illustrate by examples our results in the previous sections.

Example 1. Let $\mathsf{M} = 3$, $m = 2$, $\ell = 0$, which is with respect to the simplest case for $\mathsf{M} > 2$. Then, by Theorems 3 and 6, we can directly obtain

$$P(z) = z^{-2}\left(\frac{1+z+z^2}{3}\right)^2; \quad Q_1(z) = -\frac{1}{9}z^{-2}(1+2z); \quad Q_2(z) = -\frac{1}{9}z^{-2}(2+z).$$
$$A(z) = 1; \quad B_1(z) = (1-z)^2; \quad B_2(z) = z(1-z)^2.$$

By Eq. (4.2), we have $\nu_2(P, \mathsf{M}) = 1.5$ and $\nu_2(A, \mathsf{M}) = -0.5$. Hence by Corollary 5, the pair $(\{\phi; \psi^1, \psi^2\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2\})$ associated with $(\{P; Q_1, Q_2\}, \{A; B_1, B_2\})$ is a pair of biorthogonal M-wavelet bases in a pair of dual Sobolev spaces $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ for all $0.5 < \tau < 1.5$.

Example 2. Let $\mathsf{M} = 3$, $m = 2$, $\ell = 2$. By Algorithm 1, we obtain

$$\begin{aligned}
P(z) &= z^{-2}\left(\frac{1+z+z^2}{3}\right)^2; \\
Q_1(z) &= \frac{1}{243}z^{-5}(1-z)^2(4+16z+40z^2+50z^3+20z^4+5z^5); \\
Q_2(z) &= \frac{1}{243}z^{-5}(1-z)^2(5+20z+50z^2+40z^3+16z^4+4z^5). \\
A(z) &= -\frac{1}{3}z^{-3}\left(\frac{1+z+z^2}{3}\right)^2(4-11z+4z^2); \\
B_1(z) &= (1-z)^2; \\
B_2(z) &= z(1-z)^2.
\end{aligned}$$

For the same special case, the dual-chain method yields, as given in [4, Example 1],

$$\begin{aligned}
Q_1(z) &= -\frac{1}{27}z^{-2}(1-z)^2(1+4z+10z^2+10z^3+4z^4+z^5); \\
Q_2(z) &= -\frac{1}{81}z^{-2}(1-z)^2(1+4z+10z^2-10z^3-4z^4-z^5); \\
B_1(z) &= -\frac{1}{2}(1-z)^2(1+z); \qquad B_2(z) = -\frac{1}{6}(1-z)^3,
\end{aligned}$$

which has Q_1 and Q_2 Laurent polynomials of the same length, and with symmetry properties, but B_1 and B_2 Laurent polynomials of longer length.

By Eq. (4.2), we have $\nu_2(P, \mathsf{M}) = 1.5$ and $\nu_2(A, \mathsf{M}) \approx 0.2105$. Hence by Corollary 5, the pair $(\{\phi; \psi^1, \psi^2\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2\})$ associated with $(\{P; Q_1, Q_2\}, \{A; B_1, B_2\})$ is a pair of biorthogonal M-wavelet bases for $L_2(\mathbb{R})$. Moreover, the wavelets ψ^1, ψ^2, $\tilde{\psi}^1$, $\tilde{\psi}^2$ have vanishing moments of order 2.

Example 3. Let $\mathsf{M} = 3$, $m = 3$, $\ell = 0$. By Algorithm 1, we obtain

$$
\begin{aligned}
P(z) &= z^{-2}\left(\frac{1+z+z^2}{3}\right)^3; & A(z) &= 2 - z;\\
Q_1(z) &= -\frac{1}{27}z^{-2}(2 + 6z + 3z^2 + z^3); & B_1(z) &= (1-z)^3;\\
Q_2(z) &= -\frac{1}{27}z^{-2}(5 + 6z + 3z^2 + z^3); & B_2(z) &= z(1-z)^3.
\end{aligned}
$$

By Eq. (4.2), we have $\nu_2(P, \mathsf{M}) = 2.5$ and $\nu_2(A, \mathsf{M}) \approx -1.2325$. Hence by Corollary 5, the pair $(\{\phi; \psi^1, \psi^2\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2\})$ associated with $(\{P_3; Q_1, Q_2\}, \{A_3; B_1, B_2\})$ is a pair of biorthogonal M-wavelet bases in a pair $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ of dual Sobolev spaces for all $1.2325 < \tau < 2.5$.

Example 4. Let $\mathsf{M} = 3$, $m = 3$, $\ell = 3$. By Algorithm 1, we obtain

$$
\begin{aligned}
P(z) &= z^{-2}\left(\frac{1+z+z^2}{3}\right)^3;\\
Q_1(z) &= -\frac{1}{2187}z^{-8}(1-z)^3(7 + 42z + 147z^2 + 336z^3 + 546z^4 + 588z^5 + 378z^6 + 168z^7 + 48z^8 + 8z^9);\\
Q_2(z) &= -\frac{1}{2187}z^{-8}(1-z)^3(8 + 48z + 168z^2 + 378z^3 + 588z^4 + 546z^5 + 336z^6 + 147z^7 + 42z^8 + 7z^9);
\end{aligned}
$$

and

$$
\begin{aligned}
A(z) &= \frac{1}{3}z^{-6}\left(\frac{1+z+z^3}{3}\right)^3(7 - 34z + 57z^2 - 34z^3 + 7z^4);\\
B_1(z) &= (1-z)^3;\\
B_2(z) &= z(1-z)^3.
\end{aligned}
$$

By Eq. (4.2), we have $\nu_2(P, \mathsf{M}) = 2.5$ and $\nu_2(A, \mathsf{M}) \approx -0.5004$. Hence by Corollary 5, the pair $(\{\phi; \psi_1, \psi_2\}, \{\tilde{\phi}; \tilde{\psi}_1, \tilde{\psi}_2\})$ associated with $(\{P; Q_1, Q_2\}, \{A; B_1, B_2\})$ is a pair of biorthogonal M-wavelet bases in a pair $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ of dual Sobolev spaces for all $0.5004 < \tau < 2.5$. The cardinal spline wavelets ψ^1, ψ^2 have vanishing moments of order 3.

Example 5. Let $\mathsf{M} = 4$, $m = 3$, $\ell = 2$. By Algorithm 1, we obtain

$$
\begin{aligned}
P(z) &= z^{-3}\left(\frac{1+z+z^2+z^3}{4}\right)^3;\\
Q_1(z) &= \frac{1}{8192}z^{-7}(1-z)^2(35z^{11} + 175z^{10} + 525z^9 + 1225z^8 + 2198z^7 + 3150z^6 + 3570z^5 + 2730z^4\\
&\quad + 1575z^3 + 675z^2 + 225z + 45);\\
Q_2(z) &= \frac{1}{512}z^{-7}(1-z)^2(1+z)^5(5 + 25z^2 + 15z^4 + 3z^6);\\
Q_3(z) &= \frac{1}{8192}z^{-7}(1-z)^2(35z^{11} + 175z^{10} + 525z^9 + 1225z^8 + 2230z^7 + 3310z^6 + 4050z^5 + 3850z^4\\
&\quad + 2695z^3 + 1155z^2 + 385z + 77);
\end{aligned}
$$

and

$$A(z) = -\frac{1}{8}z^{-4}\left(\frac{1+z+z^2+z^3}{4}\right)^2(45-145z+127z^2-35z^3);$$
$$B_1(z) = (1-z)^3;$$
$$B_2(z) = z(1-z)^3;$$
$$B_3(z) = z^2(1-z)^3.$$

By Eq. (4.2), we have $\nu_2(P, \mathsf{M}) = 2.5$ and $\nu_2(A, \mathsf{M}) \approx -0.8256$. Hence by Corollary 5, the pair $(\{\phi; \psi^1, \psi^2, \psi^3\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3\})$ associated with the PR multirate system $(\{P; Q_1, Q_2, Q_3\}, \{A; B_1, B_2, B_3\})$ is a pair of biorthogonal M-wavelet bases in $(H^\tau(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ for all $0.8256 < \tau < 2.5$. The cardinal spline wavelets ψ^1, ψ^2, ψ^3 have vanishing moments of order 2.

Example 6. Finally, for $\mathsf{M} = 3$ and $m = 2, 3, 4$, we list the values

$$\nu_2(P_2, 3) = 1.5; \quad \nu_2(P_3, 3) = 2.5; \quad \nu_2(P_4, 3) = 3.5,$$

as well as, as given in Table 5.7 (see Appendix A), the values of $\nu_2(A^\ell_{m,3}, 3)$, for $\ell = 0, 1, \ldots, 9$.

Acknowledgments

The research of the first author (C.K. Chui) was supported by ARO Grant W911NF-11-1-0426. The research of the last author (X. Zhuang) and the work described in this paper was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 11304414). The authors would like to thank anonymous referees for their valuable comments and suggestions that help the improvement of the paper.

Appendix A. Tables

Table 5.1
The Laurent polynomials $A_2, \ldots, A_8$ for $\mathsf{M} = 2$.

m	$A_m(z)$
2	1
3	$\frac{1}{2}(3-z)$
4	$\frac{1}{2}z^{-1}(-1+4z-z^2)$
5	$\frac{1}{8}z^{-1}(-5+25z-15z^2+3z^3)$
6	$\frac{1}{8}z^{-2}(3-18z+38z^2-18z^3+3z^4)$
7	$\frac{1}{16}z^{-2}(7-49z+126z^2-98z^3+35z^4-5z^5)$
8	$\frac{1}{16}z^{-3}(-5+40z-131z^2+208z^3-131z^4+40z^5-5z^6)$

Table 5.2
The Laurent polynomials $A_2, \ldots, A_8$ for $\mathsf{M} = 3$.

m	$A_m(z)$
2	1
3	$2-z$
4	$\frac{1}{3}z^{-1}(-4+11z-4z^2)$
5	$\frac{1}{3}z^{-1}(-7+25z-20z^2+5z^3)$
6	$\frac{1}{3}z^{-2}(7-34z+57z^2-34z^3+7z^4)$
7	$\frac{1}{9}z^{-2}(35-201z+427z^2-392z^3+168z^4-28z^5)$
8	$\frac{1}{9}z^{-3}(-40+276z-768z^2+1073z^3-768z^4+276z^5-40z^6)$

Table 5.3
The Laurent polynomials $A_2, \ldots, A_8$ for $\mathsf{M} = 4$.

m	$A_m(z)$
2	1
3	$\frac{1}{2}(5 - 3z)$
4	$\frac{1}{2}z^{-1}(-5 + 12z - 5z^2)$
5	$\frac{1}{8}z^{-1}(-45 + 145z - 127z^2 + 35z^3)$
6	$\frac{1}{8}z^{-2}(63 - 282z + 446z^2 - 282z^3 + 63z^4)$
7	$-\frac{1}{16}z^{-2}(-273 + 1463z - 2982z^2 + 2842z^3 - 1297z^4 + 231z^5)$
8	$-\frac{1}{16}z^{-3}(429 - 2792z + 7387z^2 - 10064z^3 + 7387z^4 - 2792z^5 + 429z^6)$

Table 5.4
For $\mathsf{M} = 2$, the Laurent polynomials $\{Q_{m,1} : m = 2, 3, 4\}$.

m	$Q_{m,1}(z)$
2	$-\frac{1}{4}z^{-1}$
3	$-\frac{1}{16}z^{-1}(3 + z)$
4	$\frac{1}{16}z^{-2}(1 + 4z + z^2)$

Table 5.5
For $\mathsf{M} = 3$, the Laurent polynomials $\{Q_{m,\gamma} : m = 2, 3, 4, \gamma = 1, 2\}$.

m	$Q_{m,1}(z)$
2	$-\frac{1}{9}z^{-2}(1 + 2z)$
3	$-\frac{1}{27}z^{-2}(2 + 6z + 3z^2 + z^3)$
4	$\frac{1}{243}z^{-4}(4 + 16z + 40z^2 + 50z^3 + 20z^4 + 5z^5)$
m	$Q_{m,2}(z)$
2	$-\frac{1}{9}z^{-2}(2 + z)$
3	$-\frac{1}{27}z^{-2}(5 + 6z + 3z^2 + z^3)$
4	$\frac{1}{243}z^{-4}(5 + 20z + 50z^2 + 40z^3 + 16z^4 + 4z^5)$

Table 5.6
For $\mathsf{M} = 4$, the Laurent polynomials $\{Q_{m,\gamma} : m = 2, 3, 4, \gamma = 1, 2, 3\}$.

m	$Q_{m,1}(z)$
2	$-\frac{1}{16}z^{-3}(1 + 2z + 3z^2)$
3	$-\frac{1}{128}z^{-3}(5 + 15z + 30z^2 + 18z^3 + 9z^4 + 3z^5)$
4	$\frac{1}{512}z^{-6}(5 + 20z + 50z^2 + 100z^3 + 140z^4 + 140z^5 + 70z^6 + 28z^7 + 7z^8)$
m	$Q_{m,2}(z)$
2	$-\frac{1}{16}z^{-3}(1 + 4z + 2z^2)$
3	$-\frac{1}{256}z^{-3}(9 + 59z + 70z^2 + 42z^3 + 21z^4 + 7z^5)$
4	$\frac{1}{1024}z^{-6}(13 + 52z + 130z^2 + 260z^3 + 380z^4 + 300z^5 + 150z^6 + 60z^7 + 15z^8)$
m	$Q_{m,3}(z)$
2	$-\frac{1}{16}z^{-3}(3 + 2z + z^2)$
3	$-\frac{1}{128}z^{-3}(21 + 31z + 30z^2 + 18z^3 + 9z^4 + 3z^5)$
4	$\frac{1}{512}z^{-6}(7 + 28z + 70z^2 + 140z^3 + 140z^4 + 100z^5 + 50z^6 + 20z^7 + 5z^8)$

Table 5.7
The values of $\nu_2(A_m^\ell, \mathsf{M})$, for $\mathsf{M} = 3$, $m = 2, 3, 4$ and $\ell = 0, 1, \ldots, 9$.

$m\backslash\ell$	0	1	2	3	4
2	−0.5	−0.233	0.211	0.289	0.5
3	−1.23	−0.789	−0.711	−0.5	−0.494
4	−1.79	−1.71	−1.50	−1.49	−1.36
$m\backslash\ell$	5	6	7	8	9
2	0.506	0.641	0.642	0.745	0.745
3	−0.359	−0.358	−0.255	−0.255	−0.171
4	−1.36	−1.25	−1.25	−1.17	−1.17

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