Applied Mathematics Letters 91 (2019) 213–219

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Directional compactly supported box spline tight framelets with simple geometric structure

Bin Han [a], Tao Li [a], Xiaosheng Zhuang [b,*]

[a] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

[b] Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Hong Kong

ARTICLE INFO

Article history:
Received 5 September 2018
Received in revised form 15 December 2018
Accepted 15 December 2018
Available online 30 December 2018

Keywords:
Directional tight framelets
Tight framelet filter banks
Haar refinable functions
Refinable box splines

ABSTRACT

A *directional compactly supported d-dimensional Haar tight framelet* is constructed such that all its high-pass filters in its underlying tight framelet filter bank have only two nonzero coefficients with opposite signs and they exhibit totally $(3^d-1)/2$ directions in dimension d. Furthermore, applying the projection method to such a tight framelet, a *directional compactly supported box spline tight framelet* with simple geometric structure is built such that all the high-pass filters in its underlying tight framelet filter bank have only two nonzero coefficients with opposite signs as well. Moreover, such compactly supported box spline tight framelets can achieve arbitrarily high numbers of directions by using refinable box splines with increasing supports. Their application to pMRI with good performance is presented.

To capture singularities in high-dimensional data such as images/videos, directional representations play an important role both in theory and application. For example, see curvelets and shearlets in [1,2] and tensor product complex tight framelets in [3,4]. On the other hand, (refinable) box splines are widely used in both approximation theory and wavelet analysis. Motivated by the interesting example of a two-dimensional directional Haar tight framelet constructed in [5] which has impressive performance in parallel magnetic resonance imaging (pMRI), in this paper we construct compactly supported tight framelets with directionality and very simple geometric structures from the Haar refinable functions and all refinable box splines in all dimensions. All the high-pass filters in such directional tight framelets have only two nonzero coefficients with opposite signs. Consequently, all of them naturally exhibit directionality and their associated fast framelet transforms can be efficiently implemented through simple difference operations.

Let us first recall some definitions and notation. Let $\phi, \psi_1, \dots, \psi_s \in L_2(\mathbb{R}^d)$. We say that $\{\phi; \psi_1, \dots, \psi_s\}$ is *a (nonhomogeneous dyadic) tight framelet in* $L_2(\mathbb{R}^d)$ if

$$\|f\|^2_{L_2(\mathbb{R}^d)} = \sum_{k\in\mathbb{Z}^d} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}^d} |\langle f, 2^{jd/2}\psi_\ell(2^j\cdot - k)\rangle|^2, \qquad \forall\, f \in L_2(\mathbb{R}^d). \tag{1}$$

* Corresponding author.
E-mail addresses: bhan@ualberta.ca (B. Han), tao8@ualberta.ca (T. Li), xzhuang7@cityu.edu.hk (X. Zhuang).

https://doi.org/10.1016/j.aml.2018.12.016
0893-9659/

According to [6, Theorem 4.5.4], tight framelets are closely related to filter banks. By $l_0(\mathbb{Z}^d)$ we denote the set of all finitely supported sequences/filters $a = \{a(k)\}_{k\in\mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{C}$ on $\mathbb{Z}^d$. For a filter $a \in l_0(\mathbb{Z}^d)$, its Fourier series is defined to be $\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}^d} a(k)e^{-ik\cdot\xi}$ for $\xi \in \mathbb{R}^d$, which is a $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial. In particular, by $\boldsymbol{\delta}$ we denote *the Dirac sequence* such that $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for all $k \in \mathbb{Z}^d\backslash\{0\}$. For $\gamma \in \mathbb{Z}^d$, we also use the notation $\boldsymbol{\delta}_\gamma$ to stand for the sequence $\boldsymbol{\delta}(\cdot - \gamma)$, i.e., $\boldsymbol{\delta}_\gamma(\gamma) = 1$ and $\boldsymbol{\delta}_\gamma(k) = 0$ for all $k \in \mathbb{Z}^d\backslash\{\gamma\}$. Note that $\widehat{\boldsymbol{\delta}_\gamma}(\xi) = e^{-i\gamma\cdot\xi}$. For filters $a, b_1, \dots, b_s \in l_0(\mathbb{Z}^d)$, we say that a filter bank $\{a; b_1, \dots, b_s\}$ is *a (d-dimensional dyadic) tight framelet filter bank* if

$$\sum_{k\in\mathbb{Z}^d} a(\gamma+2k)\overline{a(n+\gamma+2k)}+\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}^d} b_\ell(\gamma+2k)\overline{b_\ell(n+\gamma+2k)} = 2^{-d}\boldsymbol{\delta}(n), \qquad \forall\, \gamma \in \{0,1\}^d,\ \forall n \in \mathbb{Z}^d. \tag{2}$$

In the frequency domain, it is equivalent to $\widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi\omega)} + \sum_{\ell=1}^{s} \widehat{b_\ell}(\xi)\overline{\widehat{b_\ell}(\xi+\pi\omega)} = \boldsymbol{\delta}(\omega)$, $\forall\, \xi \in \mathbb{R}^d, \omega \in \{0,1\}^d$. Eq. (1) is just the *perfect reconstruction property* of a tight framelet filter bank ([6, Theorems 1.1.1 and 1.1.4]).

Let $a, b_1, \dots, b_s \in l_0(\mathbb{Z}^d)$ and assume that $\widehat{a}(0) = \sum_{k\in\mathbb{Z}^d} a(k) = 1$. Then we can define compactly supported tempered distributions ϕ and $\psi_1, \dots, \psi_s$ on $\mathbb{R}^d$ through

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi) \qquad \text{and} \qquad \widehat{\psi_\ell}(\xi) = \widehat{b_\ell}(\xi/2)\widehat{\phi}(\xi/2), \qquad \xi \in \mathbb{R}^d, \ell = 1, \dots, s. \tag{3}$$

It is known in [7, Corollary 12 and Theorem 17] and [6, Theorem 4.5.4] that $\{\phi; \psi_1, \dots, \psi_s\}$ is a tight framelet in $L_2(\mathbb{R}^d)$ if and only if $\{a; b_1, \dots, b_s\}$ is a tight framelet filter bank. Also cf. [8–11] for related results. Further see [7,9–15] and many references therein for extensive investigation on tight framelets derived from refinable functions. The tempered distribution ϕ is called *a refinable function* satisfying the refinement equation $\widehat{\phi}(\xi) = \widehat{a}(\xi/2)\widehat{\phi}(\xi/2)$ for $\xi \in \mathbb{R}^d$ with the refinement filter a.

This paper is motivated by the interesting paper [5], where a two-dimensional directional Haar tight framelet has been constructed and applied with impressive performance to pMRI. Applying finite linear combinations to the standard tensor product two-dimensional Haar wavelet, the authors constructed in [5, (3.5)] a two-dimensional directional tight framelet filter bank $\{a^H; b_1, \dots, b_6\}$ with

$$\begin{aligned}
a^H &= \frac{1}{4}(\boldsymbol{\delta}_{(0,0)} + \boldsymbol{\delta}_{(0,1)} + \boldsymbol{\delta}_{(1,0)} + \boldsymbol{\delta}_{(1,1)}), \qquad & b_1 &= \frac{1}{4}(\boldsymbol{\delta}_{(0,0)} - \boldsymbol{\delta}_{(0,1)}), \qquad & b_2 &= \frac{1}{4}(\boldsymbol{\delta}_{(0,0)} - \boldsymbol{\delta}_{(1,0)}),\\
b_3 &= \frac{1}{4}(\boldsymbol{\delta}_{(0,0)} - \boldsymbol{\delta}_{(1,1)}), & b_4 &= \frac{1}{4}(\boldsymbol{\delta}_{(0,1)} - \boldsymbol{\delta}_{(1,0)}), & b_5 &= \frac{1}{4}(\boldsymbol{\delta}_{(0,1)} - \boldsymbol{\delta}_{(1,1)}),\\
b_6 &= \frac{1}{4}(\boldsymbol{\delta}_{(1,0)} - \boldsymbol{\delta}_{(1,1)}).
\end{aligned}$$

Since each high-pass filter has only two nonzero coefficients with opposite signs, it naturally has directionality and very simple structures. To deal with problems in higher dimensions such as video inpainting/denoising, it is very natural to ask

Q1. Is it possible to construct a directional Haar tight framelet for every dimension such that each high-pass filter has only two nonzero coefficients with opposite signs?

A similar construction method/argument as in [5] will quickly run into difficulty, since there are so many possible linear combinations even at the dimension three. Fortunately, adopting a geometric viewpoint, we can positively answer the question **Q1** completely as follows:

Theorem 1. *Let $a^H = 2^{-d}\sum_{\gamma\in\{0,1\}^d} \boldsymbol{\delta}_\gamma$ be the d-dimensional Haar low-pass filter. Define the high-pass filters $b_1, \dots, b_s$ with $s := \binom{2^d}{2} = 2^{d-1}(2^d - 1)$ in the following way: $2^{-d}(\boldsymbol{\delta}_{\gamma_1} - \boldsymbol{\delta}_{\gamma_2})$ for all undirected edges with endpoints $\gamma_1, \gamma_2 \in \{0,1\}^d$ and $\gamma_1 \neq \gamma_2$. Then $\{a^H; b_1, \dots, b_s\}$ is a tight framelet filter bank such that* *all the high-pass filters $b_1, \dots, b_s$ have directionality and exhibit $\frac{1}{2}(3^d - 1)$ directions in dimension d. Define functions ϕ and $\psi_1, \dots, \psi_s$ as in (3). Then $\{\phi; \psi_1, \dots, \psi_s\}$ is a d-dimensional directional compactly supported Haar tight framelet in $L_2(\mathbb{R}^d)$.*

Proof. To prove that $\{a^H; b_1, \dots, b_s\}$ is a tight framelet filter bank, we have to check the conditions in (2). Since all the filters are supported inside $\{0,1\}^d$, it is trivial to observe that all the filters $a^H, b_1, \dots, b_s$ vanish at the position $\gamma + 2k$ for all $\gamma \in \{0,1\}^d$ and for all $k \in \mathbb{Z}^d \backslash \{0\}$. Therefore, Eqs. (2) become

$$a^H(\gamma)\overline{a^H(n+\gamma)} + \sum_{\ell=1}^{s} b_\ell(\gamma)\overline{b_\ell(n+\gamma)} = 2^{-d}\boldsymbol{\delta}(n), \qquad \gamma \in \{0,1\}^d, n \in \mathbb{Z}^d. \tag{4}$$

Case 1: $n = 0$. Then $|a^H(\gamma)|^2 = 2^{-2d}$. By the definition of the high-pass filters, there are totally $2^d - 1$ filters whose supports contain the point γ. Consequently, $\sum_{\ell=1}^{s} |b_\ell(\gamma)|^2 = 2^{-2d}(2^d - 1)$. Thus, we have $|a^H(\gamma)|^2 + \sum_{\ell=1}^{s} |b_\ell(\gamma)|^2 = 2^{-2d} + 2^{-2d}(2^d - 1) = 2^{-d}$ which proves (4) with $n = 0$.

Case 2: $n \neq 0$ and $n + \gamma \notin \{0,1\}^d$. For this case, since all the filters are supported inside $\{0,1\}^d$, we trivially have $a^H(n+\gamma) = 0$ and $b_\ell(n+\gamma) = 0$ for all $\ell = 1, \dots, s$. Hence, (4) is trivially true for $n \neq 0$ and $n + \gamma \notin \{0,1\}^d$.

Case 3: $n \neq 0$ and $n + \gamma \in \{0,1\}^d$. Then n and $n + \gamma$ are two distinct points in $\{0,1\}^d$. By the definition of the high-pass filters, there exists exactly one integer j with $1 \leqslant j \leqslant s$ such that $b_j(\gamma)\overline{b_j(n+\gamma)} = -2^{-2d}$ and $b_\ell(\gamma)\overline{b_\ell(n+\gamma)} = 0$ for all $\ell \in \{1, \dots, s\} \backslash \{j\}$. Noting that $a^H(\gamma) = a^H(n+\gamma) = 2^{-d}$, we conclude

$$a^H(\gamma)\overline{a^H(n+\gamma)} + \sum_{\ell=1}^{s} b_\ell(\gamma)\overline{b_\ell(n+\gamma)} = a^H(\gamma)\overline{a^H(n+\gamma)} + b_j(\gamma)\overline{b_j(n+\gamma)} = 2^{-2d} - 2^{-2d} = 0,$$

which proves (4) for $n \neq 0$ and $n + \gamma \in \{0,1\}$.

Therefore, $\{a^H; b_1, \dots, b_s\}$ is a tight framelet filter bank. Since each high-pass filter has only two nonzero coefficients with opposite signs, all the high-pass filters $b_1, \dots, b_s$ trivially have directionality. We now count the total number of directions of all the high-pass filters. Note that the direction of a high-pass filter $2^{-d}(\boldsymbol{\delta}_{\gamma_1} - \boldsymbol{\delta}_{\gamma_2})$ can be represented by the vector $v = \gamma_1 - \gamma_2$ or $v = \gamma_2 - \gamma_1$. Such a direction vector v is unique if we additionally require that the first nonzero entry of v should be positive. Note that $v \in \{-1,0,1\}^d \backslash \{0\}$ with each entry of v belonging to $\{-1,0,1\}$. Let S be the set of all the nonzero vectors $v \in \{-1,0,1\}^d$ such that the first nonzero entry of v is positive (i.e., 1). Hence, any direction vector v of a high-pass filter belongs to S. Conversely, for every vector $v \in S$, we can uniquely write $v = \gamma_1 - \gamma_2$ with $\gamma_1, \gamma_2 \in \{0,1\}^d$ and $\gamma_1 + \gamma_2 \in \{0,1\}^d$ by separating the positive and negative entries of v. Therefore, the vector v represents the direction of the high-pass filter $2^{-d}(\boldsymbol{\delta}_{\gamma_1} - \boldsymbol{\delta}_{\gamma_2})$. Hence, the total number of directions of all the high-pass filters is equal to the cardinality of the set S. Consider the subset S_j whose elements are in S with the first nonzero entry at the position j for $j = 1, \dots, d$. Clearly, the cardinality of S_j is 3^{d-j}. Since S is the disjoint union of $S_1, \dots, S_d$, we conclude that the cardinality of S is $3^{d-1} + 3^{d-2} + \cdots + 3^{d-d} = (3^d - 1)/2$. $\square$

The tight framelet filter bank in Theorem 1 with $d = 1$ is just the standard Haar orthogonal wavelet filter bank and the case $d = 2$ recovers the directional Haar tight framelet filter bank in [5].

One possible shortcoming of Theorem 1 is that all the Haar refinable functions $\chi_{[0,1]^d}$ are discontinuous and often introduce unpleasant block effects in data/image processing. Therefore, smooth refinable functions and directional smooth tight framelets are often preferred in applications. Refinable box splines can be made arbitrarily smooth and are widely used in approximation theory and wavelet analysis. This naturally leads us to ask

Q2. Can we construct directional compactly supported tight framelets in $L_2(\mathbb{R}^d)$ with simple geometric structure from every refinable box spline in every dimension?

Applying the projection method in [16,17] to Theorem 1, we positively answer the question **Q2** painlessly. To do so, let us recall the definition of box splines which are closely linked to the projection method. Let P be a $d \times n$ real-valued matrix of rank d with $d \leqslant n$. A box spline M_P with the $d \times n$ direction matrix P is defined to be

$$\widehat{M_P}(\xi) := \prod_{k \in P} \frac{1 - e^{-ik\cdot\xi}}{ik \cdot \xi}, \qquad \xi \in \mathbb{R}^d, \tag{5}$$

where $k \in P$ means that k is a column vector of P and k goes through all the columns of P once and only once. A box spline can be also defined through the projection method. For an integrable function $f \in L_1(\mathbb{R}^n)$, we can define the projected function Pf on $\mathbb{R}^d$ by $\widehat{Pf}(\xi) := \widehat{f}(P^{\mathsf{T}}\xi), \xi \in \mathbb{R}^d$. Since $\widehat{f}$ is continuous on $\mathbb{R}^n$, the function $\widehat{Pf}$ is a well-defined continuous function on $\mathbb{R}^d$. In the spatial domain, the definition of the d-dimensional projected function Pf can be equivalently expressed as $[Pf](x) = \frac{1}{\sqrt{\det(PP^{\mathsf{T}})}} \int_{P^{-1}x} f dS, x \in \mathbb{R}^d$, where S is the surface element on the superplane $P^{-1}x := \{y \in \mathbb{R}^n \ : \ Py = x\}$. In fact, for $f \in L_1(\mathbb{R}^n)$, the projected function $Pf \in L_1(\mathbb{R}^d)$ (see [16,17]). Note that $\widehat{\chi_{[0,1]^n}}(\xi) = \prod_{k\in\{0,1\}^n} \frac{1-e^{-ik\cdot\xi}}{ik\cdot\xi}$ and $\widehat{P\chi_{[0,1]^n}}(\xi) = \widehat{\chi_{[0,1]^n}}(P^{\mathsf{T}}\xi) = \prod_{k\in\{0,1\}^n} \frac{1-e^{-ik\cdot(P^{\mathsf{T}}\xi)}}{ik\cdot(P^{\mathsf{T}}\xi)} = \prod_{k\in\{0,1\}^n} \frac{1-e^{-i(Pk)\cdot\xi}}{i(Pk)\cdot\xi} = \widehat{M_P}(\xi)$. Hence, the box spline M_P is nothing else but the projected function $P\chi_{[0,1]^n}$ of the n-dimensional Haar function along the direction matrix P, refer to the book [18] for extensive study on box splines.

The projection method can be also applied to filters on $\mathbb{Z}^n$ provided that P is a $d \times n$ integer matrix. For an n-dimensional filter $a \in l_0(\mathbb{Z}^n)$, the projected filter $Pa \in l_0(\mathbb{Z}^d)$ is defined by

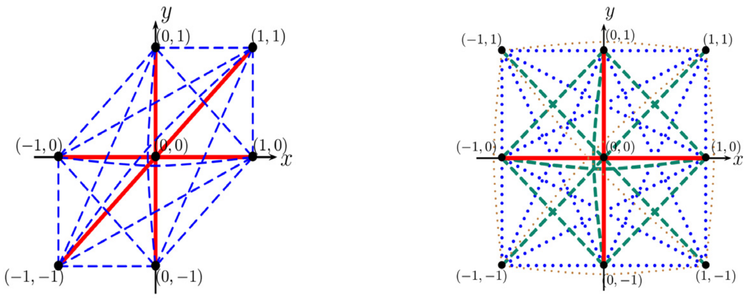
$$\widehat{Pa}(\xi) := \widehat{a}(P^{\mathsf{T}}\xi), \qquad \xi \in \mathbb{R}^d, \quad \text{or equivalently,} \quad [Pa](j) = \sum_{k \in P^{-1}j} a(k), \qquad j \in \mathbb{Z}^d, \tag{6}$$

where $P^{-1}j := \{k \in \mathbb{Z}^n \ : \ Pk = j\}$. Because $\widehat{a}$ is a $2\pi\mathbb{Z}^n$-periodic trigonometric polynomial and P is an integer matrix, $\widehat{Pa}$ is a well-defined $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial. If a^H is the n-dimensional Haar low-pass filter, then we define $a_P := Pa^H$ to be the box spline refinement filter/mask for the box spline M_P with a $d \times n$ direction matrix. For an integer projection matrix P, the box spline function M_P in (5) is refinable: $\widehat{M_P}(2\xi) = \widehat{a_P}(\xi)\widehat{M_P}(\xi)$, since $P\chi_{[0,1]^n} = M_P$ and the n-dimensional Haar function $\chi_{[0,1]^n}$ is refinable: $\widehat{\chi_{[0,1]^n}}(2\xi) = \widehat{a^H}(\xi)\widehat{\chi_{[0,1]^n}}(\xi)$.

Suppose that P is a $d \times n$ integer matrix of rank d with $d \leqslant n$ satisfying $P^{\mathsf{T}}(\mathbb{Z}^d \backslash [2\mathbb{Z}^d]) \subseteq \mathbb{Z}^n \backslash [2\mathbb{Z}^n]$. For every tight framelet $\{\phi; \psi_1, \dots, \psi_s\}$ in $L_2(\mathbb{R}^n)$ regardless of whether $\{\phi; \psi_1, \dots, \psi_s\}$ has an associated underlying filter bank or not, it is known in [16, Theorem 4] and [17, Theorem 2.3 and Corollary 5.3] that $\{P\phi; P\psi_1, \dots, P\psi_s\}$ must be a tight framelet in $L_2(\mathbb{R}^d)$. Similarly, for every n-dimensional tight framelet filter bank $\{a; b_1, \dots, b_s\}$, then $\{Pa; Pb_1, \dots, Pb_s\}$ must be a d-dimensional tight framelet filter bank. The condition that $P^{\mathsf{T}}(\mathbb{Z}^d \backslash [2\mathbb{Z}^d]) \subseteq \mathbb{Z}^n \backslash [2\mathbb{Z}^n]$ is equivalent to saying ([17, Theorem 2.5]) that the filter a_P has the sum rules of order at least one, i.e., $\sum_{k\in\mathbb{Z}^d} a_P(\gamma + 2k) = 2^{-d}$ for all $\gamma \in \{0,1\}^d$. If such a condition fails, then the box spline filter a_P does not have any sum rules and therefore, no tight framelets can be ever derived from the box spline M_P, refer to [17, Theorem 2.5] for more details as well as Han [16,17] for some applications of the projection method in wavelet analysis.

Note that if a high-pass filter b has only two nonzero coefficients with opposite signs, then either $Pb = 0$ or Pb has only two nonzero coefficients with opposite signs. Applying the projection method to the Haar tight framelets in Theorem 1, we have the following result positively answering **Q2**.

Theorem 2. *Let P be a $d \times n$ integer matrix of rank d with $d \leqslant n$ such that $P^{\mathsf{T}}(\mathbb{Z}^d \backslash [2\mathbb{Z}^d]) \subseteq \mathbb{Z}^n \backslash [2\mathbb{Z}^n]$. Let $\{a^H; b_1, \dots, b_s\}$ with $s := \binom{2^n}{2} = 2^{n-1}(2^n - 1)$ be the n-dimensional Haar tight framelet filter bank constructed in Theorem 1. Then $\{Pa^H; Pb_1, \dots, Pb_s\}$ is a d-dimensional tight framelet filter bank with Pa^H being the box spline refinement filter a_P such that all the high-pass filters have only two nonzero coefficients with opposite signs. Define ϕ and $\psi_1, \dots, \psi_s$ as in (3) with a and $b_1, \dots, b_s$ being replaced by Pa^H and*



(a) (b)

Fig. 1. Each edge connecting every two vertices indicates a high-pass filter with coefficients of the same weight but opposite signs at its endpoints. (a) Example 1. Weight for each dashed blue edge (total 15) is $\frac{1}{8}$, while weight for each solid red edge (total 6) is $\frac{\sqrt{2}}{8}$. The total number of all the high-pass filters is 21 with 6 solid red edges and 15 dashed blue edges. The total number of directions/slopes for the 21 high-pass filters is 6 with angles 0° (5 edges), 26.6° (= $\arctan(\frac{1}{2})$, 2 edges), 45° (5 edges), 63.4° (= $\arctan(2)$, 2 edges), 90° (5 edges), and −45° (2 edges). (b) Example 2. Weight for each small-dotted brown arc-edge (total 6) is $\frac{1}{16}$, weight for each dotted blue edge (total 16) is $\frac{\sqrt{2}}{16}$, weight for each dashed green edge (total 10) is $\frac{1}{8}$, and weight for each solid red edge (total 4) is $\frac{\sqrt{2}}{8}$. The total number of directions/slopes for the 36 high-pass filters is 8 with angles 0° (9 edges), ±26.6° (2 edges each), ±45° (5 edges each), ±63.4° (= $\arctan(\frac{1}{2})$, 2 edges each), and 90° (9 edges).

$Pb_1, \dots, Pb_s$, respectively. Then $\{\phi; \psi_1, \dots, \psi_s\}$ is a d-dimensional directional compactly supported tight framelet in $L_2(\mathbb{R}^d)$ with ϕ being the box spline M_P in (5) having the direction matrix P.

Let $\{a; b_1, \dots, b_s\}$ be a d-dimensional tight framelet filter bank. If $b_1 = c_1 b$ and $b_2 = c_2 b$ for some constants $c_1, c_2 \in \mathbb{C}$ and $b \in l_0(\mathbb{Z}^d)$, then it is trivial that $\{a; \sqrt{|c_1|^2 + |c_2|^2} b, b_3, \dots, b_s\}$ is a tight framelet filter bank. That is, we can combine high-pass filters which are almost the same up to a multiplicative constant. Hence, the number of high-pass filters in Theorem 2 can be reduced. We would like to point out that a further development on Theorem 2 can be found in [19]. Now we provide a geometric construction for the box spline tight framelet filter bank in Theorem 2 but with similar filters combined to reduce the number of filters. First we calculate the support $\mathrm{supp}(a_P)$ of the box spline filter a_P, which must be the set $P\{0,1\}^n \subseteq \mathbb{Z}^d$. Now the set $\{0,1\}^n$ of the vertices of the unit cube $[0,1]^n$ can be written as a disjoint union of the subsets $P^{-1}k, k \in \mathrm{supp}(a_P)$. Then all the high-pass filters are constructed in the following way: For every pair of two distinct points $\gamma_1, \gamma_2 \in \mathrm{supp}(a_P)$, construct the high-pass filter $2^{-n}\sqrt{(\#P^{-1}\gamma_1)(\#P^{-1}\gamma_2)}(\boldsymbol{\delta}_{\gamma_1} - \boldsymbol{\delta}_{\gamma_2})$, where $\#S$ is the cardinality of a set S. Clearly, there are totally $\binom{m}{2}$ with $m := \#\mathrm{supp}(a_P)$ number of high-pass filters. We provide two examples to illustrate this construction.

Example 1. Let P be the 2×3 integer matrix $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. Then M_P is the three-direction interpolating linear box spline with the refinement filter given by $\widehat{a_P}(\xi_1, \xi_2) = 2^{-3}(1 + e^{-i\xi_1})(1 + e^{-i\xi_2})(1 + e^{i(\xi_1+\xi_2)})$. Note that $\mathrm{supp}(a_P) = \{-1,0,1\}^2 \backslash \{(-1,1)^\mathsf{T}, (1,-1)^\mathsf{T}\}$ with $\#\mathrm{supp}(a_P) = 7$ and $P^{-1}(0,0)^\mathsf{T} = \{(0,0,0)^\mathsf{T}, (1,1,1)^\mathsf{T}\}$, while $P^{-1}\gamma$ contains only one point in $\mathbb{Z}^3$ for every $\gamma \in \mathrm{supp}(a_P) \backslash \{(0,0)^\mathsf{T}\}$. Consequently, there are totally 21 (by $\binom{7}{2} = 21$) high-pass filters given by $b_1 = \frac{\sqrt{2}}{8}(\boldsymbol{\delta}_{(1,0)} - \boldsymbol{\delta})$, $b_2 = \frac{\sqrt{2}}{8}(\boldsymbol{\delta}_{(1,1)} - \boldsymbol{\delta})$, $b_3 = \frac{\sqrt{2}}{8}(\boldsymbol{\delta}_{(0,1)} - \boldsymbol{\delta})$, $b_4 = \frac{\sqrt{2}}{8}(\boldsymbol{\delta}_{(-1,0)} - \boldsymbol{\delta})$, $b_5 = \frac{\sqrt{2}}{8}(\boldsymbol{\delta}_{(-1,-1)} - \boldsymbol{\delta})$, $b_6 = \frac{\sqrt{2}}{8}(\boldsymbol{\delta}_{(0,-1)} - \boldsymbol{\delta})$ and all the rest 15 filters $b_7, \dots, b_{21}$ are given by choosing a pair of two distinct points from $\mathrm{supp}(a_P) \backslash \{(0,0)^\mathsf{T}\} = \{(1,0)^\mathsf{T}, (1,1)^\mathsf{T}, (0,1)^\mathsf{T}, (-1,0)^\mathsf{T}, (-1,-1)^\mathsf{T}, (0,-1)^\mathsf{T}\}$ with value $\frac{1}{8}$ at one point and $-\frac{1}{8}$ at the other. This filter bank $\{a_P; b_1, \dots, b_{21}\}$ occupies 6 directions in dimension two. See Fig. 1a for details.

Example 2. Let P be the 2×4 integer matrix $P = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$. Then M_P is the tensor product of the piecewise linear B-spline with the refinement filter a_P given by $\widehat{a_P}(\xi_1, \xi_2) = 2^{-2}|1 + e^{-i\xi_1}|^2 |1 + e^{-i\xi_2}|^2$. Note that $\mathrm{supp}(a_P) = \{-1,0,1\}^2$ with $\#\mathrm{supp}(a_P) = 9$ and similar to the calculation of Example 1,

Table 1
pMRI performance in terms of normalized mean square error (NMSE, the smaller NMSE, the better performance) for the Shepp–Logan phantom as detailed in [5, Sec. 5.1]. "Tensor Haar" is the standard tensor product Haar orthogonal filter bank, "D-Haar" is the 2-dimensional directional Haar tight framelet filter bank in Theorem 1 and in [5], Examples 1 and 2 refer to the directional tight framelet filter banks given there.

Systems	Tensor Haar	D-Haar	Example 1	Example 2
NMSE	4.177E−04	2.326E−04	2.133E−04	2.017E−04

there are total 36 (by $\binom{9}{2} = 36$) high-pass filters that can be written in 5 groups: (i) $\frac{1}{8}(\boldsymbol{\delta}_\gamma - \boldsymbol{\delta}), \gamma \in S_1$, (ii) $\frac{\sqrt{2}}{8}(\boldsymbol{\delta}_\gamma - \boldsymbol{\delta}), \gamma \in S_2$, (iii) $\frac{\sqrt{2}}{16}(\boldsymbol{\delta}_{\gamma_1} - \boldsymbol{\delta}_{\gamma_2}), \gamma_1 \in S_1, \gamma_2 \in S_2$, (iv) $\frac{1}{16}(\boldsymbol{\delta}_{\gamma_1} - \boldsymbol{\delta}_{\gamma_2}), \gamma_1 \neq \gamma_2, \gamma_1, \gamma_2 \in S_1$, (v) $\frac{1}{8}(\boldsymbol{\delta}_{\gamma_1} - \boldsymbol{\delta}_{\gamma_2}), \gamma_1 \neq \gamma_2, \gamma_1, \gamma_2 \in S_2$, with 4, 4, 16, 6, and 6 filters for each group, respectively, and with $S_1 := \{(1,-1)^\mathsf{T}, (-1,1)^\mathsf{T}, (1,1)^\mathsf{T}, (-1,-1)^\mathsf{T}\}$ and $S_2 := \{(1,0)^\mathsf{T}, (0,1)^\mathsf{T}, (-1,0)^\mathsf{T}, (0,-1)^\mathsf{T}\}$. This tight framelet filter bank $\{a_P; b_1, \dots, b_{36}\}$ occupies 8 directions in dimension two. See Fig. 1b for details.

We finally compare the performance of our newly developed directional tight framelets in a pMRI (parallel magnetic resonance imaging) application, which is a noninvasive medical imaging technique used in radiology to investigate the anatomy and physiology of the human body. The pMRI uses an array of surface coils to acquire multiple sets of undersampled k-space data simultaneously to significantly accelerate the MRI process as well as improve the quality of reconstruction images, which can be modeled as: $g_\ell = \mathcal{F}^{-1}\mathcal{P}\mathcal{F}\mathcal{S}_\ell u + \eta_\ell, \ell = 1, \dots, p$, where u is the desired image, S_ℓ is the coil sensitivity, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform matrix and its inverse, $\mathcal{P}$ is the sampling matrix, η_ℓ is the white Gaussian noise, and p is the number of coils. We test the pMRI reconstruction process for the Shepp–Logan phantom (512×512 image) using the same experiment setting as detailed in [5, Sec. 5.1] and the same algorithm provided by [5, Algorithm 1] with several directional tight framelet filter banks constructed in this paper. The result is reported in Table 1 and one can easily draw the conclusion that directionality does improve performance in pMRI applications. We shall explore more applications of the directional tight framelets in Theorems 1 and 2 and report more details elsewhere.

Acknowledgments

We would like to thank anonymous reviewers for their valuable comments and suggestions that greatly helped us to improve the presentation of the paper. We would like to thank Dr. Yan-ran Li for kindly providing us their code used in [5]. This work appeared in arXiv as *arXiv:1708.08421*. Research of B. Han was supported in part by NSERC, Canada under Grant RGP 228051. Research of T. Li was supported by a Mitacs global link project. Research of X. Zhuang was supported in part by the Research Grants Council of Hong Kong (Project No. CityU 11302218).

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